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ON SEMI-INVARIANT SUBMANIFOLDS OF ALMOST α-COSYMPLECTIC f -MANIFOLDS ADMITTING A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract In this paper, semi-invariant submanifolds of an almost α-cosymplectic f-manifold endowed with a semi-symmetric non-metric connection are studied. Necessary and sufficient conditions are given on a submanifold of an almost α-cosymplectic f-manifold to be semi-invariant submanifold with semi-symmetric non-metric connection. Moreover, we studied the intergrability condition of the distribution on semi-invariant submanifolds of an almost α-cosymplectic f-manifold with semi-symmetric non-metric connection.

1 Introduction

The notion of CR-submanifold of a Kaehler manifold was introduced by Bejancu [6]. Later, semi-invariant (or contact CR-) submanifolds of a Sasakian manifold was studied by Shahid, Sharfuddin and Husain [20], Kobayashi [14], Matsumoto [17] and many others. Submanifolds of cosymplectic manifold have been studied by Ludden [16], A. Cabras, A.Ianus and G.H. Pitis [11]. Later, the subject was considered for Riemannian manifolds with an almost contact structure. In this sense A. Bejancu and N. Papaghiuc study semi-invariant submanifolds of a Sasakian manifold or Sasakian space form ([7], [8], [18], [19]) and M. A. Akyol, C. L. Bejan and A. Cabras et.al. study on cosymplectic manifolds in ([2], [5], [10]). B. B. Sinha and R. N. Yadav studied the integrable conditions of distributions and the geometry of leaves on a semi-invariant submanifolds in a Kenmotsu manifold [21].

In [13] Friedmann and Schouten introduced the notion of semi-symmetric linear connections. More precisely, if ∇ is a linear connection in a differentiable manifold M, the torsion tensor T of ∇ is given by $T(X,Y) = \nabla_X Y - \nabla_X Y - [X,Y]$, for any vector fields X and Y on M . The connection ∇ is said to be symmetric if the torsion tensor T vanishes, otherwise it is said to be non-symmetric. In this case, ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$, for any X, $Y \in \Gamma(TM)$, where η is a 1-form on M. Moreover, if g is a (pseudo)-Riemannian metric on M, ∇ is called a metric connection if $\nabla g = 0$, otherwise it is called non-metric. We also refer some papers ([3], [4]) related to the notion of semi-symmetric non-metric connections.

In 2014, Öztürk et.al. introduced and studied almost α-cosymplectic f-manifold [1] defined for any real number α which is defined a metric f-manifold with f-structure $(\varphi, \xi_i, \eta^i, g)$ satisfying the condition $d\eta^i = 0$, $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$.

The paper is organized as follows: In section 2, we give basic formulas and definitions for almost α-cosymplectic f-manifolds. In section 3, we defined almost α-cosymplectic f-manifold with a semi-symmetric non-metric connection and we obtained some basic results for semi-invariant submanifolds of almost α-cosymplectic f-manifold with a semi-symmetric non-metric connection. In last section, we obtained some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds of almost α-cosymplectic f-manifold with a semi-symmetric non-metric connection.

2 Preliminaries

Let $\widetilde{M}$ be a real (2n+s)-dimensional framed metric manifold [15] with a framed $(\varphi, \xi_i, \eta^i, g),\ i \in \{1, ..., s\}$, that is, φ is a non-vanishing tensor field of type (1,1) on $\widetilde{M}$ which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r = 2n$; $\xi_1, ..., \xi_s$ are s vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on $\widetilde{M}$ such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i \tag{2.1}$$

$$\eta^i(\xi_j) = \delta^i_j,\ \varphi(\xi_i) = 0,\ \eta^i o\varphi = 0, \tag{2.2}$$

$$\eta^i(X) = g(X, \xi_i), \tag{2.3}$$

$$g(X, \varphi Y) + g(\varphi X, Y) = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2.5}$$

for all $X, Y \in \Gamma(T\widetilde{M})$ and $i, j \in \{1, ..., s\}$. In above case, we say that $\widetilde{M}$ is a metric f-manifold and its associated structure will be denoted by $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ [15].
A 2-form Ω is defined by $\Omega(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\widetilde{M})$, is called the fundamental 2-form. A framed metric structure is called normal [15] if

$$[\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ. Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi_1 + \xi_2 + ... + \xi_s$ and $\overline{\delta}^j_i = \delta^1_i + \delta^2_i + ... + \delta^s_i$. In the sequel, from [1] we give the following definition.

Definition 2.1. Let $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ be a (2n+s)-dimensional a metric f-manifold for each $\eta^i, (1 \leq i \leq s)$ 1-forms and each 2-form Ω, if $d\eta^i = 0$ and $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$ satisfy, then $\widetilde{M}$ is called almost α-cosymplectic f-manifold [1].

Let $\widetilde{M}$ be an almost α-cosypmlectic f-manifold. Since the distribution D is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by:

$$\begin{aligned} 2g((\widetilde{\widetilde{\nabla}}_X \varphi)Y, Z) = 2\alpha g\left(\sum_{i=1}^{s}(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X), Z\right) \\ + g(N(Y, Z), \varphi X) \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Putting $X = \xi_i$ we obtain $\widetilde{\nabla}_{\xi_i}\varphi = 0$ which implies $\widetilde{\nabla}_{\xi_i}\xi_j \in D^\perp$ and then $\widetilde{\widetilde{\nabla}}_{\xi_i}\xi_j = \widetilde{\widetilde{\nabla}}_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$.
We put $A_i X = -\widetilde{\widetilde{\nabla}}_X \xi_i$ and $h_i = \frac{1}{2}(L_{\xi_i}\varphi)$, where L denotes the Lie derivative operator. If $\widetilde{M}$ is almost α-cosymplectic f-manifold with Kaehlerian leaves [12], we have

$$(\widetilde{\widetilde{\nabla}}_X \varphi)Y = \sum_{i=1}^{s} \left[-g(\varphi A_i X, Y)\xi_i + \eta^i(Y)\varphi A_i X\right]$$

or

$$(\widetilde{\widetilde{\nabla}}_X \varphi)Y = \sum_{i=1}^{s} \left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X\right]. \tag{2.7}$$

Proposition 2.2. *([1]) For any $i \in \{1, ..., s\}$ the tensor field A_i is a symmetric operator such that*

(i) $A_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$

(ii) $A_i o\varphi + \varphi o A_i = -2\alpha\varphi$

(iii) $tr(A_i) = -2\alpha n$

(iv) $\widetilde{\widetilde{\nabla}}_X \xi_i = -\alpha\varphi^2 X - \varphi h_i X.$

Proposition 2.3. *([9]) For any $i \in \{1, ..., s\}$ the tensor field h_i is a symmetric operator and satisfies*

(i) $h_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$

(ii) $h_i o\varphi + \varphi o h_i = 0$

(iii) $trh_i = 0$

(iv) $tr(\varphi h_i) = 0.$

Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold with respect to the curvature tensor field $\widetilde{\widetilde{R}}$ of $\widetilde{\widetilde{\nabla}}$, the following formulas are proved in [1], for all $X, Y \in \Gamma(T\widetilde{M}),\ i, j \in \{1, ..., s\}$.

$$\begin{aligned} \widetilde{\widetilde{R}}(X,Y)\xi_i &= \alpha^2 \sum_{k=1}^{s} (\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y) \\ &- \alpha \sum_{k=1}^{s} (\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X) \\ &+ (\widetilde{\widetilde{\nabla}}_Y \varphi h_i)X - (\widetilde{\widetilde{\nabla}}_X \varphi h_i)Y, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \widetilde{\widetilde{R}}(X,\xi_j)\xi_i &= \sum_{k=1}^{s} \delta_j^k (\alpha^2\varphi^2 X + \alpha\varphi h_k X) \\ &+ \alpha\varphi h_i X - h_i h_j X + \varphi(\widetilde{\widetilde{\nabla}}_{\xi_j} h_i)X, \end{aligned} \tag{2.9}$$

$$\widetilde{\widetilde{R}}(\xi_j, X)\xi_i - \varphi\widetilde{R}(\xi_j, \varphi X)\xi_i = 2(-\alpha^2\varphi^2 X + h_i h_j X). \tag{2.10}$$

Moreover, by using the above formulas, in [1] it is obtained that

$$\widetilde{\widetilde{S}}(X,\xi_i) = -2n\alpha^2 \sum_{k=1}^{s} \eta^k(X) - (div\varphi h_i)X \tag{2.11}$$

$$\widetilde{\widetilde{S}}(\xi_i,\xi_j) = -2n\alpha^2 - tr(h_j h_i) \tag{2.12}$$

for all $X, Y \in \Gamma(T\widetilde{M}),\ i, j \in \{1, ..., s\}$, where $\widetilde{\widetilde{S}}$ denote, the Ricci tensor field of the Riemannian connection. From [1], we have the following result.

Proposition 2.4. *Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold and M be an integral manifold of D. Then*

(i) when $\alpha = 0$, M is totally geodesic if and only if all the operators h_i vanish,

(ii) when $\alpha \neq 0$, M is totally umbilic if and only if all the operators h_i vanish.

3 Basic Results

Let $\tilde{\tilde{\nabla}}$ be the Levi-Civita connection of $\widetilde{M}$ with induced metric g. Then Gauss and Weingarten formulas are given by

$$\tilde{\tilde{\nabla}}_X Y = \nabla^{\star}_X Y + B(X, Y) \tag{3.1}$$

$$\tilde{\tilde{\nabla}}_X N = -A_N X + \nabla^{\star\perp}_X N \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. $\nabla^{\star\perp}$ is the connection in the normal bundle, B is the second fundamental form of $\widetilde{M}$ and A_N is the Weingarten endomorphism with associated with N. The second fundamental form B and the shape operator A related by

$$g(B(X, Y), N) = g(A_N X, Y) \tag{3.3}$$

Now, a semi-symmetric non-metric connection $\tilde{\nabla}$ is defined as

$$\tilde{\nabla}_X Y = \tilde{\tilde{\nabla}}_X Y + \sum_{i=1}^{s} \eta^i(Y) X \tag{3.4}$$

such that

$$(\tilde{\nabla}_X g)(Y, Z) = -\sum_{i=1}^{s} \{g(X, Y)\eta^i(Z) + g(X, Z)\eta^i(Y)\} \tag{3.5}$$

from (3.4), we have

$$(\tilde{\nabla}_X \varphi)Y = (\tilde{\tilde{\nabla}}_X \varphi)Y - \sum_{i=1}^{s} \eta^i(Y)\varphi X \tag{3.6}$$

and if $\widetilde{M}$ with Kaehlerian leaves

$$(\tilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X\right] - \sum_{i=1}^{s} \eta^i(Y)\varphi X. \tag{3.7}$$

Corollary 3.1. *Let M be semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with semi-symmetric non-metric connection, then*

$$\tilde{\nabla}_X \xi_i = -\alpha\varphi^2 X - \varphi h_i X + X \tag{3.8}$$

and

$$(\tilde{\nabla}_X \bar{\eta})Y = (\tilde{\tilde{\nabla}}_X \bar{\eta})Y - \bar{\eta}(X)\bar{\eta}(Y). \tag{3.9}$$

We denote by same symbol g both metrices on $\widetilde{M}$ and M. Let $\tilde{\nabla}$ be the semi-symmetric non-metric connection on $\widetilde{M}$ and ∇ be the induced connection on M with respect to unit normal N. Then,

$$(\tilde{\nabla}_X Y) = \nabla_X Y + m(X, Y) \tag{3.10}$$

where m is a tensor field of type $(0, 2)$ on semi-invariant submanifold M. Using (3.1) and (3.4) we have,

$$\nabla_X Y + m(X, Y) = \nabla^{\star}_X Y + B(X, Y) + \sum_{i=1}^{s} \eta^i(Y) X. \tag{3.11}$$

So equation tangential and normal components from both the sides, we get

$$m(X, Y) = B(X, Y)$$

and

$$\nabla_X Y = \nabla^{\star}_X Y + \sum_{i=1}^{s} \eta^i(Y) X. \tag{3.12}$$

From (3.2) and (3.12),

$$\begin{aligned} \nabla_X N &= \nabla^\star_X N + \sum_{i=1}^{s} \eta^i(N)X \\ &= -A_N X + \sum_{i=1}^{s} \eta^i(N)X \\ &= (-A_N + \sum_{i=1}^{s} \eta^i(N))X. \end{aligned}$$

Now, Gauss and Weingarten formulas for a semi-invariant submanifolds of an almost α- cosymplectic f- manifold with a semi-symmetric non-metric connection is

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{3.13}$$

and

$$\begin{aligned} \widetilde{\nabla}_X N &= (-A_N + \sum_{i=1}^{s} \eta^i(N))X + \nabla^\perp_X N \\ &= -A_N X + \nabla^\perp_N X \end{aligned} \tag{3.14}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$, B second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form B and the shape operator A related by

$$g(B(X,Y),N) = g(A_N X, Y) \tag{3.15}$$

The projection morphisms of TM to D and Q respectively. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, we have

$$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i \tag{3.16}$$

and

$$\varphi N = CN + DN \tag{3.17}$$

$$h_i X = t_i X + f_i X \tag{3.18}$$

where CN and t_iX(resp.DN and f_iX) denotes the tangential (resp. normal) of φN and h_iX, respectively.

Theorem 3.2. *The connection induced on semi-invariant submanifolds of an almost α-cosymplectic f-manifold $\widetilde{M}$ with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.*

For any $X, Y \in \Gamma(TM)$, we put

$$u(X,Y) = \nabla_X \varphi PY - A_{\varphi QY} X. \tag{3.19}$$

We start with proving the following lemma.

Lemma 3.3. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold* *with Kaehlerian leaves admitting semi-symmetric non-metric connection. Then we have*

$$P(u(X,Y)) = \varphi P\nabla_X Y - \sum_{i=1}^{s}[(\alpha+1)\eta^i(Y)\varphi PX + \eta^i(Y)Pt_iX] \tag{3.20}$$

$$Q(u(X,Y)) = QCB(X,Y) - \sum_{i=1}^{s}\eta^i(Y)Qt_iX \tag{3.21}$$

$$\begin{aligned} B(X,\varphi PY) + \nabla^{\perp}_X\varphi QY &= \varphi Q\nabla_X Y + DB(X,Y) \\ &- \sum_{i=1}^{s}[(\alpha+1)\eta^i(Y)\varphi QX - \eta^i(Y)f_iX] \end{aligned} \tag{3.22}$$

$$\begin{aligned} \eta^i(u(X,Y))\xi_i &= \sum_{i=1}^{s}[\alpha g(\varphi PX,Y)\xi_i + g(h_iX,Y)\xi_i] \\ &- \sum_{i,j=1}^{s}\eta^i(Y)\eta^j(t_iX)\xi_i. \end{aligned} \tag{3.23}$$

Proof. For any $X,Y \in \Gamma(TM)$, putting (3.6) in the equation (2.7) we get

$$\begin{aligned} (\widetilde{\nabla}_X\varphi)Y &= \sum_{i=1}^{s}[\alpha(g(\varphi PX,Y)\xi_i - \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX) + g(h_iX,Y)\xi_i \\ &- \eta^i(Y)Pt_iX - \eta^i(Y)Qt_iX - \eta^i(Y)\sum_{j=1}^{s}\eta^j(t_iX)\xi_j - \eta^j(Y)f_iX \\ &- \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX]. \end{aligned}$$

On the other hand

$$\begin{aligned} (\widetilde{\nabla}_X\varphi)Y &= \widetilde{\nabla}_X\varphi Y - \varphi\widetilde{\nabla}_X Y \\ &= \widetilde{\nabla}_X\varphi PY + \widetilde{\nabla}_X\varphi QY - \varphi(\nabla_X Y + B(X,Y)) \\ &= \nabla_X\varphi PY + B(X,\varphi PY) - A_{\varphi QY}X + \nabla^{\perp}_X\varphi QY \\ &- \varphi P\nabla_X Y - \varphi Q\nabla_X Y - CB(X,Y) - DB(X,Y) \end{aligned}$$

$$\begin{aligned} (\widetilde{\nabla}_X\varphi)Y &= P\nabla_X\varphi PY + Q\nabla_X\varphi PY + \sum_{i=1}^{s}\eta^i(\nabla_X\varphi PY)\xi_i + B(X,\varphi PY) \\ &- PA_{\varphi QY}X - QA_{\varphi QY}X + \nabla^{\perp}_X\varphi QY - \sum_{i=1}^{s}\eta^i(A_{\varphi QY}X)\xi_i \\ &- \varphi P\nabla_X Y - \varphi Q\nabla_X Y - CB(X,Y) - DB(X,Y). \end{aligned}$$

Taking the components of D, ξ_i, $D^{\perp}$ and $TM^{\perp}$ in above equations, we get desired result. □

Lemma 3.4. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with Kaehlerian leaves admitting semi-symmetric non-metric connection. Then we have*

$$\varphi P(A_NX) + P(\nabla_X CN) = P(A_{DN}X) \tag{3.24}$$

$$Q((C\nabla^{\perp}_X N) + A_{DN}X - \nabla_X CN) = 0 \tag{3.25}$$

$$\eta(A_{DN}X - \nabla_X CN) = \alpha g(X,CN) + g(h_iX,N)\xi_i \tag{3.26}$$

$$B(X,CN) + \varphi Q(A_NX) + \nabla^{\perp}_X DN = D\nabla^{\perp}_X N \tag{3.27}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

Proof. By using the decompositions (3.16), (3.17) and the equations of Gauss and Weingarten in (2.7) we have

$$(\widetilde{\nabla}_X\varphi)N = \widetilde{\nabla}_X\varphi N - \varphi\widetilde{\nabla}_X N = \sum_{i=1}^{s}[\alpha g(\varphi X, N)\xi_i + g(h_i X, N)\xi_i]$$

$$\nabla_X CN + B(X, CN) - A_{DN}X + \nabla_X^{\perp} DN + \varphi A_N X - \varphi\nabla_X^{\perp} N = \sum_{i=1}^{s}[\alpha g(\varphi X, N)\xi_i + g(h_i X, N)\xi_i]$$

$$= P\nabla_X CN + Q\nabla_X CN + \sum_{i=1}^{s}\eta^i(\nabla_X CN)\xi_i + B(X, CN) - PA_{DN}X - QA_{DN}X - \sum_{i=1}^{s}(A_{DN}X)\xi_i$$

$$+ \nabla_X^{\perp} DN + \varphi PA_N X + \varphi QA_N X - C\nabla_X^{\perp} N - D\nabla_X^{\perp} N$$

$$= -\sum_{i=1}^{s}[\alpha g(X, CN)\xi_i + g(h_i X, N)\xi_i].$$

Then (3.24)-(3.27) follows by taking the components on each of the vector bundle D, $D^{\perp}$, ξ_i and respectively $TM^{\perp}$. □

Lemma 3.5. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold admitting semi-symmetric non-metric connection. For any $X \in \Gamma(D)$ and $X \in \Gamma(D^{\perp})$, then we have*

$$\nabla_X\xi_i = (\alpha + 1)X - \varphi t_i X - Cf_i X, \quad B(X, \xi_i) = -Df_i X \tag{3.28}$$

$$\nabla_{\xi_i}\xi_j = 0, \quad B(\xi_i, \xi_j) = 0. \tag{3.29}$$

Proof. For $X \in \Gamma(TM)$, using (3.8), (3.13), (3.17) and (3.18) we have

$$\begin{aligned}
\widetilde{\nabla}_X\xi_i = \nabla_X\xi_i + B(X, \xi_i) &= -\alpha\varphi^2 X - \varphi h_i X + X \\
&= \alpha X - \alpha\sum_{i=1}^{s}\eta^i(X)\xi_i - \varphi h_i X + X \\
&= \alpha X - \alpha\sum_{i=1}^{s}\eta^i(X)\xi_i - \varphi t_i X - \varphi f_i X + X \\
&= \alpha X - \alpha\sum_{i=1}^{s}\eta^i(X)\xi_i - \varphi t_i X - Cf_i X - Df_i X + X. \qquad (3.30)
\end{aligned}$$

Thus (3.28) and (3.29) follows from (3.30). □

Lemma 3.6. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with Kaehlerian leaves admitting semi-symmetric non-metric connection. Then we have*

$$A_{\varphi X}Y = A_{\varphi Y}X \tag{3.31}$$

for all $X, Y \in \Gamma(D^{\perp})$.

Proof. By using (3.7), (3.13) and (3.15), we get

$$\begin{aligned}
g(A_{\varphi X}Y, Z) &= g(B(Y, Z), \varphi X) = g(\widetilde{\nabla}_Z Y, \varphi X) \\
&= -g(\varphi\widetilde{\nabla}_Z Y, X) = -g(\widetilde{\nabla}_Z\varphi Y - (\widetilde{\nabla}_Z\varphi)Y, X) \\
&= -g(\widetilde{\nabla}_Z\varphi Y, X) - g((\widetilde{\nabla}_Z\varphi)Y, X) \\
&= -g(\widetilde{\nabla}_Z\varphi Y, X) = g(\varphi Y, \widetilde{\nabla}_Z X) \\
&= g(\varphi Y, B(Z, X)) \\
&= g(A_{\varphi Y}X, Z)
\end{aligned}$$

for all $X, Y \in \Gamma(D^{\perp}), Z \in \Gamma(TM)$ which proves (3.31). □

Lemma 3.7. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with admitting semi-symmetric non-metric connection. Then we find*

$$\nabla_{\xi_k} U \in \Gamma(D), for\ any\ U \in \Gamma(D) \tag{3.32}$$

$$\nabla_{\xi_i} V \in \Gamma(D^{\perp}), for\ any\ V \in \Gamma(D^{\perp}). \tag{3.33}$$

Proof. From (3.5), we have

$$\begin{aligned}(\widetilde{\nabla}_X g)(Y, Z) &= -\sum_{i=1}^{s}\{g(X, Y)\eta^i(Z) + g(X, Z)\eta^i(Y)\} \\ &= Xg(Y, Z) - g(\widetilde{\nabla}_X Y, Z) - g(Y, \widetilde{\nabla}_X Z).\end{aligned}$$

Now, by taking $Y = U \in D$ and $X = \xi_k, Z = \xi_\ell,\ k, \ell \in \{1, ..., s\}$ in the above equation, we get

$$\begin{aligned}(\widetilde{\nabla}_{\xi_k} g)(U, \xi_\ell) &= -\sum_{i=1}^{s}\{g(\xi_k, U)\eta^i(\xi_\ell) + g(\xi_k, \xi_\ell)\eta^i(U)\} \\ &= \xi_k g(U, \xi_\ell) - g(\widetilde{\nabla}_{\xi_k} U, Z) - g(U, \widetilde{\nabla}_{\xi_k}\xi_\ell).\end{aligned}$$

Then we obtain,

$$g(\nabla_{\xi_k} U, \xi_\ell) = 0.$$

On the other hand, by taking $X = \xi_k, Y = U \in D, Z = V \in D^{\perp}$ we obtain

$$\begin{aligned}(\widetilde{\nabla}_{\xi_k} g)(U, V) &= -\sum_{i=1}^{s}\{g(\xi_k, U)\eta^i(V) + g(\xi_k, V)\eta^i(U)\} \\ &= \xi_k g(U, V) - g(\widetilde{\nabla}_{\xi_k} U, V) - g(U, \widetilde{\nabla}_{\xi_k} V).\end{aligned}$$

Hence,

$$\begin{aligned}g(\widetilde{\nabla}_{\xi_k} U, V) &= -g(U, \widetilde{\nabla}_{\xi_k} V) \\ &= g(\varphi^2 U, \widetilde{\nabla}_{\xi_k} V) \\ &= -g(\varphi U, \varphi\widetilde{\nabla}_{\xi_i}\varphi V) \\ &= -g(\varphi U, \widetilde{\nabla}_{\xi_k}\varphi V) \\ &= g(\widetilde{\nabla}_{\xi_i}\varphi U, \varphi V) \\ &= 0.\end{aligned}$$

So $\nabla_{\xi_k} U \in \Gamma(D)$. In a similary way is deduced (3.33). □

4 Integrability of Distribution on a Semi-Invariant Submanifolds of Almost α-Cosymplectic f-Manifolds Admitting a semi- symmetric non-metric connection

Lemma 4.1. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with admitting semi-symmetric non-metric connection. Then we have*

$$g(X, t_i Y) = g(t_i X, Y), \tag{4.1}$$

$$\varphi t_i X + t_i \varphi X + C f_i X = 0, \tag{4.2}$$

$$D f_i X + f_i \varphi X = 0 \tag{4.3}$$

for any $X, Y \in \Gamma(M)$.

Proof. Since h_i is symmetric, we get

$$g(X, h_iY) = g(h_iX, Y)$$
$$g(X, t_iY + f_iY) = g(t_iX, Y) + g(f_iX, Y)$$
$$g(X, t_iY) + g(X, f_iY) = g(t_iX, Y) + g(f_iX, Y).$$

From above equation we get (4.1). By making use of proposotion 2.3 and using (3.17), (3.18), we get

$$\varphi t_iX + t_i\varphi X + Cf_iX + Df_iX + f_i\varphi X = 0. \tag{4.4}$$

Comparing the tangential and normal part of (4.4), we get (4.2) and (4.3), respectively. □

Theorem 4.2. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with admitting semi-symmetric non-metric connection. Then the distribution D is never integrable.*

Proof. For all $X, Y \in \Gamma(D)$, we have

$$\begin{aligned}
g([X, Y], \xi_i) &= g(\nabla_X Y, \xi_i) - g(\nabla_Y X, \xi_i)\\
&= -g(Y, \nabla_X \xi_i) + g(X, \nabla_Y \xi_i)\\
&= -g(Y, \alpha X - \varphi t_iX - Cf_iX + X) + g(X, \alpha Y - \varphi t_iY - Cf_iY + Y)\\
&= g(Y, \varphi t_iX) + g(Y, Cf_iX) - g(X, \varphi t_iY) - g(X, Cf_iY)\\
&= g(Y, \varphi t_iX + Cf_iX) - g(X, \varphi t_iY + Cf_iY)\\
&= -g(Y, t_i\varphi X) + g(X, t_i\varphi Y)\\
&= -g(t_iY, \varphi X) + g(t_iX, \varphi Y)\\
&= -g(Y, t_i\varphi X) - g(\varphi t_iX, Y)\\
&= -g(Y, t_i\varphi X + \varphi t_iX)\\
&= g(Y, Cf_iX) \neq 0.
\end{aligned}$$

This follows the non-integrability of D. □

Theorem 4.3. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with Kaehlerian leaves admitting semi-symmetric non-metric connection. The distribution $D \oplus \{\xi_1, ..., \xi_s\}$ is integrable if and only if*

$$B(X, \varphi Y) = B(\varphi X, Y) \tag{4.5}$$

is satisfied.

Proof. From (3.22), the distribution $D \oplus \{\xi_1, ..., \xi_s\}$ is integrable if and only if

$$B(X, \varphi Y) - B(Y, \varphi X) = \varphi Q[X, Y] = 0$$

is satisfied so, $B(X, \varphi Y) = B(Y, \varphi X)$. □

Theorem 4.4. *Let M be a semi-invariant submanifold of an almost α-cosymplectic f-manifold $\widetilde{M}$ with Kaehlerian leaves admitting semi-symmetric non-metric connection. Then the distribution $D^\perp$ is integrable.*

Proof. From (3.19), we have for $X, Y \in \Gamma(D^\perp)$

$$U(X, Y) = -A_{\varphi QY}X$$

operating φ in (3.20) we get

$$P\widetilde{\nabla}_X Y = \varphi P(A_{\varphi Y}X) \tag{4.6}$$

for any $X, Y \in \Gamma(D^\perp)$. By virtue of Lemma 3.6, (4.6) reduce to

$$P([X, Y]) = 0$$

which is prove that $[X, Y] \in \Gamma(D^\perp)$. □

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