Palestine Journal of Mathematics

Vol. 9(2) (2020) , 908–914

CESÀRO MEANS OF GENERALIZED MITTAG-LEFFLER'S EXPANSIONS

Adnan G. AlAmoush

Communicated by S. P. Goyal

MSC Subj. Class: 30C45

Keywords and phrases: univalent function; Hypergeometric functions; Mittag Leffler's function; Cesàro means; convolution product.

Abstract The polynomial approximants which retain the zero free property of a given analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$ of the form

$$ {}^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^k, $$

$$ \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, R(\delta) > 0, z \in C $$

is found. Cesàro means of order μ by the convolution method of geometric function retain the zero free property of the derivatives of bounded convex functions in the unit disk. Also other properties are established.

1 Introduction

The function

$$ \psi_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \alpha \in C, R(\alpha) > 0, R(\beta) > 0, z \in C \tag{1.1} $$

was introduced by [1] and was investigated systimatically by several other authors (for detail, see [2]). The Mittag-Leffler's function is the direct generalisation of the exponential function to which it reduces for $\alpha = 1$.

Wiman [2] studied the generalisation of $\psi_{\alpha,\beta}(z)$, that is given by

$$ \psi_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, z \in C \tag{1.2} $$

has properties very similar to those of Mittag-Leffler's function $\psi_\alpha(z)$.

In 1971, Prabhakar [3] introduced the function in the form $\psi^{\gamma}_{\alpha,\beta}(z)$ in the following form

$$ {}^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\beta+\alpha k)}, \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, z \in C. \tag{1.3} $$

Saigo and Kilbas [4] and Raina [5] investigated several properties and applications of (1.1)-(1.3). The function defined by (1.2) gives a generalization of (1.1). This generalization was studied by Wiman [2], Agarwal [6], and others.

Let D denote the open unit disk in C. It is well known that outer functions are zero-free on the unit disk. Outer functions, which play an important role in H_p theory to find a suitable finite (polynomial) approximation for the outer infinite series f so that the approximate reduces the zero-free property of f, arise in the characteristic equation which determines the stability of certain nonlinear systems of differential equations. Recall that an outer function is a function $f \in H_p$ of the form

$$f(z) = e^{i\gamma} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{it}z}{1-e^{it}z} \log\psi(t)dt}$$

where $\psi(t) \geq 0$, $\log\psi(t)$ is in L^1 and $\psi(t)$ is in L^p. See [6] for the definitions and classical properties of outer functions. Since any function f in H^1 which has $1/f$ in H^1 is an outer function, then typical examples of outer functions can be generated by functions of the form $\prod_{k=1}^{n}(1-e^{i\theta_k}z)^{\alpha_k}$ for $-1 < \alpha_k < 1$.

By using convolution methods that the classical Cesàro means, retains the zero-free property of the derivatives of bounded convex functions in the unit disk and play an important role in geometric function theory (see [6], [7], [8], [9], [10]).

Recently, Shukla and Prajapati [11], investigated and studied the following function

$${}^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^k, \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, R(\delta) > 0, z \in C \tag{1.4}$$

with

$$0 < (\gamma)_{k\delta} \leq \Gamma(\beta + \alpha k).$$

Where $(\gamma)_{\delta k}$ is the Pochhammer symbol defined by $(\gamma)_{\delta k} = \frac{\Gamma(\gamma+\delta k)}{\Gamma(\gamma)}$. whenever $\Gamma(\gamma)$ is defined, $(\gamma)_0 = 1$, $\gamma \neq 0$. It is an entire function of order $\rho = R[(\alpha)]^{-1}$ and type $\sigma = \frac{1}{\rho} R[(\alpha)^{R(\alpha)}]^{-\rho}$. It is a special case of Wright's generalized hypergeometric functions, Wright ([12], [13]) as well as the H-function (see in detail [14]) . Some special cases of this function are enumerated below

$${}^{\gamma,1}_{\alpha,\beta}(z) = {}^{\gamma}_{\alpha,\beta}(z), \psi_{\alpha,1}(z) = \psi^1_{\alpha}(z),$$

$${}^{1,1}_{\alpha,\beta}(z) = \psi^1_{\alpha,\beta}(z) = \psi_{\alpha,\beta}(z),$$

$${}^{1,1}_{1,1}(z) = \psi_{1,1}(z), \psi_1(z) = e^z, \psi_2(z^2) = cosh(z).$$

Lemma 1.1. *[8] Let* $0 < \alpha \leq \beta$. *if* $\beta \geq 2$ *or* $\alpha + \beta \geq 3$, *then the function of the form* $f(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1}$ *is convex.*

Lemma 1.2. *[15] Assume that* $a_1 = 1$ *and* $a_k \geq 0$ *for* $k \geq 0$ *such that* a_k *is a convex decreasing sequence i.e.:*

$$a_k - 2a_{k+1} + a_{k+2} \geq 0 \text{ and } a_{k+1} - a_{k+2} \geq 0.$$

Then

$$R\{\sum_{k=1}^{\infty} a_k z^{k-1}\} > \frac{1}{2}, z \in U.$$

We denote by S^*, C, QS^* and QC the subclasses of A consisting of functions which are, respectively, starlike in U, convex in U, close-to-convex and quasi-convex in U Thus by definition, we have

$$S^* := \{\psi \in A : R(\frac{z\psi'(z)}{\psi(z)}) > 0, z \in U\},$$

$$C := \{\psi \in A : R(1 + \frac{z\psi''(z)}{\psi'(z)}) > 0, z \in U\},$$

$$QS^* := \{\psi \in A : \exists g \in S^* s.t. R(\frac{z\psi'(z)}{g(z)}) > 0, z \in U\},$$

and

$$QC := \{\psi \in A : \exists g \in C s.t. R(\frac{(z\psi'(z))'}{g'(z)}) > 0, z \in U\}.$$

It is easily observed from the above definitions that

$$\psi(z) \in C \Leftrightarrow z\psi' \in S^* \tag{1.5}$$

$$\psi(z) \in QC \Leftrightarrow z\psi^{'} \in QS^{*}. \tag{1.6}$$

Note that $\psi \in QS^{*}$ if and only if there exists a function $g \in S^{*}$ such that:

$$z\varphi'(z) = g(z)p(z) \tag{1.7}$$

where $p(z) \in P$, the class of all analytic functions of the form

$$p(z) = 1 + p_1 + z + p_2 z^2 + ... \text{, with } p(0) = 1.$$

Let be given two functions $\psi, g \in A$, s.that:

$$\psi(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^k$$

and

$$g(z) = \sum_{k=0}^{\infty} \frac{(\varrho)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^k.$$

Then their convolution or Hadamard product $\psi(z) * g(z)$ is defined by:

$$\psi(z) * g(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{k\delta}(\varrho)_{k\delta}}{k!k![\Gamma(\beta+\alpha k)]^2} z^k.$$

We can verify the following result for $f \in A$ and takes the form (1).

Lemma 1.3. *[9] $(i) If \psi \in C and\ g \in S^{*}$, then $\psi * g \in S^{*}$*
$(ii) If \psi \in C$ and $g \in S^{*}$, $p \in P$ with $p(0) = 1$, then $\psi * gp = (\psi * g)p_1$

where $p_1(U) \subset$ close convex hull of $p(U)$.

2 CESÀRO APPROXIMANTS FOR OUTER FUNCTIONS

The Cesàro sums of order μ where $\mu \in N \cup \{0\}$ of series of the form (1.1) can be defined as

$$\sigma_n^{\mu}(z,\psi) = \sigma_n^{\mu} * \psi(z) = \sum_{k=0}^{n} \frac{\begin{pmatrix} n-k+\mu \\ n-k \end{pmatrix}}{\begin{pmatrix} n+\mu \\ n \end{pmatrix}} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^{k+1}$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. We have the following result:

Theorem 2.1. *Let $\psi \in A$ be convex in U. Then the Cesàro means $\sigma_n^{\mu}(z,\psi)$, $z \in U$ of order $\mu > 1$, of $\psi'(z)$ are zero-free on U for all k.*

Proof. In view of Lemma 1.1, the analytic function f of the form (1) is convex in U if $\Gamma(\beta + \alpha k) \geq 2$ or $\Gamma(\beta + \alpha k) + (\gamma)_{k\delta} \geq 3$. where $0 < (\gamma)_{k\delta} \leq \Gamma(\beta + \alpha k)$.

Let $\varphi(z) = \sum_{k=0}^{\infty}(1 + k)z^{k+1}$ be defined such that:

$$z\psi'(z) = \phi(z) * \psi(z) = \sum_{k=0}^{\infty} \frac{(k+1)(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^{k+1}.$$

Then

$$\begin{aligned} \sigma_n^{\mu}(z,\psi') &= \sigma_n^{\mu}(z) * \psi'(z) \\ &= \frac{z\sigma_n^{\mu} * z\psi'(z)}{z} \\ &= \frac{z\sigma_n^{\mu} * \psi(z) * \phi(z)}{z} \end{aligned}$$

$$= \frac{(z\sigma_n^\mu)' * \psi(z)}{z}.$$

In view of Lemma 1.3, the relation (1.4) and the fact that is $z\sigma_n^\mu$ convex yield that there exists a function $g \in S^*$, and $p \in P$ with $p(0) = 1$ such that

$$= \frac{(z\sigma_n^\mu)' * \psi(z)}{z} = \frac{gp(z) * \psi(z)}{z} = \frac{[g(z) * \psi(z)]p_1(z)}{z} \neq 0.$$

We know that $R(p_1(z)) > 0$ and that $\psi(z) * g(z) = 0$ if and only if $z = 0$. Hence, $\sigma_n^\mu(z, \psi') \neq 0$ and the proof is complete.

Corollary 2.2. *If $f(U)$ is bounded convex domain, then the Cesàro means $\sigma_n^\mu(z)$, $z \in U$ for the outer function $\psi'(z)$ are zero-free on U for all k.*

Proof. It comes from the fact that the derivatives of bounded convex functions are outer function [16]. The next result shows the upper and lower bound for $\sigma_n^\mu(z, \psi')$.

Theorem 2.3. *Let $\in A$, Assume that $0 < (\gamma)_{k\delta} \leq \Gamma(\beta + \alpha k)$ with $\Gamma(\beta + \alpha k) \geq 2$ or $\Gamma(\beta + \alpha k) + (\gamma)_{k\delta} \geq 3$. Then*

$$\frac{1}{2}|z| < |\sigma_n^\mu(z, \psi')| \leq n(n+1), n \geq 1, z \in U, z \neq 0.$$

Proof. Under the conditions of the theorem, we have that f is convex Lemma 1.1, then in virtue of Theorem 2.1, we obtain that

$\sigma_n^\mu(z, \psi') \neq 0$, thus $|\sigma_n^\mu(z, \psi')| > 0$. Now by applying Lemma 1.2, on $\sigma_n^\mu(z, \psi') \neq 0$, and using the fact that $R(z) \leq |z|$ and since:

$$\left(\begin{array}{c} k - n + \mu \\ k - n \end{array}\right) = \frac{k!(k - n + \mu)!}{(k - n)!(k + \mu)!} \leq 1 \tag{2.1}$$

for $\mu \geq 1$ and $k = 1, 2, 3, ..., n$ yield

$$\frac{1}{2} < R\left(\frac{\sigma_n^\mu(z, \psi')}{z}\right) \leq \frac{|\sigma_n^\mu(z, \psi')|}{|z|}, |z| > 0 \text{ and } z \in U.$$

For the other side, we pose that

$$|\sigma_n^\mu(z, \psi')| = |\psi'(z) * \sigma_n^\mu(z)| = |\sum_{k=0}^n (k+1) \frac{\left(\begin{array}{c} n - k + \mu \\ n - k \end{array}\right)}{\left(\begin{array}{c} n + \mu \\ n \end{array}\right)} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta + \alpha k)} z^k|$$

$$\leq \sum_{k=0}^n (k+1) \frac{\left(\begin{array}{c} n - k + \mu \\ n - k \end{array}\right)}{\left(\begin{array}{c} n + \mu \\ n \end{array}\right)} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta + \alpha k)} |z^k|$$

$$\leq \sum_{k=0}^n (k+1) \frac{\left(\begin{array}{c} n - k + \mu \\ n - k \end{array}\right)}{\left(\begin{array}{c} n + \mu \\ n \end{array}\right)} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta + \alpha k)}$$

$$\leq \sum_{k=0}^n (k+1) \leq k(k+1)$$

when $k \to n$: Hence the proof.

Theorem 2.4. *Let $\psi \in A$ and $0 < (\gamma)_{k\delta} \leq \Gamma(\beta + \alpha k)$. Then:*

$$\lim_{n\to\infty} \sigma_n^{\alpha}(z,\psi) = \frac{z}{(1-z)^{\lambda}}, \lambda > 1, z \in U.$$

Proof. By the assumption and the fact (1.7), we obtain

$$\left| \sigma_n^{\alpha}(z,\psi) - \frac{z}{(1-z)^{\lambda}} \right|$$

$$= \left| \sum_{k=0}^{n} \frac{\begin{pmatrix} n-k+\mu \\ n-k \end{pmatrix}}{\begin{pmatrix} n+\mu \\ n \end{pmatrix}} \frac{(\gamma)_{k\delta}}{k!\Gamma(\beta+\alpha k)} z^{k+1} - \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} z^{k+1} \right|$$

$$= \frac{1}{k!} \left| \left[\sum_{k=0}^{n} \frac{\begin{pmatrix} n-k+\mu \\ n-k \end{pmatrix}}{\begin{pmatrix} n+\mu \\ n \end{pmatrix}} \frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)} - (\lambda)_k \right] z^{k+1} - \sum_{k=n+1}^{\infty} (\lambda)_k z^{k+1} \right|$$

$$\leq \left| \sum_{k=n+1}^{\infty} \frac{(\lambda)_k}{k!} - \sum_{k=1}^{n} \frac{(\lambda)_k}{k!} \right|$$

=0, as $n \to \infty$.

3 BOUNDED TURNING OF $\sigma_n^{\mu}(z,\psi)$

For $0 < \upsilon < 1$, let $B(\upsilon)$ denote the class of functions f of the form (1.1) so that $R(\psi') > \upsilon$ in U. The functions in $B(\upsilon)$ are called functions of bounded turning (see [17]). By the Nashiro-Warschowski Theorem, the functions in $B(\upsilon)$ are univalent and also close-to-convex in U. In the sequel we need to the following results.

Lemma 3.1. *[18] For $f \in U$ we have*

$$R\left\{ \sum_{k=1}^{j} \frac{z^k}{k+2} \right\} > \frac{-1}{3}, (z \in U).$$

Lemma 3.2. *[19] Let $P(z)$ be analytic in U; such that $P(0) = 1$; and $R(P(z)) > \frac{1}{2}$ in U. For functions Q analytic in U the convolution function $P * Q$ takes values in the convex hull of the image on U under Q.*

Theorem 3.3. *Let $g(z) \in H$ the class of normalized function takes the form $R\left\{ \sum_{k=2}^{\infty} a_k z^k \right\}, (z \in U)$. Denoted by $\omega_k = \frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)} \geq 1$ such that $\gamma_1 = 1$, if $\frac{1}{2} < \upsilon < 1$ and $g(z) \in B(\upsilon)$, then*

$$\sigma_n^{\mu}(z,\psi) g(z) \in B\left(\frac{3(n+\mu)! - (\mu+1)! n! (1-\upsilon)}{3(n+\mu)!} \right).$$

Proof. Let $g(z) \in B(\nu)$ that is

$$R(g'(z)) > \nu, (0 < \upsilon < 1, z \in U).$$

Implies

$$R\left\{ 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \right\} > \upsilon > \frac{1}{2}.$$

Now for $\frac{1}{2} < \upsilon < 1$ we have

$$R\left\{ 1 + \sum_{k=2}^{\infty} \frac{k}{1-\upsilon} a_k z^{k-1} \right\} > R\left\{ 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \right\} > \upsilon > \frac{1}{2}.$$

It is clear

$$R\left\{1+\sum_{k=2}^{\infty}\frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)}\frac{k}{1-\upsilon}a_k z^{k-1}\right\}>\frac{1}{2}. \tag{3.1}$$

Applying the convolution properties of power series to $[\sigma_n^{\mu}(z,\psi)g(z)]'$, we may

$$[\sigma_n^{\mu}(z,\psi)g(z)]' =$$

$$= 1+\ \sum_{k=2}^{\infty}\frac{\begin{pmatrix} n-(k-1)+\mu \\ n-(k-1)\end{pmatrix}}{\begin{pmatrix} n+\mu \\ n\end{pmatrix}}\frac{k!(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)}$$

$$= [1+\ \sum_{k=2}^{\infty}\frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)}\frac{k}{1-\upsilon}a_k z^{k-1}] * [\sum_{k=2}^{\infty}\frac{\begin{pmatrix} n-(k-1)+\mu \\ n-(k-1)\end{pmatrix}}{\begin{pmatrix} n+\mu \\ n\end{pmatrix}}(1-\upsilon)z^{k-1}]$$

$$:= P(z)*Q(z). \tag{3.2}$$

In virtue of Lemma 1.2 and for $j=n-1$; we receive

$$R\left\{\sum_{k=2}^{n}\frac{z^{k-1}}{k+1}\right\}>\frac{-1}{3}, \tag{3.3}$$

since

$$R\left\{\sum_{k=2}^{n}z^{k-1}\right\}\geq R\left\{\sum_{k=2}^{n}\frac{z^{k-1}}{k+1}\right\}, \tag{3.4}$$

and in view of (3.2),

$$R\left\{\sum_{k=2}^{n}z^{k-1}\right\}\geq\frac{-1}{3}. \tag{3.5}$$

Thus when $n\to k$, a computation gives

$$R(Q(z)) = R\left\{1+\ \sum_{k=2}^{\infty}\frac{\begin{pmatrix} n-(k-1)+\mu \\ n-(k-1)\end{pmatrix}}{\begin{pmatrix} n+\mu \\ n\end{pmatrix}}(1-\upsilon)z^{k-1}\right\}$$

$$> \left(\ \frac{3(n+\mu)-!(\mu+1)!n!(1-\upsilon)}{3(n+\mu)!}\ \right).$$

On the other hand, the power series

$$P(z)=\left\{1+\sum_{k=2}^{\infty}\frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)}\frac{k}{1-\upsilon}a_k z^{k-1}\right\},(z\in U).$$

Therefore, by Lemma 3.2, we have

$$R\{P(z)\}=R\left\{1+\sum_{k=2}^{\infty}\frac{(\gamma)_{k\delta}}{\Gamma(\beta+\alpha k)}\frac{k}{1-\upsilon}a_k z^{k-1}\right\}>\tfrac{1}{2},(z\in U).$$

$$R\{[\sigma_n^{\mu}(z,\psi)g(z)]'\}>\left(\ \frac{3(n+\mu)!(\mu+1)!n!(1-\upsilon)}{3(n+\mu)!}\ \right),(z\in U).$$

This completes the proof of Theorem 3.3.

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold. Then for*

$$\left\{ \frac{\binom{n-(k-1)+\mu}{n-(k-1)}}{\binom{n+\mu}{n}} \to 1 \right\}, \sigma_n^{\mu}(z,\psi)g(z) \in B(\tfrac{2+\upsilon}{3}).$$

References

[1] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris*, **137** 554–558 (1903).

[2] A. Wiman, Uber den fundamental Satz in der Theorie der Funktionen $E_\alpha(x)$, *Acta Math.* **29** 191–201 (1905).

[3] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler, *Yokohana. Math. J.*, **19** 7–15 (1971).

[4] M. Saigo, A. A. Kilbas, On Mittag-Leffler type function and applications, *Int. Trans. Spec. Funs.*, **7** 97–112 (1998).

[5] R. K. Raina, On generalized Wright's hypergeometric functions and fractional calculus operators, *East. Asian. Math. J.*, **21 (2)** 191–203 (2005).

[6] R. P. Agarwal, A product d'ume note de M. pierre Humbert, *C.R. Acad. Sci. Paris*, **236** 2031–2032 (1953).

[7] P. L. Duren *Univalent Functions,* Springer-Verlag, Berlin, 1983.

[8] St. Ruscheweyh, Geometric properties of Cesáro means, *Results Math.*, **22** 739–748 (1992).

[9] St. Ruscheweyh, *Convolutions in Geometric Function Theory*, Les Presses de'l Universit'e de Montr'eal, Montr'eal. 1982.

[10] M. Darus , R. W. Ibrahim, On Cesaro Means of Order μ for Outer Functions, *Int. Jour. of Nonlinear Sci.*, **9 (4)** 455–460 (2010).

[11] A. K. Shukla, J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math Anal and Appl.*, **336** 797–811 (2007).

[12] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, *J. London Math. Soc.*, **10** 286–293 (1935).

[13] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, *J. London Math. Soc.*, **46** 389–408 (1940).

[14] M. M. Arak, K. S. Ram, and J. H. Hans, The H-function: Theory and Applications, 0.1007/978-1-4419-0916-9, (2007).

[15] St. Ruscheweyh, L. Salinas, Subordination by Cesáro means, *Complex Var. Theory Appl.*, **21** 279–285 (1993).

[16] R.W. Barnard, J. Cima, and K. Pearce, Cesáro sum approximation of outer functions, *Ann. Univ. Maria Curie-Sklodowska Sect.*, **A52 (1)** 1–7 (1998).

[17] B. Muckenhoupt, D. W. Webb, Two-weight norm inequalities for Cesáro means of Laguerre expansions, *Trans.Amer. Math. Soc.*, **353 (3)** 1119–1149 (2001).

[18] J. M. Jahangiri, K. Farahmand, Partial sums of functions of bounded turning, *J. Inequal. Pure and Appl. Math.*, **4(4) Art. 79** 1–3 (2003).

[19] A. W. Goodman, *Univalent Functions*, Vols. I and II, Polygonal Publishing House, Washington, New Jersey. 1983.

Author information

Adnan G. AlAmoush, Faculty of Science, Taibah University, Saudi Arabia, www.taibahu.edu.sa.

E-mail: `adnan-omoush@yahoo.com`

Received: August 29, 2018.

Accepted: December 13, 2018.