Palestine Journal of Mathematics

Vol. 9(2)(2020) , 740–748

HYPER TERMINAL WIENER INDEX OF SOME DENDRIMER GRAPHS AND DETOUR SATURATED TREES

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C12;

Keywords and phrases: Dendrimers, Topological indices, Wiener index, Terminal Wiener index and Hyper Terminal, claw, double claw.

Abstract The hyper terminal Wiener index of graph G is defined as $HTW(G) = \frac{1}{2}\sum_{1\leq i<j\leq k}\left[d(v_i,v_j)^2 + d(v_i,v_j)\right]$. In this paper the hyper terminal Wiener index of two kinds of dendrimer graphs and detour saturated trees are computed

1 Introduction

Let G be a simple connected graph with vertex set V and edge set E. The degree of the vertex u is the number of edges incident to the vertex. It is denoted by $deg(u)$. If $deg(u)$ =1 then u is called pendant vertex or terminal vertex. The length of the path between two vertices is the number of edges on that path. The distance between two vertices denoted by $d(u,v)$ and it is the length of shortest path between them. For the standard on its terminology and notation we follow [1]. The graphs consider here are simple, finite and connected unless mentioned otherwise. A topological index is a numeric quantity of a molecule that is mathematically derived in an unambiguous way from the structural graph of a molecule. In theoretical chemistry, topological indices are used for modelling physical, pharmacological, biological and other properties of chemical compounds. The usage of topological indices in chemistry began in 1947 when chemist Harold Wiener developed the most widely known topological descriptor, Wiener index, and used it to determine physical properties of types of alkanes known as paraffin [2]. Wiener index is defined as the sum of distance between all pair vertices of a graph.

$$W(G) = \sum_{(u,v)\subseteq V(G)} d(u,v\backslash G)$$

For details on its chemical applications of Wiener index one may refer to [3,4,5,6]. Among all the trees on n vertices $K_{1,n-1}$, on n vertices has the lowest Wiener index and P_n on n vertices has largest Wiener index. If graph has k-pendant vertices then the terminal Wiener index $TW(G)$ is defined as the sum of distance between all pair of pendant vertices[7,8].

$$TW(G) = \sum_{1\leq i<j\leq k} d(v_i, v_j\backslash G)$$

Recently Shirkol et. al [9] have introduced new indices namely terminal type Wiener index $TW_\lambda(G)$ and Hyper terminal Wiener index $HTW(G)$ as

$$TW_\lambda = TW_\lambda(G) = \sum_{k\geq 1} d_T(G,k)\cdot k^\lambda$$

Where $d_T(G,k)$ is the number of pairs of pendant vertices of the graph G whose distance is k, λ is some real number.

$$HTW = HTW(G) = \frac{1}{2}\sum_{1\leq i<j\leq k}\left[d(v_i,v_j)^2 + d(v_i,v_j)\right] = \frac{1}{2}\sum_{1\leq i<j\leq k} d(v_i,v_j)[d(v_i,v_j)+1]$$

where $d(v_i,v_j)$is the distance between pair of pendant vertices . The Hyper terminal Wiener index shows good correlation, with Wiener index having correlation coefficient 0.74 for alkane isomers, compared to Wiener index vs terminal Wiener index with correlation coefficient 0.56. The comparison is as shown in Figure1.

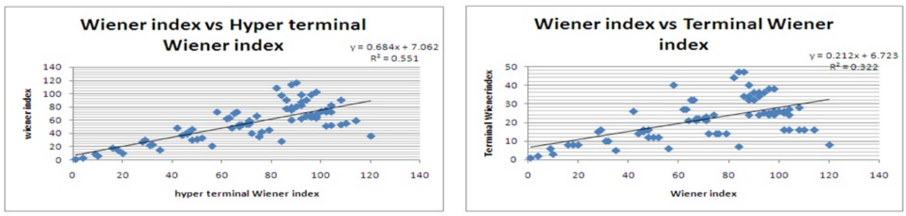

Figure 1:Correlation between Wiener index *vs* Hyper terminal Wiener index and Correlation between Wiener index *vs* terminal Wiener index

2 Dendrimer Graphs

2.1 THE HYPER TERMINAL WIENER INDEX OF COMPLETE BINARY TREE.

Let $G_n = (V, E)$ be the graph with vertex set V and edge set E as shown in Figure 2. This graph starts with one vertex u_0 which connects to two other vertices such that each one of these two vertices connects to two other vertices and so on. The vertices at equal distance from u_0 are located in the same level .Thus level of u_0 is 0. The level of $u_1$is 1 and so on. The maximum number of pendant vertices in the level l is 2^l .This graph is called complete binary tree.

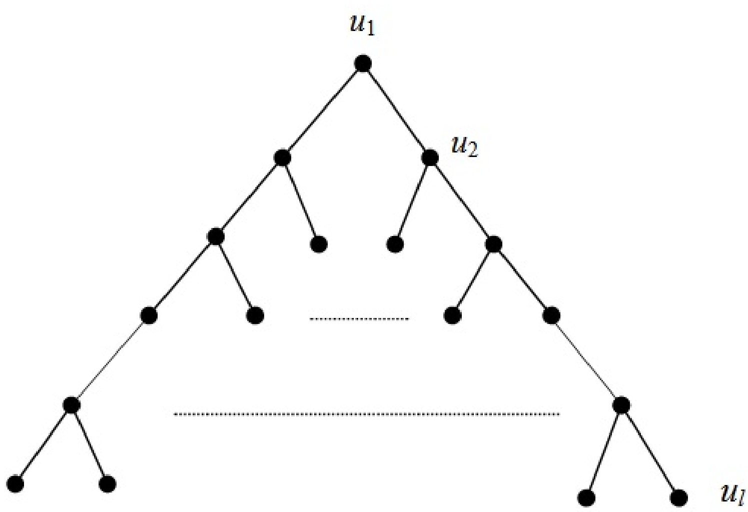

Figure 2. The first dendrimer graph G_n

Proposition 2.1. *For a complete binary tree $T = G_n = (V, E)$*

$$HTW(G_n) = 2^{l-2} \sum_{m=0}^{l-1} 2^{m+1}(2m^2 + 5m + 3)$$

Proof. By the definition of hyper terminal Wiener index

$$HTW(G) = \frac{1}{2} \sum_{1 \le i < j \le k} [d(v_i, v_j)^2 + d(v_i, v_j)]$$

At level 1,

$$HTW(G_n) = \frac{1}{2}\left[2^2 + 2\right] = 3$$

At level 2

$$HTW(G_n) = \frac{1}{2}\left[\underbrace{(2^2+2)}_{2times} + \underbrace{(4^2+4)}_{4times}\right] = 46$$

At level 3

$$HTW(G_n) = \frac{1}{2}\left[\underbrace{(2^2+2)}_{4times} + \underbrace{(4^2+4)}_{8times} + \underbrace{(6^2+6)}_{16times}\right] = 428$$

and so on. The graph has 2^l pendant vertices in

$$l^{th}$$

level. Now we prove the same by mathematical induction
Let at level 1
$HTW(G_n) = \frac{1}{2}[2(3)] = 3$
Thus the result is true for level 1.
At level 2

$$HTW(G_n) = 2^0 \sum_{m=0}^{1} 2^{m+1}(2m^2+5m+3)$$
$$= 2(3) + 4(10) = 46$$

Thus the result is true for level 2.
Let result is true for level l

$$HTW(G_n) = 2^{l-2} \sum_{m=0}^{l-1} 2^{m+1}(2m^2+5m+3)$$
$$= \frac{2^{l-1}}{2} \sum_{m=0}^{l-1} 2^{m+1}(m+1)(2m+3)$$
$$= \frac{2^{l-1}}{2}\left[\sum_{m=0}^{l-1} 2^m(2m+2)(2m+3)\right]$$
$$= \frac{2^{l-1}}{2}\left[2^0(2\cdot 3) + 2^1(4\cdot 5)+, ...2^{l-1}(2l\cdot(2l+1))\right]$$
$$= \frac{2^{l-1}}{2}\left[2^0(2^2+2) + 2^1(4^2+4) + ... + 2^{l-1}((2l)^2+2l)\right]$$
$$= \frac{1}{2}\left[\underbrace{(2^2+2)}_{2^{l-1}times} + \underbrace{(4^2+4)}_{2^l times} + ... + \underbrace{((2l)^2+2l)}_{2^{2l-2}times}\right]$$

Now we will prove that it is true for level $l+1$

$$HTW(G_n) = \frac{1}{2}\left[\underbrace{(2^2+2)}_{2^l times} + \underbrace{(4^2+4)}_{2^{l+1}times} + ... + \underbrace{((2l)^2+2l)}_{2^{2l-1}times} + \underbrace{(2(l+1))^2+(2(l+1))}_{2^{2l}times}\right]$$
$$= \frac{2^l}{2}\left[2^0(2^2+2) + 2^1(4^2+4) + ... + 2^{l-1}((2l)^2+2l) + 2^l((2(l+1))^2+(2(l+1))\right]$$
$$= 2^{l-1} \sum_{m=0}^{l} 2^m((2(l+1))^2+(2(l+1))$$
$$= 2^{l-1} \sum_{m=0}^{l} 2^m 2(l+1)(2l+3)$$
$$== 2^{l-1} \sum_{m=0}^{l} 2^{m+1}(m+1)(2m+3)$$
$$= 2^{l-1} \sum_{m=0}^{l} 2^{m+1}(2m^2+5m+3)$$

Thus the result is true for level $l+1$
Hence

$$HTW(G_n) = 2^{l-2} \sum_{m=0}^{l-1} 2^{m+1}(2m^2+5m+3)$$

2.2 THE HYPER TERMINAL WIENER INDEX OF SECOND DENDRIMER GRAPH D_n

Let D_0,D_1,D_2,... be the series of dendrimer graphs. The dendrimer graph $D_h$obtained by attaching p pendant vertices to each pendant vertex of D_{h-1}, h=1,2,..., For an illustration see Figure 3. Thus the dendrimer graph D_h $3\cdot 2^n$ has pendant vertices.

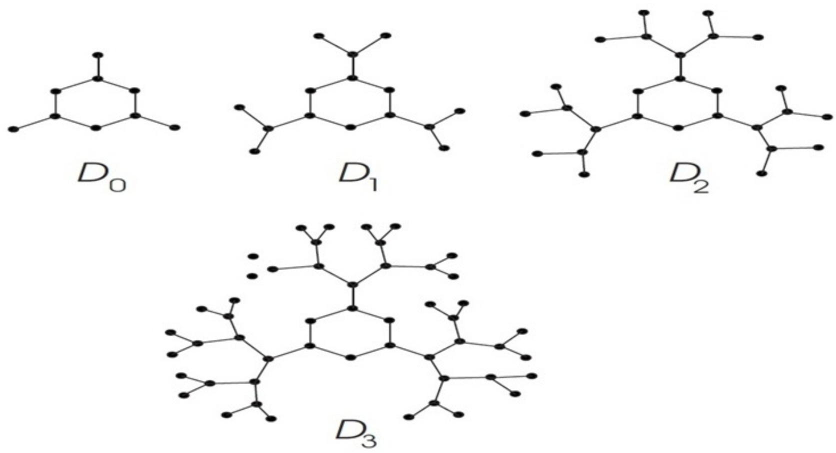

Figure 3. The second dendrimer graph D_n

Proposition 2.2. *The Hyper terminal Wiener index of Dendrimer graph D_n is*

$$HTW(D_n) = 3 \cdot 2^{n-2} \left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3) + 2^{n+2}(2n^2+9n+10) \right]$$

Proof. The dendrimer graph D_n has pendant $3 \cdot 2^n$ vertices
By the definition of hyper terminal Wiener index

$$HTW(G) = \frac{1}{2} \sum_{1 \le i < j \le k} \left[d(v_i, v_j)^2 + d(v_i, v_j) \right]$$

For n=0

$$HTW(D_0) = \frac{1}{2} \left[3(4^2+4) \right] = 30$$

n=1

$$HTW(D_1) = \frac{1}{2} \left[\underbrace{(2^2+2)}_{3 times} + \underbrace{(6^2+6)}_{12 times} \right] = 261$$

n=2

$$HTW(D_2) = \frac{1}{2} \left[\underbrace{(2^2+2)}_{6 times} + \underbrace{(4^2+4)}_{12 times} + \underbrace{(8^2+8)}_{48 times} \right]$$
$$= 3 \cdot 2^0 \left[2^0(2^2+2) + 2^1(4^2+4) + 2^3(8^2+8) \right] = 1866$$

n=3

$$HTW(D_3) = \frac{1}{2} \left[\underbrace{(2^2+2)}_{12 times} + \underbrace{(4^2+4)}_{24 times} + \underbrace{(6^2+6)}_{48 times} + \underbrace{(10^2+10)}_{192 times} \right]$$
$$= 3 \cdot 2^1 \left[2^0(2^2+2) + 2^1(4^2+4) + 2^2(6^2+6) + 2^4(10^2+10) \right] = 11844$$

and so on.
We will prove the same from above results using mathematical induction

$$HTW(D_n) = 3 \cdot 2^{n-2} \left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3) + 2^{n+2}(2n^2+9n+10)) \right]$$

Let $n = 0$

$$HTW(D_0) = \frac{3}{4} \left[2^2 \cdot 2 \cdot 5) \right] = 30$$

The result true for $n = 0$
Let $n = 1$

$$HTW(D_1) = \frac{3}{2} \left[2 \cdot 3 + 2^3(21) \right] = 261$$

The result true for $n = 1$
If n=2

$$HTW(D_2) = 3 \left[2 \cdot 3 + 2^2(10) + 2^4(36) \right] = 1866$$

The result true for n=2
Let assume the result is true for n=l

$$HTW(D_l) = 3\cdot 2^{l-2}\left[\sum_{m=0}^{l-1} 2^{m+1}(2m^2+5m+3) + 2^{l+2}(2l^2+9l+10))\right]$$
$$= 3.\frac{2^{l-1}}{2}\left[\sum_{m=0}^{l-1} 2^{m}(2m+2)(2m+3) + 2^{l+1}(2l+3)(2l+4))\right]$$
$$= \frac{1}{2}\left[\underbrace{(2^2+2)}_{3\cdot 2^{l-1}times} + \underbrace{(4^2+4)}_{3\cdot 2^{l}times} + ... + \underbrace{((2l)^2+2l)}_{3\cdot(2l-2)times} + \underbrace{((2l+4)^2+(2l+4))}_{3\cdot 2^{2l}}\right]$$

To prove result true for$n$=l+1

Let

$$HTW(D_{l+1}) = \frac{1}{2}\left[\underbrace{(2^2+2)}_{3\cdot 2^{l}times} + \underbrace{(4^2+4)}_{3\cdot 2^{l+1}times} + ... + \underbrace{((2l)^2+2l)}_{3\cdot 2^{(2l-1)}times} + \underbrace{((2l+4)^2+(2l+4))}_{3\cdot 2^{2(l+1)}times}\right]$$
$$= \frac{1}{2}3\cdot 2^{l}\left[2^0(2^2+2) + 2^1(4^2+4) + ...2^l((2l)^2+2l) + 2^{l+2}((2l+4)^2+2l+4)\right]$$
$$= 3\cdot 2^{l-1}\left[\sum_{m=0}^{l} 2^{m}(2m+2)(2m+3) + 2^{l+2}((2l+4)^2+2l+4)\right]$$
$$= 3\cdot 2^{l-1}\left[\sum_{m=0}^{l} 2^{m+1}(2m^2+5m+3) + 2^{l+3}(2l^2+9l+10))\right]$$

Thus the result true for n=l+1.

$$HTW(D_n) = 3\cdot 2^{n-2}\left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3) + 2^{n+2}(2n^2+9n+10)\right]$$

3 DENTOUR SATURATED TREES

3.1 THE HYPER TERMINAL WIENER INDEX OF DENTOUR SATURATED TREES

A graph is said to be detour–saturated if the addition of any edge results in an increased greatest path length. A benzenoid graph is called catacondensed if its characteristic graph is a tree. The characteristic graph of a hexagonal chain is isomorphic to the path. Cata-condensed species have dualist graphs, which are detour saturated trees, while those Peri-condensed species contain at least one cycle. The dualist graph of Cata-condensed species is a claw.

The family T_k is obtained by starting with one tree in T_k(called a skeletal tree) and then adding branches and leaves to it, while maintaining membership in T_k. The detour tree T_3 is called claw and T_4 is called the double claw. The general detour-saturated tree $T_3(n)$ for odd $n \geq$ 5is obtained from $T_3(n-1) by attaching two new pendant vertices to each of the old pendant vertices, as shown in Figu$

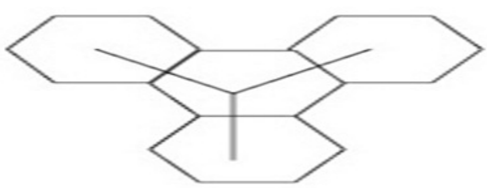

Figure 4. Cata-Condensed and its dualist graph T_0

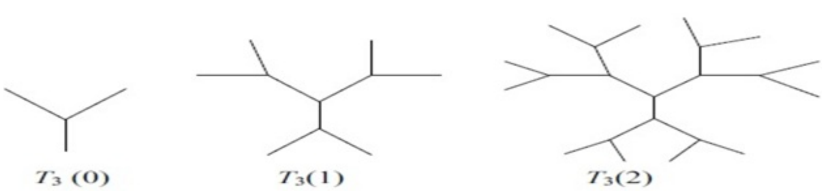

Figure 5. Detour saturated tree $T_3(0)$, $T_3(1)$ and $T_3(2)$

Double claw $T_4(n)$ constructed by adding two new pendant vertices to each of the old pendant vertices of $T_4(n-1)$, $n \geq 6$, [10,11,12,13,14]. The double saturated tree of double claw and detour saturated tree for $T_4(n)$ is shown in Figure 6 and Figure 7.

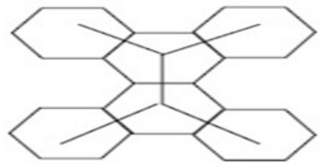

Figure 6. Detour saturated tree of double claw

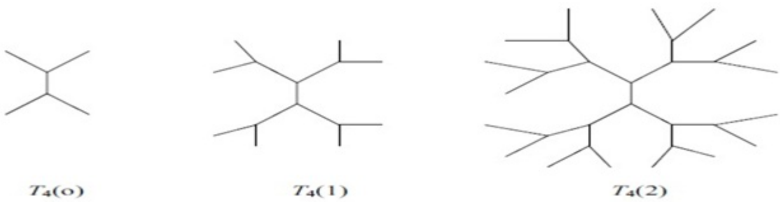

Figure 7. Detour saturated tree $T_4(0)$, $T_4(1)$ and $T_4(2)$

Proposition 3.1. *The hyper terminal Wiener index of dentour saturated tree $T_3(n)$ is given by*

$$HTW(T_3(n)) = 3 \cdot 2^{n-2} \left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3) + 2^{n+2}(2n^2+5n+3) \right]$$

Proof. If n=0 , then $T_3(0)$ has three pendant vertices. $T_3(1)$ is obtained by attaching two pendant vertices each of these pendant vertices of $T_3(0)$. Then $T_3(1))$ has six pendant vertices and $T_3(2)$ is obtained by attaching two pendant vertices each of these pendant vertices of $T_3(1)$ and so on. Let n denote the number of steps in the formation of dentour saturated trees. Then clearly the number of pendant verticesin claw $T_3(n)$ are $3 \cdot 2^n$

By the definition of hyper terminal Wiener index

$$HTW(G) = \frac{1}{2} \sum_{1 \leq i < j \leq p} \left[d(v_i, v_j)^2 + d(v_i, v_j) \right]$$

Let n=0

$$HTW(T_3(0)) = \frac{1}{2}\left[3(2^2+2)\right] = 9$$

Let n=1

$$\begin{aligned} HTW(T_3(1)) &= \frac{1}{2}\left[3(2^2+2) + 12(4^2+4)\right] \\ &= \frac{3}{2}\left[2^0(2^2+2) + 2^2(4^2+4)\right] \\ &= \frac{3}{2}\left[2^0 \cdot 2 \cdot 3 + 2^2 \cdot 4 \cdot 5\right] \\ &= 129 \end{aligned}$$

n=2

$$\begin{aligned} HTW(T_3(2)) &= \frac{1}{2}\left[6(2^2+2) + 12(4^2+4) + 48(6^2+6)\right] \\ &= \frac{3 \cdot 2}{2}\left[2^0(2^2+2) + 2^1(4^2+4) + 2^3(6^2+6)\right] \\ &= \frac{3 \cdot 2^1}{2}\left[2^0 \cdot 2 \cdot 3 + 2^1 \cdot 4 \cdot 5 + 2^3 \cdot 4 \cdot 5\right] = 1146 \end{aligned}$$

n=3

$$\begin{aligned} HTW(T_3(3)) &= \frac{1}{2}\left[12(2^2+2) + 24(4^2+4) + 48(6^2+6) + 192(8^2+8)\right] \\ &= \frac{3 \cdot 2^{2^2}}{2}\left[2^0(2^2+2) + 2^1(4^2+4) + 2^2(6^2+6) + 2^4(8^2+8)\right] \\ &= \frac{3 \cdot 2^2}{2}\left[2^0 \cdot 2 \cdot 3 + 2^1 \cdot 4 \cdot 5 + 2^2 \cdot 4 \cdot 5 + 2^4 \cdot 8 \cdot 9\right] = 8196 \end{aligned}$$

and so on .

With all these background, we will prove result by mathematical induction.

Let n = 1

$$HTW(T_3(n=1)) = \frac{3}{2}\left[2^1 \cdot 1 \cdot 3 + 2^3 \cdot 2 \cdot 5\right] = 129$$

The result is true for n=1
For n=2

$$HTW(T_3(n=2)) = 3\cdot 2^0\left[2^1\cdot 1\cdot 3 + 2^2\cdot 2\cdot 5 + 2^4\cdot 2\cdot 5\right] = 1146$$

The result is true for n=2
Let us assume result is true for n = k-1

$$\begin{aligned} HTW(T_3(n=k-1)) &= 3\cdot 2^{k-3}\left[\sum_{m=0}^{k-2} 2^{m+1}(2m^2+5m+3) + 2^{k+1}(2(k-1)^2+5(k-1)+3)\right] \\ &= 3\cdot 2^{k-3}\left[\sum_{m=0}^{k-2} 2^{m+1}(m+1)(2m+3) + 2^{k+1}(k)(2k+1)\right] \\ &= \frac{3\cdot 2^{k-2}}{2}\left[\begin{array}{l} 2^0(2^2+2)+2^1(4^2+4)+2^2(6^2+6)+... \\ \quad ...+2^{k-1}(2(k-1))^2+2(k-1))+2^{k+1}((2k)^2+2k) \end{array}\right] \end{aligned}$$

To prove the result is true for n =k

$$\begin{aligned} HTW(T_3(n=k)) &= \frac{1}{2}\left[\begin{array}{l} 3\cdot 2^{k-1}(2^2+2)+3\cdot 2^k(4^2+4)+3\cdot 2^{k+1}(6^2+6)+.... \\ +3\cdot 2^{2k}((2k+2)^2+(2k+2)) \end{array}\right] \\ &= \frac{3\cdot 2^{k-1}}{2}\left[2^0(2^2+2)+2^1(4^2+4)+2^2(6^2+6)+....+2^{k+1}(2k+2)^2+(2k+2)\right] \\ &= \frac{3\cdot 2^{k-1}}{2}\left[2^0\cdot 2\cdot 3+2^1\cdot 4\cdot 5+2^2\cdot 6\cdot 7+....+2^{k+2}(k+1)\cdot(2k+3)\right] \\ &= 3\cdot 2^{k-2}\left[\sum_{m=1}^{k-1}(m+1)(2m+3)+2^{k+2}(k+1)\cdot(2k+3)\right] \\ &= 3\cdot 2^{k-2}\left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3)+2^{k+2}(2k^2+5k+3)\right] \end{aligned}$$

Hence the result is true for n=k

$$HTW(T_3(n)) = 3\cdot 2^{n-2}\left[\sum_{m=0}^{n-1} 2^{m+1}(2m^2+5m+3)+2^{n+2}(2n^2+5n+3)\right]$$

Proposition 3.2. *The hyper terminal Wiener index of dentour saturated tree*

$$HTW(T_4(n)) = 2^n\left[\sum_{m=0}^{n} 2^{m+1}(2m^2+5m+3)+2^{n+2}(2n^2+7n+6)\right]$$

Proof. If n = 0, then $T_4(0)$ has four pendant vertices. $T_4(1)$ obtained by attaching two pendant vertices each of these pendant vertices of $T_4(0)$. Then $T_4(1)$ has eight pendant vertices. $T_4(2)$ obtained by attaching two pendant vertices each of these pendant vertices of $T_4(1)$ and so on. Let n denote the number of steps in the formation of dentour saturated trees. Then clearly double claw $T_4(n)$ contains $2n$+2 pendant vertices. By the definition of hyper terminal Wiener index $HTW(G) = \frac{1}{2}\sum_{1\le i<j\le p}[d(v_i,v_j)^2 + d(v_i,v_j)]$

Let n=0

$$\begin{aligned} HTW(T_4(0)) &= \frac{1}{2}\left[2(2^2+2)+4(3^2+3)\right] \\ &= \frac{2}{2}\left[2^0(2^2+2)+2^1(4^2+4)\right] \\ &= 2^0\left[2^0\cdot 2\cdot 3+2^1\cdot 3\cdot 4\right] \\ &= 30 \end{aligned}$$

Let n=1

$$\begin{aligned} HTW(T_4(1)) &= \frac{1}{2}\left[4(2^2+2)+8(4^2+4)+16(5^2+5)\right] \\ &= \frac{2^2}{2}\left[2^0(2^2+2)+2^1(4^2+4)+2^2(5^2+5)\right] \\ &= 2^1\left[2^0\cdot 2\cdot 3+2^1\cdot 4\cdot 5+2^2\cdot 5\cdot 6\right] = 332 \end{aligned}$$

Let n=2

$$\begin{aligned} HTW(T_4(2)) &= \tfrac{1}{2}\left[8(2^2+2)+16(4^2+4)+32(6^2+6)+64(7^2+7)\right] \\ &= \tfrac{2^3}{2}\left[2^0(2^2+2)+2^1(4^2+4)+2^2(6^2+6)+2^3(7^2+7)\right] \\ &= 2^2\left[2^0\cdot 2\cdot 3+2^1\cdot 4\cdot 5+2^2\cdot 6\cdot 7+2^3\cdot 7\cdot 8\right]=2648 \end{aligned}$$

Let n=3

$$\begin{aligned} HTW(T_4(3)) &= \tfrac{1}{2}\left[16(2^2+2)+32(4^2+4)+64(6^2+6)+128(8^2+8)+256(9^2+9)\right] \\ &= \tfrac{2^4}{2}\left[2^0(2^2+2)+2^1(4^2+4)+2^2(6^2+6)+2^3(8^2+8)+2^4(9^2+9)\right] \\ &= 2^3\left[2^0\cdot 2\cdot 3+2^1\cdot 4\cdot 5+2^2\cdot 6\cdot 7+2^3\cdot 8\cdot 9+2^4\cdot 9\cdot 10\right]=16816 \end{aligned}$$

so on, with all these calculations we prove above expression will prove by mathematical induction
For n=1

$$\begin{aligned} HTW(T_4(n=1)) &= 2^1\left[\sum_{m=0}^{1}2^{m+1}(2m^2+5m+3)+2^3\cdot 15\right] \\ &= 2^1\left[2^1\cdot 3+2^2\cdot 10+2^3\cdot 15\right] \\ &= 332 \end{aligned}$$

The result is true for n=1 Let n=2

$$\begin{aligned} HTW(T_4(n=2)) &= 2^2\left[\sum_{m=0}^{2}2^{m+1}(2m^2+5m+3)+2^4\cdot 28\right] \\ &= 2^2\left[2^1\cdot 3+2^2\cdot 10+2^3\cdot 21+2^4\cdot 28\right]=2648 \end{aligned}$$

The result is true for n=2
Let assume result true for n=k-1

$$\begin{aligned} &= 2^{k-1}\left[\sum_{m=0}^{k-1}2^m\cdot(2m+2)(2m+3)+2^k\cdot(k+1)(2k+1)\right] \\ &= \tfrac{1}{2}\left[\begin{array}{r} 2^k(2^2+2)+2^{k+1}(4^2+4)+2^{k+2}(6^2+6)+\ldots.. \\ 2^{2k-2}((2k)^2+2k)+2^k(((2k+1)^2+2k+1) \end{array}\right] \end{aligned}$$

Now we show that is true for n=k
We have

$$\begin{aligned} HTW(T_4(n=k)) &= \tfrac{1}{2}\left[\begin{array}{r} 2^{k+1}(2^2+2)+2^{k+2}(4^2+4)+2^{k+3}(6^2+6)+\ldots+2^{2k}((2(k+1))^2 \\ +2(k+1))+2^{2k+1}(2(k+1)+1)^2+(2(k+1)+2)) \end{array}\right] \\ &= \tfrac{2^{k+1}}{2}\left[\begin{array}{r} 2^0(2^2+2)+2^1(4^2+4)+2^2(6^2+6)+\ldots+2^k(2(k+1))^2 \\ +2(k+1))+2^{k+1}(2(k+1)+1)^2+(2(k+1)+2)) \end{array}\right] \\ &= 2^k\left[\sum_{m=0}^{k}2^m\cdot\left((2(m+1))^2+2(m+1)\right)+2^{k+1}(2(k+1)+1)^2+(2(k+1)+2))\right] \\ &= 2^k\left[\sum_{m=0}^{k}2^{m+1}(m+1)(2m+3)+2^{k+2}(2k+3)(k+2)\right] \\ &= 2^k\left[\sum_{m=0}^{k}2^{m+1}(2m^2+5m+3)+2^{k+2}(2k^2+7k+6)\right] \end{aligned}$$

Hence the result is true for n=k
$HTW(T_4(n))=2^n\left[\sum_{m=0}^{n}2^{m+1}(2m^2+5m+3)+2^{n+2}(2n^2+7n+6)\right]$

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Received: May 22, 2018.
Accepted: March 17, 2019.