Palestine Journal of Mathematics

Vol. 9(2)(2020) , 785–791

An Extended $p - k$ Mittag-Leffler Function

Pablo I. Pucheta

Communicated by P. K. Banerji

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords $p - k$ Mittag-Leffler Function, $p - k$ Beta Function, $p - k$ Pochhammer Symbol.

Abstract In this paper, we will present a extension of the $p - k$ Mittag-Leffler function recently introduced by (R. A. Cerutti and K. S. Gehlot see [5], [7]) using relationship between the $p - k$ Beta function with $p - k$ symbol Pochhammer $\frac{{}_p(\gamma)_{n,k}}{{}_p(\xi)_{n,k}} = \frac{{}_pB_k(\gamma+nk,\xi-\gamma)}{{}_pB_k(\gamma,\xi-\gamma)}$ introduced by (Kuldeep Gehlot see [6]). We will obtain some integral representations, study some basic properties and also evaluate the Laplace transform.

1 Introduction

The Mittag-Leffler function along with the Wright function plays prominent role in the theory of the partial differential equations of the fractional order that are actively used nowadays for modeling of many physical phenomena including the anomalous diffusion, the telegraph equation, random walks (see [3]), besides this, the Mittag-Leffler function appears in the solution of certain boundary value problem involving fractional integro-differential equations of Volterra type (see [2]).
The Mittag-Leffler one parameters function is defined by the following series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{1.1}$$

where $\Gamma(,)$ denotes the classical Gamma function.
The two parameters Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{1.2}$$

Prabhakar (see [9]) introduced the Mittag-Lefler function defined by:

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{1.3}$$

with α ,β and $\delta \in \mathbb{C}$, and $(\delta)_n$ denotes the Pochhammer symbol. For more details of the Mittag-Leffler function see([1]).
Dorrego G and Cerutti R (see [4]) introduced the k-Mittag-Leffler function by:

$$E^{\delta}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!} \tag{1.4}$$

with $k > 0$, α , β and $\delta \in \mathbb{C}$ $R_e(\alpha) > 0$, $R_e(\beta) > 0$ and $z \in \mathbb{C}$, where $\Gamma_k(,)$ is the k-Gamma function and $(\delta)_{n,k}$ is the k-Pochhammer symbol due to Pariguan and Diaz (see [10])
As it is knows 2017 K.S. Gehlot (see [6]) has introduced a modification of the k-Gamma function by:

$${}_p\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{p}} t^{z-1} dt \tag{1.5}$$

where $z \in \mathbb{C} - k\mathbb{Z}^-$, k, p, $\in \mathbb{R}^+$ and $R_e(z) > 0$ The following property is verified (recurrence formula):

$$ {}_p\Gamma_k(z+k) = \frac{zp}{k}\, {}_p\Gamma_k(z) \tag{1.6} $$

Also, defined the $p-k$ Beta function as:

$$ {}_pB_k(x,y) = \frac{{}_p\Gamma_k(x)\,{}_p\Gamma_k(y)}{{}_p\Gamma_k(x+y)} \tag{1.7} $$

where $R_e(x) > 0$, $R_e(y) > 0$
The $p-k$ Beta function satisfies the following identities:

$$ {}_pB_k(x,y) = \frac{1}{k}\int_0^1 \frac{t^{\frac{t}{k}-1} + t^{\frac{y}{k}-1}}{(t+1)^{\frac{x+y}{k}}} dt \tag{1.8} $$

$$ {}_pB_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt \tag{1.9} $$

$$ {}_pB_k(x,y) = \frac{1}{k} B(\frac{x}{k}, \frac{y}{k}) \tag{1.10} $$

He has also defined a new Pochhammar symbol

$$ \begin{aligned} {}_p(z)_{n,k} &= \left(\frac{zp}{k}\right)\left(\frac{zp}{k}+p\right)\left(\frac{zp}{k}+2p\right)\ldots\left(\frac{zp}{k}+(n-1)p\right) \\ &= \frac{{}_p\Gamma_k(z+nk)}{{}_p\Gamma_k(z)} \end{aligned} \tag{1.11} $$

And, for the ${}_p(z)_{n,k}$ Pochhammer symbol, we have the following properties:

$$ {}_p(z)_{n,k} = \left(\frac{p}{k}\right)^n (z)_{n,k} = p^n \left(\frac{z}{k}\right)_n \tag{1.12} $$

$$ {}_p(z)_{n+j,k} = {}_p(z)_{j,k} \times {}_p(z+jk)_{n,k} \tag{1.13} $$

Lemma 1.1. *For any $a \in \mathbb{C}$, $p,\ k > 0$ and $|x| < \frac{1}{p}$, the following identity holds*

$$ \sum_{n=0}^{\infty} \frac{{}_p(a)_{n,k} x^n}{n!} = (1-xp)^{-\frac{a}{k}} \tag{1.14} $$

Recently R A. Cerutti, G A. Dorrego and L L. Luque (see [5]) have introduced the $p-k$ Mittag-Leffler function by:

$$ {}_pE^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \tag{1.15} $$

where ${}_p\Gamma_k(;)$ is the $p-k$ Gamma function and ${}_p(\gamma)_{n,k}$ is the $p-k$ Pochhammer symbol

Lemma 1.2. *For p-k Mittag-Leffler function satisfies the following properties*

$$ \frac{1}{p}\, {}_pE^{\gamma}_{k,\alpha,\beta}(z) = \frac{\beta}{k}\, {}_pE^{\gamma}_{k,\alpha,\beta+k}(z) + \frac{\alpha}{k} z \frac{d}{dz}\left({}_pE^{\gamma}_{k,\alpha,\beta+k}(z)\right) \tag{1.16} $$

$$ \frac{d^m}{dz^m} = \left(z^{\frac{\beta}{k}-1}\, {}_pE^{\gamma}_{k,\alpha,\beta}(z^{\frac{\alpha}{k}})\right) = p^{-m} z^{\frac{\beta}{k}-m-1}\, {}_pE^{\gamma}_{k,\alpha,\beta-mk}(z^{\frac{\alpha}{k}}) \tag{1.17} $$

The proof could be seen in ([5])

2 Main Result

In this paper, we extended the $p-k$ Mittag-Leffler function ${}_pE^{\gamma}_{k,\alpha,\beta}(z)$ in the following way since

$$
\begin{aligned}
{}_pE^{\gamma}_{k,\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}\ {}_p(\xi)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)\ {}_p(\xi)_{n,k}} \frac{z^n}{n!}
\end{aligned}
$$

using the fact that

$$\frac{{}_p(\gamma)_{n,k}}{{}_p(\xi)_{n,k}} = \frac{{}_pB_k(\gamma+nk,\xi-\gamma)}{{}_pB_k(\gamma,\xi-\gamma)}$$

Obtaing the following

Definition 2.1. *Let* $\alpha,\ \beta,\ \gamma,\ \xi \in \mathbb{C}$ *such that* $R_e(\xi) > R_e(\gamma) > 0,\ R_e(\alpha) > 0,\ R_e(\beta) > 0$ *and* $p,\ k\ \in \mathbb{R}^{+}-\{0\}$. *The extended Mittag-Leffler function is defined as follows series*

$$ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \tag{2.1}$$

where $R_e(\gamma+nk) > 0$ and $R_e(\xi-\gamma) > 0$
Note that if ${}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z) \to {}_pE^{\gamma}_{k,\alpha,\beta}(z)$ as $\xi \to 1$

Remark 2.2.
- *For* $p=k,\ \xi=1$ *we have:*

$$ {}_kE^{\gamma,1}_{k,\alpha,\beta}(z) = E^{\gamma}_{k,\alpha,\beta}(z) \tag{2.2}$$

- *For* $p=k=\xi=1$ *we have:*

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z) \tag{2.3}$$

- *For* $p=k=\gamma=\xi=1$ *we have:*

$$ {}_1E^{1,1}_{1,\alpha,\beta}(z) = E_{\alpha,\beta}(z) \tag{2.4}$$

Theorem 2.3. *Let* $\alpha,\ \beta,\ \gamma,\ \xi \in \mathbb{C}$ *such that* $R_e(\xi) > R_e(\gamma) > 0,\ R_e(\alpha) > 0,\ R_e(\beta) > 0$ *and* $p,\ k\ \in \mathbb{R}^{+}-\{0\}$. *Then*

$$ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z) = \frac{1}{k\ {}_pB_k(\gamma,\xi-\gamma)} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{\xi-\gamma}{k}-1}\ {}_pE^{\xi}_{k,\alpha,\beta}(tz)dt \tag{2.5}$$

Proof. From (2.1) and (1.9) we have

$$
\begin{aligned}
{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{k} \int_0^1 t^{\frac{\gamma+nk}{k}-1}(1-t)^{\frac{\xi-\gamma}{k}-1}dt \right) \frac{{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}
\end{aligned} \tag{2.6}
$$

Interchanging the order of integration and summation in (2.6), we have

$$
\begin{aligned}
{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z) &= \frac{1}{k} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{\xi-\gamma}{k}-1} \sum_{n=0}^{\infty} \frac{{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)} \frac{(tz)^n}{n!}dt \\
&= \frac{1}{k\ {}_pB_k(\gamma,\xi-\gamma)} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{\xi-\gamma}{k}-1} \sum_{n=0}^{\infty} \frac{{}_p(\xi)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(tz)^n}{n!}dt
\end{aligned}
$$

using (2.12) we have:

$$
{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)=\frac{1}{k\;{}_pB_k(\gamma,\xi-\gamma)}\int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{\xi-\gamma}{k}-1}\;{}_pE^{\xi}_{k,\alpha,\beta}(tz)dt \tag{2.7}
$$

□

Corollary 2.4. *Taking $t=\frac{u}{1+u}$ in theorem 2.3, we have*

$$
{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)=\frac{1}{k\;{}_pB_k(\gamma,\xi-\gamma)}\int_0^\infty \frac{u^{\frac{\gamma}{k}-1}}{(1+u)^{\frac{\xi}{k}}}\;{}_pE^{\xi}_{k,\alpha,\beta}\left(\left(\frac{u}{1+u}\right)z\right)du \tag{2.8}
$$

Corollary 2.5. *Taking $t=sin^2\theta$ in theorem 2.3, we have*

$$
{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)=\frac{1}{k\;{}_pB_k(\gamma,\xi-\gamma)}\int_0^{\frac{\pi}{2}} sin^{\frac{2\gamma}{k}-1}cos^{\frac{2(\xi-\gamma)}{k}-1}\;{}_pE^{\xi}_{k,\alpha,\beta}(zsin^2\theta)d\theta \tag{2.9}
$$

Theorem 2.6. *Let $\alpha,\ \beta,\ \gamma,\ \xi\in\mathbb{C}$ such that $R_e(\xi)>R_e(\gamma)>0$, $R_e(\alpha)>0$, $R_e(\beta)>0$ and $p,\ k\ \in\mathbb{R}^+-\{0\}$. Then*

$$
\int_0^z t^{\frac{\beta}{k}-1}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left(at^{\frac{\alpha}{k}}\right)dt=p\;z^{\frac{\beta}{k}}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta+k}(az^{\frac{\alpha}{k}}) \tag{2.10}
$$

Proof. From definition (2.1) and interchanging the order of integration and summation, we have

$$
\begin{aligned}
\int_0^z t^{\frac{\beta}{k}-1}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left(at^{\frac{\alpha}{k}}\right)dt &= \int_0^z t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}a^n\frac{t^{\frac{\alpha n}{k}}}{n!}dt\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}\frac{a^n}{n!}\int_0^z t^{\frac{\beta+\alpha n}{k}-1}dt\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}\frac{a^n}{n!}\frac{z^{\frac{\beta+\alpha n}{k}}}{\left(\frac{\beta+\alpha n}{k}\right)}
\end{aligned}
$$

Taking into account (1.6) and $\left(\frac{\beta+\alpha n}{k}\right)\frac{p}{p}=\frac{1}{p}\;{}_p\Gamma_{n,k}(\alpha n+(\beta+k))$, we obtain

$$
\begin{aligned}
\int_0^z t^{\frac{\beta}{k}-1}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left(at^{\frac{\alpha}{k}}\right)dt &= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta+k)\frac{1}{p}}z^{\frac{\beta}{k}}\frac{(az^{\frac{\alpha}{k}})^n}{n!}\\
&= p\;z^{\frac{\beta}{k}}\;{}_pE^{\gamma,\xi}_{k,\alpha,(\beta+k)}\left(az^{\frac{\alpha}{k}}\right)
\end{aligned}
$$

□

Remark 2.7. *Note that if $p=k=\gamma=\xi=1$, we obtain*

$$
\int_0^z t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})dt=z^{\beta}E_{\alpha,\beta+1}(az^{\alpha}) \tag{2.11}
$$

Theorem 2.8. *Let $\alpha,\ \beta,\ \gamma,\ \xi\in\mathbb{C}$ such that $R_e(\xi)>R_e(\gamma)>0$, $R_e(\alpha)>0$, $R_e(\beta)>0$ and $p,\ k\ \in\mathbb{R}^+-\{0\}$. Then*

$$
\frac{1}{p}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)=\frac{\beta}{k}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta+k}(z)+\frac{\alpha}{k}z\frac{d}{dz}\left({}_pE^{\gamma,\xi}_{k,\alpha,\beta+k}(z)\right) \tag{2.12}
$$

Proof. Starting for the right member of (2.12), using definition (2.1) and taking account (1.6), we have

$$\begin{aligned}
\frac{\beta}{k}\,{}_pE^{\gamma,\xi}_{k,\alpha,\beta+k}(z)+\frac{\alpha}{k}z\frac{d}{dz}\left({}_pE^{\gamma,\xi}_{k,\alpha,\beta+k}(z)\right) &= \frac{\beta}{k}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta+k)}\frac{z^n}{n!}\\
&+\frac{\alpha}{k}z\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+(\beta+k))}\frac{z^n}{n!}\right)\\
&= \frac{\beta}{k}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+(\beta+k)}\frac{z^n}{n!}\\
&+\frac{\alpha n}{k}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+(\beta+k))}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\left(\frac{\alpha n+\beta}{k}\right)\;{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta+k))}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)p\;{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!}\\
&= \frac{1}{p}\;{}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)
\end{aligned}$$

□

Remark 2.9. *Note that if* $p=k=\gamma=\xi=1$ *(2.12) is*

$$E_{\alpha,\beta}(z)=\beta E_{\alpha,\beta+1}(z)+\alpha z\frac{d}{dz}\left(E_{\alpha,\beta+1}(z)\right) \tag{2.13}$$

Theorem 2.10. *Let* $\alpha,\ \beta,\ \gamma,\ \xi\in\mathbb{C}$ *such that* $R_e(\xi)>R_e(\gamma)>0,\ R_e(\alpha)>0,\ R_e(\beta)>0$ *and* $p,k\in\mathbb{R}^+-\{0\}$ *and* $m\in\mathbb{N}$*. Then*

$$\frac{d^m}{dz^m}\left({}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)\right)={}_p(\gamma)_{m,k}\;{}_pE^{\gamma+m,\xi+m}_{k,\alpha,\alpha n+\beta}(z) \tag{2.14}$$

Proof. From definition (2.1) and taking into account the property of the Pochhammer symbol ${}_p(\xi)_{n+m,k}={}_p(\xi)_{m,k}\;{}_p(\xi+mk)_{n,k}$, we obtain

$$\begin{aligned}
\frac{d^m}{dz^m}\left({}_pE^{\gamma,\xi}_{k,\alpha,\beta}(z)\right) &= \frac{d^m}{dz^m}\left(\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}\frac{d^m}{dz^m}\frac{z^n}{n!}\\
&= \sum_{n=m}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\;{}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{(n-m)!}\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+(m+n)k,\xi-\gamma)\;{}_p(\xi)_{m+n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha(m+n)+\beta)}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{{}_pB_k((\gamma+mk)+nk,\xi-\gamma)\;{}_p(\xi)_{m,k}\;{}_p(\xi+mk)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+(\alpha m+\beta))}\frac{z^n}{n!}\\
&= {}_p(\xi)_{m,k}\sum_{n=0}^{\infty}\frac{{}_pB_k((\gamma+mk)+nk,\xi-\gamma)\;{}_p(\xi+mk)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\;{}_p\Gamma_k(\alpha n+(\alpha m+\beta))}\frac{z^n}{n!}\\
&= {}_p(\xi)_{m,k}\;{}_pE^{\gamma+mk,\xi+mk}_{k,\alpha,\alpha m+\beta}(z)
\end{aligned}$$

□

Definition 2.11. *Let $f:\mathbb{R}^+\to\mathbb{R}$ an exponential order and piecewise continuous function, then the Laplace transform of f is defined*

$$\mathcal{L}\left\{f(t)\right\}(s)=\int_0^{\infty}e^{-st}f(t)dt \tag{2.15}$$

the integral exist for $R_e(s)>0$

Definition 2.12. *(see [6]) Given $z\in\mathbb{C}$; $k,p\in(\mathbb{R}^+)^r$; $s,t\in(\mathbb{R}^+)^q$, $b=(b_1,b_2\dots,b_q)\in\mathbb{C}^q$ such that $b_i\in\mathbb{C}-s_i\mathbb{Z}^-$. The $p-k$ hypergeometric function is given by:*

$$F(a,p,k,b,t,s,z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{n,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{n,s_j}}\frac{z^n}{n!} \tag{2.16}$$

Definition 2.13. *Let $a,b\in\mathbb{C}$, $c\in\mathbb{C}-\mathbb{Z}_0^-$ and $p,k\in\mathbb{R}^+-\{0\}$. The Gauss hypergeometric function ${}_2F_1$ is given by:*

$$\begin{aligned}{}_2F_1(a,b,c,p,k,z) &= \sum_{n=0}^{\infty}\frac{{}_p(a)_{n,k}\ {}_p(b)_{n,k}}{{}_p(c)_{n,k}}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{{}_pB_k(b+nk,c-b)}{{}_pB_k(b,c-b)}\ {}_p(a)_{n,k}\frac{z^n}{n!}\end{aligned} \tag{2.17}$$

Theorem 2.14. *Let α, β, γ, $\xi\in\mathbb{C}$ such that $R_e(\xi)>R_e(\gamma)>0$, $R_e(\alpha)>0$, $R_e(\beta)>0$ and $p,k\in\mathbb{R}^+-\{0\}$. Then*

$$\mathcal{L}\left\{z^{\frac{\beta}{k}-1}\ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left((bz)^{\frac{\alpha}{k}}\right)\right\}(s)=\frac{1}{(sp)^{\frac{\beta}{k}}}\ {}_2F_1\left(\xi,\gamma,\xi,p,k,\left(\frac{b}{sp}\right)^{\frac{\alpha}{k}}\right) \tag{2.18}$$

Proof. From definitions (2.15), (2.1) and taking into account (2.17), we have

$$\begin{aligned}\mathcal{L}\left\{z^{\frac{\beta}{k}-1}\ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left((bz)^{\frac{\alpha}{k}}\right)\right\}(s) &= \int_0^{\infty}e^{-sz}z^{\frac{\beta}{k}-1}\ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left((bz)^{\frac{\alpha}{k}}\right)dz\\ &= \int_0^{\infty}e^{-sz}z^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)}\frac{b^{\frac{\alpha}{k}n}z^{\frac{\alpha}{k}n}}{n!}\end{aligned} \tag{2.19}$$

interchanging the order integration and summation in (2.19) and application Laplace transform of the potential function, we obtain

$$\begin{aligned}\mathcal{L}\left\{z^{\frac{\beta}{k}-1}\ {}_pE^{\gamma,\xi}_{k,\alpha,\beta}\left((bz)^{\frac{\alpha}{k}}\right)\right\}(s) &= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)}\frac{b^{\frac{\alpha}{k}}n}{n!}\int_0^{\infty}e^{-sz}z^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}dz\\ &= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)}\frac{b^{\frac{\alpha}{k}}n}{n!}\frac{\Gamma\left(\frac{\alpha n+\beta}{k}\right)}{s^{\frac{\alpha}{k}n+\frac{\beta}{k}}}\frac{p^{\frac{\alpha n+\beta}{k}}}{p^{\frac{\alpha n+\beta}{k}}}\\ &= \sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)\ {}_p\Gamma_k(\alpha n+\beta)}\frac{b^{\frac{\alpha}{k}}n}{n!}\frac{{}_p\Gamma_k(\alpha n+\beta)}{s^{\frac{\alpha n+\beta}{k}}p^{\frac{\alpha n+\beta}{k}}}\\ &= \frac{1}{(sp)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{{}_pB_k(\gamma+nk,\xi-\gamma)\ {}_p(\xi)_{n,k}}{{}_pB_k(\gamma,\xi-\gamma)}\frac{\left(\left(\frac{b}{sp}\right)^{\frac{\alpha}{k}}\right)^n}{n!}\\ &= \frac{1}{(sp)^{\frac{\beta}{k}}}\ {}_2F_1\left(\xi,\gamma,\xi,p,k,\left(\frac{b}{sp}\right)^{\frac{\alpha}{k}}\right)\end{aligned}$$

□

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Author information

Pablo I. Pucheta, Faculty of Agricultural Sciences. Department of Mathematics, National University of Nordeste, Corrientes, Argentina.
E-mail: pablo.pucheta@hotmail.com

Received: June 06, 2018.
Accepted: October 07, 2018.