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Differential Equation of p-k Mittag-Leffler Function

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Abstract In this paper we introduce a homogeneous linear differential equation whose one of the solution is the p-k Mittag-Leffler function and deduce this differential equation for earlier defined different Mittag-Leffler functions.

1 Introduction

The different Mittag-Leffler function has been given by different authors in last century.
The Mittag-Leffler function $E_\alpha(z)$ introduced by Gosta Mittag-Leffler [5] in 1903, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1.1}$$

Here $z \in C, \alpha \geqslant 0$.

Wiman [3] generalized $E_\alpha(z)$ in 1905 and gave $E_{\alpha,\beta}(z)$ known as Wiman function, defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \tag{1.2}$$

Here $z, \alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.

Prabhakar [12] in 1971, gave next generalization of Mittag-Leffler function and denoted as $E^{\gamma}_{\alpha,\beta}(z)$ and defined as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.3}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$.

Shukla and Prajapati [2] in 2007, gave second generalization of Mittag-Leffler function and denoted it as $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.4}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < Re(\alpha) + 1$ and for $|z| < 1$ if $q = Re(\alpha) + 1$. It is entire function of order $\frac{1}{Re(\alpha)}$.

K.S.Gehlot [7], introduce Generalized k- Mittag-Leffler function in 2012, denoted as $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R^+; z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$, as,

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\, z^n}{\mathit{\Gamma}_k(n\alpha + \beta)(n!)}, \tag{1.5}$$

where$(\gamma)_{nq,k}$ is the k- pochhammer symbol and $\Gamma_k(x)$ is the k-gamma function given by [11]. The generalized Pochhammer symbol is given as,

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn}\prod_{r=1}^{q}(\frac{\gamma+r-1}{q})_n,\ if\ q\in N. \tag{1.6}$$

Recentely K.S.Gehlot [9] in the year 2018, introduce p-k Mittag-Leffler function,
Let $k,p \in R^+ - \{0\}; \alpha,\beta,\gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$.
The p - k Mittag-Leffler function denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{n!}. \tag{1.7}$$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by [8].

The two parameter pochhammer symbol is recently introduce by ([8], equation (2.1)), the p - k Pochhammer Symbol ${}_p(x)_{n,k}$ is given by,

$${}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k}+p)(\frac{xp}{k}+2p).........(\frac{xp}{k}+(n-1)p). \tag{1.8}$$

Where $x \in C; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$.
Two Parameter Gamma Function is given by ([8], equation (2.6), (2.7) and (2.14)), the p - k Gamma Function ${}_p\Gamma_k(x)$ is given by,

$${}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!p^{n+1}(np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \tag{1.9}$$

or

$${}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!p^{n+1}(np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \tag{1.10}$$

Where $x \in C/kZ^-; k,p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$.
The integral representation of p - k Gamma Function is given by

$${}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}}t^{x-1}dt,\ (k,p\in R^+-\{0\}; Re(x)>0). \tag{1.11}$$

The Generalized Hypergeometric function representation of ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$
for $\alpha = km, m \in N$ and $q \in N$, is given by ([10], equation 2.1),

$${}_pE^{\gamma,q}_{k,km,\beta}(z) = B\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i)_n}{\prod_{j=1}^{m}(b_j)_n(n!)}(Az)^n. \tag{1.12}$$

$${}_pE^{\gamma,q}_{k,km,\beta}(z) = B\ {}_qF_m[\ (a_i)_{i=1,2,...,q};\quad (b_j)_{j=1,2,..,m};\quad Az\ ]. \tag{1.13}$$

Where,

$$A = \frac{zp^qq^q}{p^mm^m},\ B = \frac{kp^{-\frac{\beta}{k}}}{\Gamma(\frac{\beta}{k})},\ a_i = (\frac{\frac{\gamma}{k}+i-1}{q})\ and\ b_j = (\frac{\frac{\beta}{k}+j-1}{m}). \tag{1.14}$$

Convergent criteria for Generalized Hypergeometic function,
(i) If $q \leq m$, the function ${}_qF_m(.)$ converge for all finite z.
(ii) If $q > m+1$, the function ${}_qF_m(.)$ converge for all $|z| < 1$ and diverge for $|z| > 1$.
(iii)If $q \leq m$, the function ${}_qF_m(.)$ diverge for $z \neq 0$.
(iv) If $q = m+1$, the function ${}_qF_m(.)$ absolutely convergent on the circle $|z| = 1$
if $Re(\sum_{j=1}^{m}\frac{\frac{\beta}{k}+j-1}{m} - \sum_{i=1}^{q}\frac{\frac{\gamma}{k}+i-1}{q}) > 0$

2 Main result

In this section we introduce a linear homogeneous differential equation known as p-k Mittag-Leffler differential equation. One of its solution is p-k Mittag-Leffler function [9]. Finally we deduce this differential equation whose one of the solution is earlier known different Mittag-Leffler functions.

Theorem 2.1. *The p-k Mittag-Leffler differential equation is defined as,*

$$[\theta\prod_{j=1}^{m}(\theta+b_j-1)-Az\prod_{i=1}^{q}(\theta+a_i)]W=0, q\leq m+1. \tag{2.1}$$

When no b_j is a negative integer or zero and no two b_j's differ by an integer or zero, then the solution is

$$W=\sum_{r=0}^{m}C_rW_r. \tag{2.2}$$

Where C_r is arbitrary constant, and

$$W_0={}_pE_{k,km,\beta}^{\gamma,q}(z). \tag{2.3}$$

and for $r=1,2,3,....,m$.

$$W_r=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i-b_r+1)_n}{\prod_{j=1}^{m}(b_j-b_r+1)_n(2-b_r)_n}(Az)^{n+1-b_r}. \tag{2.4}$$

Where $\theta=z\frac{d}{dz}$ and a_i, b_j and A are given by (1.14).

Proof. Whenever, in addition to the above restrictions, no b_j is a positive integer, then the linear combination (2.2) is the general solution of equation (2.1) around $z=0$. Note also that if $q\leq m$, then the series for $W_r; r=0,1,2,...,m$, converge for all finite z and that for $q=m+1$, the series for W_r converge for $\mid z\mid<1$.
First we will verify that W_0, satisfies equation (2.1).
Since $\theta(Az)^n=n(Az)^n$ and using (1.12), it follows that

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_0=B\sum_{n=0}^{\infty}\frac{n\prod_{j=1}^{m}(n+b_j-1)\prod_{i=1}^{q}(a_i)_n}{\prod_{j=1}^{m}(b_j)_n(n!)}(Az)^n,$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_0=B\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{q}(a_i)_n}{\prod_{j=1}^{m}(b_j)_{n-1}(n-1)!}(Az)^n,$$

now we replace n by $n+1$, we have

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_0=B\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i)_{n+1}}{\prod_{j=1}^{m}(b_j)_n(n)!}(Az)^{n+1},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_0=B\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i)_n\prod_{i=1}^{q}(n+a_i)}{\prod_{j=1}^{m}(b_j)_n(n)!}(Az)^{n+1},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_0=Az\prod_{i=1}^{q}(\theta+a_i)W_0,$$

Thus we have shown that $W_0={}_pE_{k,km,\beta}^{\gamma,q}(z)$ is a solution of the differential equation (2.1).
Now we will verify that $W_r, r=1,2,...,m$, satisfies equation (2.1).
From (2.4), we get immediately

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=\sum_{n=0}^{\infty}\frac{(n+1-b_r)\prod_{j=1}^{m}(n+1-b_r+b_j-1)\prod_{i=1}^{q}(a_i-b_r+1)_n}{\prod_{j=1}^{m}(b_j-b_r+1)_n(2-b_r)_n}(Az)^{n+1-b_r},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{q}(a_i-b_r+1)_n}{\prod_{j=1}^{m}(b_j-b_r+1)_{n-1}(2-b_r)_{n-1}}(Az)^{n+1-b_r},$$

now we replace n by $n+1$, we have

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=Az\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i-b_r+1)_{n+1}}{\prod_{j=1}^{m}(b_j-b_r+1)_{n}(2-b_r)_{n}}(Az)^{n+1-b_r},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=Az\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i-b_r+1)_{n}\prod_{i=1}^{q}(a_i-b_r+1+n)}{\prod_{j=1}^{m}(b_j-b_r+1)_{n}(2-b_r)_{n}}(Az)^{n+1-b_r},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=Az\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i-b_r+1)_{n}\prod_{i=1}^{q}(\theta+a_i)}{\prod_{j=1}^{m}(b_j-b_r+1)_{n}(2-b_r)_{n}}(Az)^{n+1-b_r},$$

$$\theta\prod_{j=1}^{m}(\theta+b_j-1)W_r=Az\prod_{i=1}^{q}(\theta+a_i)W_r.$$

Thus we have shown that, $W_r, r=1,2,...,m$ is the solutions of the differential equation (2.1).

Particular Cases: For some particular values of the parameters p,q,k,α,β and γ, we can obtain certain differential equations for different Mittag-Leffler functions, here we have chosen $\alpha=km; m,q\in N$ and $p,k\in R$.

[A] Put $p=k$ in (2.1), we have the differential equation.

$$[\theta\prod_{j=1}^{m}(\theta+b_j-1)-Az\prod_{i=1}^{q}(\theta+a_i)]W=0, q\le m+1. \tag{2.5}$$

Here $a_i=(\frac{\frac{\gamma}{k}+i-1}{q}), b_j=(\frac{\frac{\beta}{k}+j-1}{m})\ and\ A=\frac{q^q k^q}{m^m k^m},\ B=\frac{k^{1-\frac{\beta}{k}}}{\Gamma(\frac{\beta}{k})}.$
Equation (2.5), is the differential equation of Mittag-Leffler function
$W_0=GE^{\gamma,q}_{k,km,\beta}(z)$, defined by [7] and it is known result of ([6], equation (10)).
[B] Put $p=k, q=1$ in (2.1), we have the differential equation.

$$[\theta\prod_{j=1}^{m}(\theta+b_j-1)-Az(\theta+a_1)]W=0. \tag{2.6}$$

Here $a_1=\frac{\gamma}{k}, b_j=(\frac{\frac{\beta}{k}+j-1}{m})\ and\ A=\frac{k^{1-m}}{m^m},\ B=\frac{k^{1-\frac{\beta}{k}}}{\Gamma(\frac{\beta}{k})}.$
Equation (2.6), is the differential equation of K-Mittag-Leffler function
$W_0=E^{\gamma}_{k,km,\beta}(z)$, defined by [4].

[C] Put $p=k=1$ in (2.1), we have the differential equation.

$$[\theta\prod_{j=1}^{m}(\theta+b_j-1)-Az\prod_{i=1}^{q}(\theta+a_i)]W=0, q\le m+1. \tag{2.7}$$

Here $a_i=(\frac{\gamma+j-1}{q}), b_j=(\frac{\beta+j-1}{m})\ and\ A=\frac{q^q}{m^m},\ B=\frac{1}{\Gamma(\beta)}.$
Equation (2.7), is the differential equation of Mittag-Leffler function
$W_0=E^{\gamma,q}_{m,\beta}(z)$, defined by [2].
[D] Put $p=k=1, q=1$ in (2.1), we have the differential equation.

$$[\theta\prod_{j=1}^{m}(\theta+b_j-1)-Az(\theta+a_1)]W=0. \tag{2.8}$$

Here $a_1 = \gamma, b_j = (\frac{\beta+j-1}{m})$ and $A = \dfrac{1}{m^m},\ B = \dfrac{1}{\Gamma(\beta)}.$
Equation (2.8), is the differential equation of Mittag-Leffler function
$W_0 = E^{\gamma}_{m,\beta}(z)$, defined by [12].
[E] Put $p = k = 1, q = 1$ and $\gamma = 1$ in (2.1), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \tag{2.9}$$

Here $a_1 = 1, b_j = (\frac{\beta+j-1}{m})$ and $A = \dfrac{1}{m^m},\ B = \dfrac{1}{\Gamma(\beta)}.$
Equation (2.9), is the differential equation of Mittag-Leffler function
$W_0 = E_{m,\beta}(z)$, defined by [3].
[F] Put $p = k = 1, q = 1, \gamma = 1$ and $\beta = 1$ in (2.1), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \tag{2.10}$$

Here $a_1 = 1, b_j = (\frac{j}{m})$ and $A = \dfrac{1}{m^m},\ B = 1.$
Equation (2.10), is the differential equation of Mittag-Leffler function
$W_0 = E_m(z)$, definded by [5].

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