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ON A GENERALIZED FRACTIONAL FOURIER TRANSFORM

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Abstract Luchko et al. [8] studied the fractional Fourier transform. In the present paper a generalization of the fractional Fourier transform is introduced and studied. Its inversion formula is also established. As application, we obtain generalized fractional Fourier transform of V-function and k-Mittag-Leffler function. Since the V-function is reducible to very important functions being widely used in Mathematics, Engineering and Mathematical Physics, the generalized fractional Fourier transform of several special functions is obtained as the special cases.

1 Introduction

Definition 1.1. *V*-function
The author [5] introduced a general class of functions called V-function defined in the following form (also see [7, p. 23]):

$$V_n(x) = V_n^{h_m,d,g_j}[p,\tau,k,w,q,k_m,a_j,b_r,\alpha,\beta,\delta;x]$$
$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}(x/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]} \tag{1.1}$$

where
(i) $p,\ k,w,q,\beta\ ,\ \delta,k_m\ ,a_j\ ,b_r$ ($m = 1,\ldots, t$; $j = 1,\ldots, s$; $r = 1,\ldots, u$) are real numbers.
(ii) t, s and u are natural numbers.
(iii) $h_m\ ,g_j \geq 1$ ($m = 1,\ldots, t$; $j = 1,\ldots, s$).
(iv) $\alpha > 0$, Re $(\tau) > 0$, Re $(d) > 0$, x is a real number and λ is an arbitrary constant.
(v) The series on the r. h. s. of (1.1) converges absolutely if $t < s$ or $t = s$ with $|\ p(x/2)^k\ | \leq 1$.

For details of convergence conditions of the series on the r. h. s. of (1.1) one may refer to the paper [6].
The V-function defined by (1.1) is quite general in nature as it unifies and extends a number of useful functions such as unified Riemann-Zeta function [3], generalized hypergeometric function [1], Bessel function [2], generalized Bessel function [2], Struve's function [2], Lommel's function [2], generalized Mittag-Leffler function [4, 9], exponential function, sine function, cosine function and Macrobert's E-function [1] etc.(see, e.g.[5, 7]).

Definition 1.2. *K*- Mittag-Leffler function
Let $k\ \in$ R; α, β, $\gamma\ \in$ C; Re$(\alpha) > 0$, Re$(\beta) > 0$ and $\tau\ \in$C, then the generalized k-Mittag-leffler function [11] is defined as

$$E_{k,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}\, z^n}{\Gamma_k(n\alpha+\beta)\, n!}, \tag{1.2}$$

where

$$\Gamma_k(\gamma) = k^{(\gamma/k)-1}\Gamma(\gamma/k), \tag{1.3}$$

$$(\gamma)_{nq,k} = k^{(nq)}(\gamma/k)_{nq} \tag{1.4}$$

and $(x)_\tau$, (x, $\tau \in$ C) denotes the Pochhammer symbol.

Definition 1.3. Lizorkin space
Let ψ (R) be the set of functions

$$\psi(R) = \{v \in S(R) : v^{(n)}(0) = 0, n = 0, 1, 2, ...\}. \tag{1.5}$$

The Lizorkin space of function ϕ (R)is defined as

$$\phi(R) = \{\varphi \in S(R) : \Im(\varphi) \in \psi(R)\}, \tag{1.6}$$

where R is a set of real numbers, S(R) is Schwartzian space of functions and $\Im[\varphi]$ denotes the fractional Fourier transform of the function φ.

Definition 1.4. Generalized fractional Fourier transform
Let f be a function belonging to ϕ (R). The generalized fractional Fourier transform of order μ, $\mu \leq \rho$ is defined by

$$\Phi_\mu(\omega) = \Omega_\mu [f] (\omega) = \int_{-\infty}^{\infty} \text{exp}(i\omega^{\rho/\mu}x) f(x)\, dx, \quad \omega > 0 \tag{1.7}$$

where μ and ρ are positive real numbers, and $\Omega[f] \in \psi$ (R).
If we put ρ = 1, equation (1.7) reduces to the following fractional Fourier transform [8, 11]:

$$\Upsilon_\mu(\omega) = \Im_m [f] (\omega) = \int_{-\infty}^{\infty} \text{exp}(i\omega^{1/\mu}x) f(x) dx, \quad \omega > 0 \tag{1.8}$$

where $0 < \mu \leq 1$.
If we put ρ = 1 and μ = 1, equation (1.7) reduces to the conventional Fourier transform.

2 Inversion formula for generalized fractional Fourier transform

The inversion formula for the generalized fractional Fourier transform is defined by

$$f(x) = (\rho/2\pi\mu) \int_{-\infty}^{\infty} exp(-i\omega^{\rho/\mu}x)\omega^{(\rho-\mu)/\mu}\, \Phi_\mu(\omega)d\omega, \tag{2.1}$$

where μ and ρ are positive real numbers, and $\mu \leq \rho$.

Proof. Let $f(x)$ be absolutely integrable in ($-\infty$, ∞), then from Fourier integral formula we have

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} f(z) \left[\int_{-\infty}^{\infty} \text{exp}\{iv(x-z)\}dv\right] dz \tag{2.2}$$

We put $v = -\,\omega^{\rho/\mu}$ and we get

$$f(x) = (\rho/2\pi\mu) \int_{-\infty}^{\infty} \text{exp}\{-i\omega^{\rho/\mu}\, x\}\omega^{(\rho-\mu)/\mu} \left[\int_{-\infty}^{\infty} \text{exp}\{i\omega^{\rho/\mu}\, z\} f(z) dz\right] d\omega \tag{2.3}$$

Now, we use the equation (1.7) and arrive at the desired result (2.1).

3 Generalized fractional Fourier transform of V-function

Theorem 1.4. *The generalized fractional Fourier transform of the V-function for $x < 0$ is given by*

$$\Omega_\mu[V_n\,(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; {V_n}^{h_m,d,g_j}[p,\tau,k,w,q,k_m,a_j,b_r,\alpha,\beta,\delta;x]dx$$

$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau} \exp\{i\pi(nk+dw+q-1)/2\}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]2^{nk+dw+q}} \tag{3.1}$$

$$\frac{\Gamma(nk+dw+q+\,1)}{(\omega)^{\rho(nk+dw+q+\,1)/\mu}},$$

where μ and ρ are positive real numbers, $\mu \le \rho$, Re$(nk + dw + q) > -1$ and the conditions mentioned with (1.1) are satisfied.

Proof. We first express V-function occurring in the l. h. s. of (3.1) in series form and then interchange the order of integration and summation which is permissible since the series occurring in (1.1) is absolutely convergent. Now we put $i\omega^{\rho/\mu}x = -\xi$ and we get l. h. s. (say Δ) as follows:

$$\triangle = \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau} \exp\{i\pi(nk+dw+q-1)/2\}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]2^{nk+dw+q}} \tag{3.2}$$

$$\frac{1}{(\omega)^{\rho(nk+dw+q+\,1)/\mu}} \int_0^{\infty} e^{-\xi}\;\; \xi^{nk+dw+q}\; d\xi.$$

Now, using the following result in (3.2)

$$\int_0^{\infty} e^{-x}x^{\lambda-1}dx = \Gamma(\lambda), \quad [Re(\lambda) > 0] \tag{3.3}$$

we arrive at the desired result (3.1).

Theorem 1.5. *The generalized fractional Fourier transform of the generalized k-Mittag-Leffler function for $x < 0$ is given by*

$$\Omega_\mu[E^{\gamma,\tau}_{k,\alpha,\beta}\,(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; E^{\gamma,\tau}_{k,\alpha,\beta}(x)dx$$

$$= \frac{k^{1-(\beta/k)}}{\Gamma(\gamma/k)} \sum_{n=0}^{\infty} \frac{(k)^{(\tau-\alpha/k)n}\Gamma(\gamma/k+n\tau)\exp\{i\pi(n-1)/2\}}{\Gamma\{(n\alpha+\beta)/k\}(\omega)^{\rho(n+1)/\mu}}, \tag{3.4}$$

where μ and ρ are positive real numbers, $\mu \le \rho$, Re$(\gamma/k + n\tau) > 0$ and the conditions mentioned with (1.2) are satisfied.

Proof. We first express generalized k-Mittag-Leffler function occurring in the l. h. s. of (3.4) in series form and then interchange the order of integration and summation which is permissible since the series occurring in (1.2) is absolutely convergent. Now we put $i\omega^{\rho/\mu}x = -\xi$ and we get l. h. s. (say Δ) as follows:

$$\triangle = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}\exp\{i\pi(n-1)/2\}}{\Gamma_k(n\alpha+\beta)(\omega)^{\rho(n+1)/\mu}n!} \int_0^{\infty} e^{-\xi}\xi^n d\xi. \tag{3.5}$$

Now, using the results (1.3), (1.4) and (3.3) in (3.5) we arrive at the desired result (3.4).

4 Special Cases

(i) If we take $\rho = 1$ in (3.4), we get the known result (33) due to Saxena et al. [11].

(ii) If we take m = 1, j = 2, r = 1, h_1 = 1, $g_1 = 1, g_2 = 1$, p = 2, τ = 1, k =1, w = 0, q = 0, k_1 = 0, a_1 = 0, a_2 = 0, b_1 = 0, β = 0, δ = 1 and λ = 1/Γ(d) in (3.1), the V-function reduces to the Wright's generalized Bessel function [2] and we get

$$\Omega_\mu[J_d^\alpha(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; J_d^\alpha(x)dx, \qquad x<0$$
$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(d+\alpha n+1)\exp\{i\pi(n+1)/2\}\omega^{\rho(n+1)/\mu}}, \tag{4.1}$$

where μ and ρ are positive real numbers,$\mu \leq \rho$ and the conditions mentioned with (1.1) are satisfied.

(iii) If we take m = 1, j = 2, r = 1, h_1 = 1, $g_1 = 3/2, g_2 = 1$, p = 1, τ = 1, k =2, w = 1, q = 1, k_1 = 0, a_1 = 0, a_2 = 0, b_1 = 1/2, α =1, β = 1/2, δ = 1 and λ = 1/{Γ(d) Γ(3/2)} in (3.1), the V-function reduces to the Struve's function [2] and we get

$$\Omega_\mu[H_d(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; H_d(x)dx, \qquad x<0$$
$$= \sum_{n=0}^{\infty} \frac{\exp\{i\pi(4n+d)/2\}\Gamma(2n+d+2)}{\Gamma(3/2+n)\Gamma(d+n+3/2)(2)^{2n+d+1}(\omega)^{\rho(2n+d+2)/\mu}}, \tag{4.2}$$

where μ and ρ are positive real numbers, $\mu \leq \rho$, Re(2n + d) $> -$ 2 and the conditions mentioned with (1.1) are satisfied.

(iv) If we take m = 1, $j = 2$, $r = 1$,$h_1 = 1$, $g_1 = (\mu' + v' + 3)/2$,$g_2 = (\mu' - v' + 3)/2$, $p = 1, \tau = 1, k = 2, w = \mu', q = 1, k_1 = 0, a_1 = 0, a_2 = 0, b_1 = -1, d = 1, \alpha = 1, \beta = -1, \delta = 1$ and $\lambda = 2^{\mu'+1}/(\mu' \pm v' + 1)$ in (3.1), the V-function reduces to the Lommel's function [2] and we get

$$\Omega_\mu[s_{\mu',v'}(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; s_{\mu',v'}(x)dx, \qquad x<0$$
$$= \frac{1}{(\mu' \pm v'+1)} \sum_{n=0}^{\infty} \frac{\exp\{i\pi(4n+\mu')/2\}\,\Gamma(2n+\mu'+2)}{\left(\frac{\mu'\pm v'+3}{2}\right)_n (4)^n\,(\omega)^{\rho(2n+\mu'+2)/\mu}}, \tag{4.3}$$

where μ and ρ are positive real numbers, $\mu \leq \rho$, Re(2n + μ') $> -$ 2 and the conditions mentioned with (1.1) are satisfied.

(v) If we take m = 1, j = 1, r = 1, h_1 = h, $q_1 = 1$ $p = -2$, τ = 1, k =1, w = 0, q = 0, k_1 = 0, a_1 = 0, b_1 = -1, β = -1, δ = 1 and λ = 1/Γ(d) in (3.1), the V-function reduces to the generalized Mittag-Leffler function introduced and studied by Prabhakar [10] and we get

$$\Omega_\mu[E_{\alpha,d}^h(x)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\; E_{\alpha,d}^h(x)dx, \qquad x<0$$
$$= \sum_{n=0}^{\infty} \frac{(h)_n\;\exp\{i\pi(n-1)/2\}}{\Gamma(d+\alpha n)\;\omega^{\rho(n+1)/\mu}}, \tag{4.4}$$

where μ and ρ are positive real numbers, $\mu \leq \rho$ and the conditions mentioned with (1.1) are satisfied.
If we put h = 1 in (4.4), the generalized Mittag-Leffler function reduces to the generalized Mittag-Leffler function $E_{\alpha,\,d}(x)$ studied by Wiman [12] which reduces to the Mittag-Leffler function $E_\alpha(x)$ when d = 1.

(vi) If we take m = 1, j = 1, r = 1, h_1 = h, $g_1 = 1, p$ =-2, $k = 1, w = 0, q = 0, k_1$ = 0, a_1 = 0, b_1= 0, α =1, β = 0, δ = 0 and λ = 1in (3.1), the V-function reduces to the unified Riemann-zeta function [3] and we get

$$\Omega_\mu[\phi_h(x,\,\tau,\,d)](\omega) = \int_{-\infty}^{\infty} \exp(i\omega^{\rho/\mu}x)\;\phi_h(x,\,\tau,\,d)dx, \qquad x<0$$
$$= \sum_{n=0}^{\infty} \frac{(h)_n\;\exp\{i\pi(n-1)/2\}\,(d+n)^{-\tau}}{\omega^{\rho(n+1)/\mu}}, \tag{4.5}$$

where μ and ρ are positive real numbers, $\mu \leq \rho$ and the conditions mentioned with (1.1) are satisfied.
If we put h =1 in (4.5), the unified Riemann-zeta function reduces to the Hurwitz-Lerch zeta function which reduces to the generalized zeta function when we put x =1 and Riemann-zeta function when $x = 1$ and $d = 1$.

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