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ON FINSLER- COSMOLOGICAL MODELS IN EINSTEIN AND SCALAR-TENSOR THEORIES

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Abstract In this paper, we investigate the Finsler-Randers cosmological model. Then, to find the solution of this modified models with help of corresponding scalar factor. Further, we analyzed the behavior of such models like Einstein theory and scalar-tensor theory.

1 Introduction

The geometrical dark energy models acts as an important alternative to the scalar-field dark energy models, since they can explain the accelerated expansion of the universe. Such an approach is an attempt to evade coincidence and cosmological constant problems of the standard ΛCDM model. In this frame work, one may consider that the dynamical effects attributed to dark energy can resembled by the effects of a non standard gravity theory implying that the present accelerating stage of the universe can be driven only by cold dark matter, under a modification of the nature of gravity.

In [17], the author G. Randers introduced the Finsle-Randers cosmological model. In general, metrical extensions of Riemannions geometry can provide a Finslerian geometrical structure in a manifold which leads to generalized gravitational field theories. There are rapid developments of applications of Finsler geometry in its FR context, mainly in the topics of general relativity, astrophysics and cosmology ([17]-[1]). Then the authors P. C. Stavrinos, A. Koretsis and M. Stathakopoulos [19] were found the FR field equations provide a Hubble parameter that contains a extra geometrical term which can be used as a possibilities for dark energy.

The spatially homogeneous cosmological models allow extension of cosmological investigation to distorting and rotating universe, giving estimates of effects of anisotropy on primordial element production and on the measured $CMBR$ spectrum anisotropy [8]. Apart from the authors Hawking S. W. and Ellis G. F. R [10] were gave the observational reasons, there are various theoretical considerations that have motivated the study of anisotropic cosmologies.

There exists wide class of anisotropic cosmological models which are often studied in cosmology [15]. There are theoretical arguments that sustain the existence of an anisotropic case in [13]. Also, anisotropic cosmological models to avoid the assuption of specific initial conditions in FRW models. The universe could also be characterized by irregular expansion mechanism. Therefore, it would be useful to explore cosmological models in which anisotropies existing at early stage of expansion are damped out in the course of evolution [9].

In [19], the authors P. C. Stavrinos, et al...have studied the Friedman-like Robertson-walker model in generalized metric space time with weak anisotropy. Recently, the authors Basilakos and P. C. Stavrinos [4] were studied the cosmological equivalence between the Finsler-Randers space time and the DGP gravity model.
Bases on these, we propose to study the evolution of the universe with in the frame work of Finsler-Randers cosmology. In this paper, we study the Friedman-like Robertson-walker model in the Finsler- Randers cosmology.

2 Einstein theory of Finsler-Randers cosmological model

The energy conditions of general relativity permit one to deduce very powerful and general theorems about the behavior of strong gravitational fields and cosmological geometries. However, in this section we investigate the Finsler -Randers cosmological model and then discussed the energy conditions to such model.

The FR cosmic scenario is based on the Finalerian geometry which extends the Riemannian geometry. Notice that a Riemannian geometry is also a particular case of Finslerian. Here, we discuss only the main features of the theory (see [18], [3], [20]). Generally, a Finsler space is derived from a generating function $F(x, y)$ on the tangent bundle TM of a manifold M. The generating function F is differentiable on $TM_0 = TM\backslash\{0\}$ and continuous on the zero cross section. The function F is also positively homogeneous of degree one in y. In other words, F introduces a structure on the space-time manifold M is called Finsler space time. In the case of a FR space-time is

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i, \tag{2.1}$$

where a_{ij} are component of a Riemannian metric and $b_i = (b_0, 0, 0, 0)$ is weak primordial vector field with $|b_i| << 1$. Now, the Finslerian metric tensor g_{ij} is constructed by Hessian of $\frac{F^2}{2}$

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.2}$$

The Cartan tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{2}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$ is a significant ingredient of the Finsler geometry. It indeed, the authors P. C. Stavrinos et al..[19] has been found that $b_0 = 2C_{000}$.

The Finsler-Randers field equation is given by

$$R_{ij} - \frac{1}{2}g_{ij}T = -\frac{8\pi G}{c^4}T_{ij}, \tag{2.3}$$

where R_{ij} is Ricci tensor, T_{ij} is the energy momentum tensor and T is the trace of energy momentum tensor. Modelling the expanding universe as Finslerian perfect fluid that induces radiation and matter with four velocity U_i for comoving observers, we have

$$T_{ij} = -pg_{ij} + (\rho + p)U_i U_j, \tag{2.4}$$

where ρ and p are the total energy density and pressure of the cosmic fluid respectively.
Thus, the energy momentum tensor becomes

$$T_{ij} = diag(\rho, -pg_{11}, -pg_{22}, -pg_{33}). \tag{2.5}$$

In view of ([12], [6], [2]), we use the weak, dominant and strong energy conditions in the context of Finslerian cosmology for our model as: $T_0^0 = \rho$, $T_1^1 = T_2^2 = T_3^3 = -p$ in the locally Minkowski frame. Obviously the roots of matrix equation is

$$|T_{ij} - rg_{ij}| = diag((\rho - r), (r + p), (r + p), (r + p)). \tag{2.6}$$

It gives the eigne values r for the energy momentum tensor as $r_0 = \rho$ and $r_1 = r_2 = r_3 = -p$.

The energy conditions for our model are as follows:

(i) Null energy condition (NEC);

$$\rho + p \geq 0. \tag{2.7}$$

(ii) Weak energy condition (WEC);

$$r_0 \geq 0 \Rightarrow \rho \geq 0, \; r_0 - ri \geq 0 \Rightarrow \rho + p \geq 0. \tag{2.8}$$

(iii) Strong energy condition (SEC);

$$r_0 - \sum r_i \geq 0 \Rightarrow \rho + 3p \geq 0 \text{ and } \rho + p \geq 0. \tag{2.9}$$

(iv) Dominant energy condition (DEC);

$$r_0 \geq 0 \Rightarrow \rho \geq 0, \quad -r_0 \leq -r_i \leq r_0 \Rightarrow \rho \pm p \geq 0. \tag{2.10}$$

In the context of a FRW metric is

$$a_{ij} = diag\left(1, -\frac{a^2}{1-kr^2}, -a^2r^2, -a^2r^2\sin^2\theta\right), \tag{2.11}$$

where a is a function of time t only and k is the curvature parameter having the values $-1, 0, +1$ for open, flat and closed models respectively.
The non-zero components of the Ricci tensors are

$$R_{00} = 3\left(\frac{\ddot{a}}{a} - \frac{3\dot{a}}{4a}\dot{u}_0\right), \tag{2.12}$$

and

$$R_{ii} = -\left(\frac{a\ddot{a} + 2\dot{a}^2 + 2k + \frac{11}{4}a\dot{a}\dot{u}_0}{\Delta_{ii}}\right), \tag{2.13}$$

where $\Delta_{11} = 1 - kr^2$, $\Delta_{22} = r^2$ and $\Delta_{33} = r^2\sin^2\theta$.
From gravitational FR field equation (2.1) for comoving observers then the FRW Einstein field equations are

$$\frac{\ddot{a}}{a} + \frac{3\dot{a}}{4a}Z_t = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.14}$$

and

$$\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2(\frac{k}{a^2}) + \frac{11\dot{a}}{4a}Z_t = 4\pi G(\rho - p), \tag{2.15}$$

where over the dot denotes the derivative with respect to the cosmic time t and $Z_t = b_0 < 0$ [19]. Then, from equations (2.2) and (2.3) we get;

$$H^2 + \frac{k}{a^2} + HZ_t = \frac{8\pi G}{3}\rho. \tag{2.16}$$

The equation (2.16) is the modified Friedamann equation , in this the extra term $H(t)Z_t$ shows the affects of the dynamics of the universe. If we consider $b_0 = 0$ or $(C_{000} = 0)$ which implies that $Z_t = 0$, then the field equations (2.14) and (2.15) reduces to the nominal Einstein equations, it solution of which is usual Friedaman equation.

Here, the solution of which two cases aries to the two different physically variable cosmologies, which have physical interests to describe the decelerating and accelerating phases of universe.

Case 1:de Sittere solution

It is well known in cosmology because the current epoch, where in the universe expansion is being accelerated, can be described approximately. This kind of solution consists of an exponential expansion of the scalar factor, which yields a constant Hubble parameter.
From [1], we use the scalar factor as: $a = ce^{\sigma t}$, where c and σ are constants. For $\sigma^2 > 0$, it gives an accelerating universe. Using this scalar factor, the Hubble parameter becomes

$$H(t) = \frac{\dot{a}}{a} = \frac{\sigma c e^{\sigma t}}{c e^{\sigma t}} = \sigma. \tag{2.17}$$

From this, using (2.4) and (2.5) we obtain the energy density as

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 + \sigma Z_t + \frac{k}{c^2 e^{2\sigma t}}\right). \tag{2.18}$$

With the scalar factor and from equations (2.2) and (2.6), we get the pressure as

$$p = -\frac{3\sigma^2}{8\pi G} - \frac{5}{16\pi G}\sigma Z_t - \frac{k}{8\pi G c^2 e^{2\sigma t}}. \tag{2.19}$$

The vales of ρ and p in (2.18) and (2.19), we obtain

$$\rho + p = \frac{\sigma Z_t}{16\pi G} + \frac{k}{4\pi G c^2 e^{2\sigma t}}, \tag{2.20}$$

and

$$\rho - p = \frac{3\sigma^2}{4\pi G} + \frac{11\sigma Z_t}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}}. \tag{2.21}$$

Again, from (2.18) and (2.19), one can get

$$\rho + 3p = -\frac{3\sigma^2}{4\pi G} - \frac{9}{16\pi G}\sigma Z_t. \tag{2.22}$$

Here, notice that if $Z_t = 0$ the equation (2.16) reduces to standard Friedmann equation. By above observations says that the universe is anisotropic at early stage and becomes isotropic at late time. If the physically variable choices of $Z_t < 0$, it arise two different scenario as: (a) $Z_t = -e^{-t}$ and (b) $Z_t = -t^{-n}$.
(a) If $Z_t = -e^{-t}$:
Substituting the values of Z_t in (2.18) and (2.19), we can determine ρ and p respectively as

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 - \sigma e^{-t} + \frac{k}{c^2 e^{2\sigma t}}\right). \tag{2.23}$$

$$p = \frac{5\sigma e^{-t}}{16\pi G} - \frac{3\sigma^2}{8\pi G} - \frac{k}{8\pi G c^2 e^{2\sigma t}}. \tag{2.24}$$

Add and subtract above equations, we get

$$\rho + p = -\frac{\sigma e^{-t}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}}. \tag{2.25}$$

and

$$\rho - p = \frac{3\sigma^2}{4\pi G} - \frac{11\sigma e^{-t}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}}. \tag{2.26}$$

Again, from equations (2.23) and (2.24) in condition (2.9), one can easily obtain

$$\rho + 3p = -\frac{3\sigma^2}{4\pi G} + \frac{9\sigma e^{-t}}{16\pi G}. \tag{2.27}$$

We observed the equations (2.23) to (2.27), the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{\sigma e^{(2\sigma+1)t}} = N1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(\sigma e^{-t} - \sigma^2)}, \frac{4k}{\sigma e^{(2\sigma+1)t}}\right\} = N2.$$

The dominant energy condition is satisfied if

$$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(\sigma e^{-t} - \sigma^2)}, \frac{4k}{\sigma e^{(2\sigma+1)t}}, \frac{8k}{e^{2\sigma t}(11\sigma e^{-t} - 12\sigma^2)}\right\} = N3$$

and strong energy condition is satisfied if $0 < \sigma \leq \frac{3}{4e^t}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \leq min\{N1, N2, N3\}$, whereas SEC is satisfied in this model if $0 < \sigma \leq \frac{3}{4e^t}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, whereas SEC is violated, which is responsible for current accelerated expansion of universe.
(b) If $Z_t = -t^{-n}$:

Substituting the value of Z_t in equations (2.18) and (2.19), we can determine ρ and p respectively as;

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 - \frac{\sigma}{t^n} + \frac{k}{c^2 e^{2\sigma t}}\right), \tag{2.28}$$

and

$$p = \frac{5\sigma}{16\pi G t^n} - \frac{3\sigma^2}{8\pi G} - \frac{k}{8\pi G c^2 e^{2\sigma t}}. \tag{2.29}$$

Add and subtract the equation (2.28) and (2.29), we obtain

$$\rho + p = -\frac{\sigma t^{-n}}{16\pi G} + \frac{k}{4\pi G c^2 e^{2\sigma t}} \tag{2.30}$$

and

$$\rho - p = \frac{3\sigma^2}{4\pi G} - \frac{11\sigma t^{-n}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}}. \tag{2.31}$$

Again, from equations (2.28) and (2.29) in condition (2.9), we have

$$\rho + 3p = \frac{9\sigma t^{-n}}{16\pi G} - \frac{3\sigma^2}{4\pi G}. \tag{2.32}$$

We observed the equations (2.28) to (2.32), the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{\sigma e^{2\sigma t} t^n} = T1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(\sigma t^{-n} - \sigma^2)}, \frac{4k}{\sigma t^n e^{2\sigma t}}\right\} = T2.$$

The dominant energy condition is satisfied if

$$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(\sigma t^{-n} - \sigma^2)}, \frac{4k}{\sigma t^n e^{2\sigma t}}, \frac{8k}{e^{2\sigma t}(11\sigma t^{-n} - 12\sigma^2)}\right\} = T3$$

and strong energy condition is satisfied if $0 < \sigma \leq \frac{3}{4t^n}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \leq min\{T1, T2, T3\}$, whereas SEC is satisfied in this model if $0 < \sigma \leq \frac{3}{4t^n}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.

Case 2: Power law solution

The power law solutions are very important in the standard cosmology, because this type of solution provides a frame work for establishing the behavior of more general cosmological solutions in different histories of our universe, such as radiation dominant, matter dominant, or dark energy dominators. Let us consider a universe with power law for scalar factor [**?**] as: $a = ct^\delta$, where c and δ are constants. For $\delta > 1$ it gives an accelerating universe.
Now, the Hubble parameter becomes

$$H = \frac{\dot{a}}{a} = \frac{c\delta t^{\delta-1}}{ct^\delta} = \frac{\delta}{t}. \tag{2.33}$$

From this, using (2.16) and (2.33), the energy density is given by

$$\rho = \frac{3}{8\pi G}\left(\frac{\delta^2}{t^2} + \frac{\delta}{t}Z_t + \frac{k}{c^2 t^{2\delta}}\right). \tag{2.34}$$

From equations (2.14) and (2.34), the pressure is given by

$$p = \frac{-3\delta^2 + 2\delta}{8\pi G t^2} - \frac{5\delta}{16\pi G t}Z_t - \frac{k}{8\pi G c^2 t^{2\delta}}. \tag{2.35}$$

Add and subtract the equation (2.34) and (2.35), we obtain

$$\rho+p=\frac{\delta}{4\pi Gt^2}+\frac{\delta}{16\pi Gt}Z_t+\frac{k}{4\pi Gc^2t^{2\delta}} \tag{2.36}$$

and

$$\rho-p=\frac{\delta(3\delta-1)}{4\pi Gt^2}+\frac{11\delta}{16\pi Gt}Z_tZ_t+\frac{k}{2\pi Gc^2t^{2\delta}}. \tag{2.37}$$

Again, from equations (2.34) and (2.35) in condition (2.9), we have

$$\rho+3p=\frac{3\delta(1-\delta)}{4\pi Gt^2}-\frac{9\delta}{16\pi Gt}Z_t. \tag{2.38}$$

In this case also, we discuss , as same in case 1 two different scenario:
(i) When $Z_t=-e^{-t}$:
Substituting the value of Z_t in equations (2.34) and (2.35), we can determine ρ and p respectively as

$$\rho=\frac{3}{8\pi G}\left(\frac{\delta^2}{t^2}-\frac{\delta}{te^t}+\frac{k}{c^2t^{2\delta}}\right) \tag{2.39}$$

and

$$p=\frac{2\delta-3\delta^2}{8\pi Gt^2}+\frac{5\delta}{16\pi Gte^t}-\frac{k}{8\pi Gc^2t^{2\delta}}. \tag{2.40}$$

From equations (2.39) and (2.40), we obtain

$$\rho+p=\frac{\delta}{4\pi Gt^2}-\frac{\delta}{16\pi Gte^t}+\frac{k}{4\pi Gc^2t^{2\delta}} \tag{2.41}$$

and

$$\rho-p=\frac{\delta(3\delta-1)}{4\pi Gt^2}-\frac{11\delta}{16\pi Gte^t}+\frac{k}{2\pi Gc^2t^{2\delta}}. \tag{2.42}$$

Again, from equations (2.39) and (2.40), we get the condition $\rho+3p$ value as:

$$\rho+3p=\frac{3\delta(1-\delta)}{4\pi Gt^2}+\frac{9\delta}{16\pi Gte^t}. \tag{2.43}$$

We observed the equations (2.39) to (2.43), the null energy condition is satisfied if

$$c^2\leq\frac{4k}{\delta t^{(2\delta-1)}e^{-t}-4\delta^2t^{(2\delta-2)}}=P1.$$

The weak energy condition is satisfied for

$$c^2\leq min\left\{\frac{k}{\delta t^{2\delta-1}e^{-t}-\delta^2t^{2\delta-2}},\frac{4k}{\delta t^{2\delta-1}e^{-t}-4\delta^2t^{(2\delta-2)}}\right\}=P2.$$

The dominant energy condition is satisfied if

$$c^2\leq min\left\{\frac{k}{\delta t^{2\delta-1}e^{-t}-\delta^2t^{2\delta-2}},\frac{4k}{\delta t^{2\delta-1}e^{-t}-4\delta^2t^{(2\delta-2)}},\frac{8k}{11\delta t^{2\delta-1}e^{-t}-4\delta(3\delta-1)t^{2\delta-2}}\right\}=P3$$

and strong energy condition is satisfied if $1<\delta\leq1+\frac{3t}{4e^t}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2\leq min\{P1,P2,P3\}$, whereas SEC is satisfied in this model if $1<\delta\leq1+\frac{3t}{4e^t}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.
(ii) When $Z_t=-t^{-n}$:
Substituting the value of Z_t in equations (2.34) and (2.35), we can determine ρ and p respectively as

$$\rho=\frac{3}{8\pi G}\left(\frac{\delta^2}{t^2}-\frac{\delta}{t^{n+1}}+\frac{k}{c^2t^{2\delta}}\right) \tag{2.44}$$

and

$$p = \frac{2\delta - 3\delta^2}{8\pi G t^2} + \frac{5\delta}{16\pi G t^{n+1}} - \frac{k}{8\pi G c^2 t^{2\delta}}. \tag{2.45}$$

From equations (2.44) and (2.45), we obtain

$$\rho + p = \frac{\delta}{4\pi G t^2} - \frac{\delta}{16\pi G t^{n+1}} + \frac{k}{4\pi G c^2 t^{2\delta}} \tag{2.46}$$

and

$$\rho - p = \frac{\delta(3\delta - 1)}{4\pi G t^2} - \frac{11\delta}{16\pi G t^{n+1}} + \frac{k}{2\pi G c^2 t^{2\delta}}. \tag{2.47}$$

Again, from equations (2.44) and (2.45), we get the condition $\rho + 3p$ value as:

$$\rho + 3p = \frac{3\delta(1 - \delta)}{4\pi G t^2} + \frac{9\delta}{16\pi G t^{n+1}}. \tag{2.48}$$

From equations (2.44) to (2.48), it is observed that the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{\delta t^{(2\delta - n - 1)} - \delta t^{(2\delta - 2)}} = W1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{\delta t^{2\delta - n - 1} - \delta^2 t^{2\delta - 2}}, \frac{4k}{\delta t^{2\delta - n - 1} - \delta t^{(2\delta - 2)}}\right\} = W2.$$

The dominant energy condition is satisfied if

$$c^2 \leq min\left\{\frac{k}{\delta t^{2\delta - n - 1} - \delta^2 t^{2\delta - 2}}, \frac{4k}{\delta t^{2\delta - n - 1} - \delta t^{(2\delta - 2)}}, \frac{8k}{11\delta t^{2\delta - n - 1} - 4\delta(3\delta - 1)t^{2\delta - 2}}\right\} = W3$$

and strong energy condition is satisfied if $1 < \delta \leq 1 + \frac{3}{4t^{n-1}}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \leq min\{W1, W2, W3\}$, whereas SEC is satisfied in this model if $1 < \delta \leq 1 + \frac{3}{4t^{n-1}}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.

3 Scalar-Tensor thoery of Finsler-Randers cosmological model

The field equation of this theory are given by

$$G_{ij} - \frac{1}{2}g_{ij}R = -8\pi G T_{ij} + 2\left(\phi_i\phi_j - \frac{1}{2}g_{ij}\phi_k\phi^k\right), \tag{3.1}$$

where G is the gravitational constant and ϕ is the scalar field.
In this theory, field equations are

$$3\left(H^2 + \frac{k}{a^2} + HZ_t\right) = 8\pi G\rho + \dot{\phi}^2, \tag{3.2}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} + \frac{5}{2}HZ_t = -8\pi G\rho - \dot{\phi}^2. \tag{3.3}$$

From equation (3.2) and (3.3), we obtain

$$3\left(\frac{\ddot{a}}{a} + \frac{3\dot{a}}{4a}Z_t\right) = -4\pi G(\rho + 3p) - 2\dot{\phi}^2. \tag{3.4}$$

Here, we discuss two different physically variable cosmologies, which have physical interests to describe the decelerating and accelerating phases of universe.
Case 1: de Sitter solution

Let us consider the scalar factor $a = ce^{\gamma t}$, where $\gamma^2 > 0$.
Now, the Hubble parameter becomes

$$H = \frac{\dot{a}}{a} = \frac{\gamma c e^{\gamma t}}{c e^{\gamma t}} = \gamma. \tag{3.5}$$

From equations (3.4) and (3.5), the density is

$$\rho = \frac{3}{8\pi G}\left(\gamma^2 + \gamma Z_t + \frac{k}{c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{3}\right). \tag{3.6}$$

From equation (3.4) and (3.6), we have the pressure as

$$p = -\frac{3\gamma^2}{8\pi G} - \frac{5\gamma}{16\pi G} Z_t - \frac{k}{8\pi G c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{8\pi G}. \tag{3.7}$$

Add and subtract the equations (3.6) and (3.7), we have

$$\rho + p = \frac{\gamma}{16\pi G} Z_t + \frac{k}{4\pi G c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{4\pi G} \tag{3.8}$$

and

$$\rho - p = \frac{3\gamma^2}{4\pi G} + \frac{11\gamma}{16\pi G} Z_t + \frac{k}{2\pi G c^2 e^{2\gamma t}}. \tag{3.9}$$

Again, from (3.6) and (3.7), we obtain

$$\rho + 3p = -\frac{3\gamma^2}{4\pi G} - \frac{9\gamma}{16\pi G} Z_t + \frac{\dot{\phi}^2}{2\pi G}. \tag{3.10}$$

Now, we discuss two different physical variable cosmologies as same as in previous section but here we take the positive quantity of physical variables.
(i) If $Z_t = e^t$:
Substituting the value of Z_t in equations (3.6) and (3.7), we can determine ρ and p respectively as

$$\rho = \frac{3}{8\pi G}\left(\gamma^2 + \gamma e^t + \frac{k}{c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{3}\right). \tag{3.11}$$

$$p = -\frac{3\gamma^2}{8\pi G} - \frac{5\gamma}{16\pi G} e^t - \frac{k}{8\pi G c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{8\pi G}. \tag{3.12}$$

Add and subtract the equations (3.11) and (3.12), we have

$$\rho + p = -\frac{\gamma e^t}{16\pi G} + \frac{k}{4\pi G c^2 e^{2\gamma t}} - \frac{\dot{\phi}^2}{4\pi G} \tag{3.13}$$

and

$$\rho - p = \frac{3\gamma^2}{4\pi G} + \frac{11\gamma}{16\pi G} e^t + \frac{k}{2\pi G c^2 e^{2\gamma t}}. \tag{3.14}$$

Again, from (3.6) and (3.7), we obtain

$$\rho + 3p = -\frac{3\gamma^2}{4\pi G} - \frac{9\gamma}{16\pi G} e^t - \frac{\dot{\phi}^2}{2\pi G}. \tag{3.15}$$

We observed the equations (3.11) to (3.15), the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{e^{2\gamma t}(\gamma e^t + 4\dot{\phi}^2)} = E1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{e^{2\gamma t}(\gamma e^t - \gamma^2 + \dot{\phi}^2/3)}, \frac{4k}{e^{2\gamma t}(\gamma e^t + 4\dot{\phi}^2)}\right\} = E2.$$

The dominant energy condition is satisfied if

$$c^2 \le min\left\{\frac{k}{e^{2\gamma t}(\gamma e^t-\gamma^2+\dot{\phi}^2/3)}, \frac{4k}{e^{2\gamma t}(\gamma e^t+4\dot{\phi}^2)}, \frac{8k}{e^{2\gamma t}(11\gamma e^t-12\gamma^2)}\right\} = E3$$

and strong energy condition is satisfied if $0<\dot{\phi}^2 \le \frac{9\gamma}{8e^t}-\frac{3}{2}\gamma^2$.
It is observed that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \le min\{E1, E2, E3\}$, whereas SEC is satisfied in this model if $0<\dot{\phi}^2 \le \frac{9\gamma}{8e^t}-\frac{3}{2}\gamma^2$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.
(ii) If $Z_t = t^n$:

Substituting the value of Z_t in equations (3.6) and (3.7), we can determine ρ and p respectively as

$$\rho = \frac{3}{8\pi G}\left(\gamma^2+\gamma t^n+\frac{k}{c^2e^{2\gamma t}}-\frac{\dot{\phi}^2}{3}\right). \tag{3.16}$$

$$p = -\frac{3\gamma^2}{8\pi G}-\frac{5\gamma}{16\pi G}t^n-\frac{k}{8\pi Gc^2e^{2\gamma t}}-\frac{\dot{\phi}^2}{8\pi G}. \tag{3.17}$$

Add and subtract the equations (3.11) and (3.12), we have

$$\rho+p = \frac{\gamma t^n}{16\pi G}+\frac{k}{4\pi Gc^2e^{2\gamma t}}-\frac{\dot{\phi}^2}{4\pi G} \tag{3.18}$$

and

$$\rho-p = \frac{3\gamma^2}{4\pi G}+\frac{11\gamma}{16\pi G}t^n+\frac{k}{2\pi Gc^2e^{2\gamma t}}. \tag{3.19}$$

Again, from (3.16) and (3.17), one can obtain

$$\rho+3p = -\frac{3\gamma^2}{4\pi G}-\frac{9\gamma}{16\pi G}t^n-\frac{\dot{\phi}^2}{2\pi G}. \tag{3.20}$$

We observed the equations (3.16) to (3.20), the null energy condition is satisfied if

$$c^2 \le \frac{4k}{e^{2\gamma t}(\gamma t^n+4\dot{\phi}^2)} = S1.$$

The weak energy condition is satisfied for

$$c^2 \le min\left\{\frac{k}{e^{2\gamma t}(\gamma t^n-\gamma^2+\dot{\phi}^2/3)}, \frac{4k}{e^{2\gamma t}(\gamma t^n+4\dot{\phi}^2)}\right\} = S2.$$

The dominant energy condition is satisfied if

$$c^2 \le min\left\{\frac{k}{e^{2\gamma t}(\gamma t^n-\gamma^2+\dot{\phi}^2/3)}, \frac{4k}{e^{2\gamma t}(\gamma t^n+4\dot{\phi}^2)}, \frac{8k}{e^{2\gamma t}(11\gamma t^n-12\gamma^2)}\right\} = S3$$

and strong energy condition is satisfied if $0<\dot{\phi}^2 \le \frac{9\gamma}{8t^n}-\frac{3}{2}\gamma^2$.
It is observed that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \le min\{S1, S2, S3\}$, whereas SEC is satisfied in this model if $0<\dot{\phi}^2 \le \frac{9\gamma}{8t^n}-\frac{3}{2}\gamma^2$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.

Case 2: Power-law solution
Let us consider the scalar factor $a = ce^\delta$, where $\delta > 0$.
Now, the Hubble parameter becomes

$$H = \frac{\dot{a}}{a} = \frac{c\delta t^{\delta-1}}{ct^\delta} = \frac{\delta}{t}. \tag{3.21}$$

From equations (3.4) and (3.21), the density is

$$\rho = \frac{3}{8\pi G}\left(\frac{\delta^2}{t^2} + \frac{\delta}{t}Z_t + \frac{k}{c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{3}\right). \tag{3.22}$$

From equations (3.4) and (3.22), the pressure is given by

$$p = \frac{-3\delta^2 + 2\delta}{8\pi G t^2} - \frac{5\delta}{16\pi G t}Z_t - \frac{k}{8\pi G c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{8\pi G}. \tag{3.23}$$

Add and subtract the equation (3.22) and (3.23), we obtain

$$\rho + p = \frac{\delta}{4\pi G t^2} + \frac{\delta}{16\pi G t}Z_t + \frac{k}{4\pi G c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{4\pi G} \tag{3.24}$$

and

$$\rho - p = \frac{\delta(3\delta - 1)}{4\pi G t^2} + \frac{11\delta}{16\pi G t}Z_t + \frac{k}{2\pi G c^2 t^{2\delta}}. \tag{3.25}$$

Again, from equations (3.22) and (3.23) in condition (2.9), we have

$$\rho + 3p = \frac{3\delta(1-\delta)}{4\pi G t^2} - \frac{9\delta}{16\pi G t}Z_t - \frac{\dot{\phi}^2}{2\pi G}. \tag{3.26}$$

In this case also, we discuss , as same in case 1 two different scenario:
(i) When $Z_t = e^t$:
Substituting the value of Z_t in equations (3.22) and (3.23), we can determine ρ and p respectively as

$$\rho = \frac{3}{8\pi G}\left(\frac{\delta^2}{t^2} - \frac{\delta}{t}e^t + \frac{k}{c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{3}\right) \tag{3.27}$$

and

$$p = \frac{2\delta - 3\delta^2}{8\pi G t^2} - \frac{5\delta}{16\pi G t}e^t - \frac{k}{8\pi G c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{8\pi G}. \tag{3.28}$$

From equations (3.27) and (3.28), we obtain

$$\rho + p = \frac{2\delta}{8\pi G t^2} - \frac{\delta}{16\pi G t}e^t + \frac{k}{4\pi G c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{4\pi G} \tag{3.29}$$

and

$$\rho - p = \frac{\delta(3\delta - 1)}{4\pi G t^2} + \frac{11\delta}{16\pi G t e^t} + \frac{k}{4\pi G c^2 t^{2\delta}}. \tag{3.30}$$

Again, from equations (3.27) and (3.28), we get the condition $\rho + 3p$ value as:

$$\rho + 3p = \frac{3\delta(2-\delta)}{8\pi G t^2} - \frac{9\delta}{16\pi G t}e^t - \frac{\dot{\phi}^2}{2\pi G}. \tag{3.31}$$

We observed the equations (3.27) to (3.31), the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{\delta t^{(2\delta-1)}e^t - 4\delta^2 t^{(2\delta-2)} + 4\dot{\phi}^2 t^{2\delta}} = C1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{\delta t^{2\delta-1}e^t - \delta^2 t^{2\delta-2} + t^{2\delta}\dot{\phi}^2/3}, \frac{4k}{\delta t^{2\delta-1}e^t - 4\delta^2 t^{(2\delta-2)} + 4\dot{\phi}\phi^2 t^{2\delta}}\right\} = C2.$$

The dominant energy condition is satisfied if

$$\begin{aligned} c^2 \quad \leq \quad & min\left\{\frac{k}{\delta t^{2\delta-1}e^t - \delta^2 t^{2\delta-2} + t^{2\delta}\dot{\phi}^2/3}, \frac{4k}{\delta t^{2\delta-1}e^t - 4\delta^2 t^{(2\delta-2)} + 4\dot{\phi}^2 t^{2\delta}}\right\}, \\ & \left\{\frac{8k}{11\delta t^{2\delta-1}e^t - 4\delta(3\delta-1)t^{2\delta-2}}\right\} = C3 \end{aligned}$$

and strong energy condition is satisfied if $0 < \dot{\phi}^2 \leq \frac{3\delta(1-\delta)}{t^2} + \frac{9\delta}{8te^t}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \leq min\{C1, C2, C3\}$, whereas SEC is satisfied in this model if $0 < \dot{\phi}^2 \leq \frac{3\delta(1-\delta)}{t^2} + \frac{9\delta}{8te^t}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.
(ii) When $Z_t = t^n$:
Substituting the value of Z_t in equations (3.22) and (3.23), we can determine ρ and p respectively as

$$\rho = \frac{3}{8\pi G}\left(\frac{\delta^2}{t^2} - \frac{\delta}{t}t^n + \frac{k}{c^2 t^{2\delta}} - \frac{\dot{\phi}^2}{3}\right) \tag{3.32}$$

and

$$p = \frac{2\delta - 3\delta^2}{8\pi G t^2} - \frac{5\delta}{16\pi G}t^{n-1} - \frac{k}{8\pi G c^2 t^{2\delta}} - -\frac{\dot{\phi}^2}{8\pi G}. \tag{3.33}$$

From equations (3.32) and (3.33), we obtain

$$\rho + p = \frac{\delta}{4\pi G t} + \frac{\delta}{16\pi G}t^{n-1} + \frac{k}{4\pi G c^2 t^{2\delta}} - -\frac{\dot{\phi}^2}{4\pi G} \tag{3.34}$$

and

$$\rho - p = \frac{\delta(3\delta - 1)}{4\pi G t^2} + \frac{11\delta}{16\pi G}t^{n-1} + \frac{k}{2\pi G c^2 t^{2\delta}}. \tag{3.35}$$

Again, from equations (3.32) and (3.33), we get the condition $\rho + 3p$ value as:

$$\rho + 3p = \frac{3\delta(1-\delta)}{4\pi G t^2} - \frac{9\delta}{16\pi G}t^{n-1} - -\frac{\dot{\phi}^2}{2\pi G}. \tag{3.36}$$

From equations (3.32) to (3.36), it is observed that the null energy condition is satisfied if

$$c^2 \leq \frac{4k}{\delta t^{(2\delta-n-1)} - 4\delta t^{(2\delta-2)} + 4\dot{\phi}^2 t^{2\delta}} = L1.$$

The weak energy condition is satisfied for

$$c^2 \leq min\left\{\frac{k}{\delta t^{2\delta-n-1} - \delta^2 t^{2\delta-2} + t^{2\delta}\dot{\phi}^2/3}, \frac{4k}{\delta t^{2\delta-n-1} - 4\delta^2 t^{(2\delta-2)} + 4\dot{\phi}^2 t^{2\delta}}\right\} = L2.$$

The dominant energy condition is satisfied if

$$\begin{aligned} c^2 \quad \leq \quad & min\left\{\frac{k}{\delta t^{2\delta-n-1} - \delta^2 t^{2\delta-2} + t^{2\delta}\dot{\phi}^2/3}, \frac{4k}{\delta t^{2\delta-n-1} - 4\delta^2 t^{(2\delta-2)} + 4\dot{\phi}^2 t^{2\delta}}\right\}, \\ & \left\{\frac{8k}{11\delta t^{2\delta-n-1} - 4\delta(3\delta-1)t^{2\delta-2}}\right\} = L3 \end{aligned}$$

and strong energy condition is satisfied if $0 < \dot{\phi}^2 \leq \frac{3\delta(1-\delta)}{2t^2} + \frac{9\delta}{8t^{1-n}}$.
From these observations that, for any value of t, NEC, WEC and DEC are satisfied in this case if $c^2 \leq min\{L1, L2, L3\}$, whereas SEC is satisfied in this model if $0 < \dot{\phi}^2 \leq \frac{3\delta(1-\delta)}{t^2} + \frac{9\delta}{8te^t}$. However, we also observed that for large cosmic time t, SEC, WEC and DEC are satisfied, but SEC is violated.

4 Conclusion

In Present paper, we have investigated the Finsler-Randers cosmological models in modified theories of gravity. Then, obtained the solution with the corresponding models of scalar factor . Further, we studied the behavior of model in Einstein theory and scalar tensor theory by considering the physical variables $Z_t = -e^{-t}$, $Z_t = -t^{-n}$ and this positive variables. With obtained solutions, we have also discussed null energy condition (NEC),Weak energy condition (WEC), dominant energy condition (DEC) and Strong energy condition (SEC) and find under what conditions our FR cosmological model is physically stable in different modify theories of gravitation. It is seen that all energy conditions are satisfied for some suitable value of constant but for large cosmic time t, NEC, WEC and DEC are satisfied but SEC is violated in all modify gravity theories, which is responsible for current accelerated expansion of Universe. At $t \to \infty$ we obtained $Z_t = 0$, Finsler-Randers cosmological model tend to Friedman Robertson-Walker model. The model represents an expanding Universe, which approaches isotropy for large values of t. The results of this paper are in favor of the observational features of the Universe.

References

[1] Arbab, A I, Viscous dark energy models with variable G and lambda, *Chin. Phys. Letter*, **25 (10)**, 3834-3836, (2008).

[2] Bali, R and Saraf, S, C-field cosmological model for barotropic fluid distribution with bulk viscosity and decaying vacuum energy (Λ) in FRW space time, *Canadian Journal of Physics*, **91 (9)**, 728-732, (2013).

[3] Bao, D, Chern, S S and Shen, Z, An Introduction to Riemann-Finsler Geometry, *Springer* , New York, (2000).

[4] Basilakos, S and Stavrinos, P, Cosmological equivalence between the Finsler-Randers space-time and the DGP gravity model, *arXiv:1301.4327v2*, [gr-qc] (2013).

[5] Belinski, V A, Khalatnikov, I M and Lifshits, E M, Oscillatory Approach to a Singular Point in the Relativistic Cosmology,*Advances in Physics*, **19**, 525, (1970).

[6] Chatterjee, S and Banerjee, A, C-Field Cosmology in Higher Dimensions, *General Relativity and Gravitation*, **36 (2)**, 303-313, (2004).

[7] Ellis, G F R, Exact and inexact solutions of the Einstein field equations, *The Renaissance of General Relativity and Cosmology* , **20**, (1993).

[8] Ellis, G F R and Elst, H V, Cosmological models, *Proceedings of the NATO Advanced Study Institute on Theoretical and Observational Cosmology*, Kluwer Academic, **541** , 1-116, (1999).

[9] Hu, B L and Parker, L, Anisotropy damping through quantum effects in the early universe, *Phys. Rev. D*, **17**, 933-945, (1978).

[10] Hawking, S W and Ellis, G F R, The large scale structure of space-time,*Cambridge University Press, U.K*, (1973).

[11] Kolb, E W and Turner, M S, The Early Universe, *Addison-Wesley*, (1990).

[12] Kolassis, C A, Santos, N O and Tsoubelis, D, Energy conditions for an imperfect fluid, *Classical and Quantum Gravity*, **5**, 1329, (1988).

[13] Misner, CW, The Isotropy of the Universe, *Astrophysical Journal*, **151**, 431- 457, (1968).

[14] MacCallum, M A H, Anisotropic and inhomogeneous relativistic cosmologies, *General Relativity: An Einstein Centenary Survey, Cambridge University Press, U.K.*, 533-580, (1979).

[15] Misner, CW, Thorne, K S and Wheeler, J A, Gravitation, *W.H. Freeman and Co., San Francisco: New York*, 1279, (1973).

[16] Padmanabhan T, Accelarated expansion of the universe driven by tachyonic matter, *Phys. Rev. D*, **66**, 1301, 2002.

[17] Randers, G, On an asymmetric metric in the four-space of general relativity, *Phys. Rev*, **59**, 195-199, (1941).

[18] Rund, H, The Differential Geometry of Finlser Spaces, *Springer Berlin*, (1959).

[19] Stavrinos, P C, Kouretsis, A P and Stathakopoulos, M, Friedman-like Robertson Walker model in generalized metric space-time with weak anisotropy, *General Relativity and Gravitation*, **40 (7)**, 1403-1425, (2008).

[20] Vacaru, S, Stavrinos, P C, Gaburov, E and Gonta, D, Clifford and Riemann- Finsler structures in geometric mechanics and gravity, *Geometry Balkan Press*, (2005).

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