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Some Euler type generalized beta function involving extended Mittag-Leffler function

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Abstract In this paper, we establish a new class of generalized beta type integral operators involving generalized Mittag-Leffler function. Some special cases are deduced.

1 Introduction

The Swedish Mathematician Mittag-Leffler [10] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

The Mittag-Leffler function is a direct generalization of exp(z) in which $\alpha = 1$. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

A generalization of $E_\alpha(z)$ was studied by Wiman [19] where he defined the function $E_{\alpha,\beta}(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$ which is also known as Mittag-Leffler function or Wiman's function.

Prabhakar [12] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form (see also Killbas et al. [7]):

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

Shukla and Prajapati [17](see also Srivastava and Tomovaski [18]) defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\gamma_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.4}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_{qn} = \dfrac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol which in particular reduces to $q^{qn} \prod\limits_{r=1}^{q} (\dfrac{\gamma + r - 1}{q})_n$ if $q \in \mathbb{N}$.

A new generalized Mittag-Leffler function was defined by Salim [16] as

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{\gamma_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_n}, \tag{1.5}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0$.
In 1997, Chaudhary et al. [4] presented the following extension of Euler's Beta function

$$B_p(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\exp\left[-\frac{p}{t(1-t)}\right]dt. \tag{1.6}$$

For $p=0$, the extended beta function reduces to the classical beta function.

Now we recall the classical beta function denoted by $B(a,b)$ and is defined (see [9], see also [14]) by Euler's integral

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},(\Re(a)>0,\Re(b)>0). \tag{1.7}$$

In this paper, we consider the following generalizations of Euler beta functions as

$$B(x,y;p;m)=\int_0^1 u^{x-1}(1-u)^{y-1}\exp\left(\frac{-p}{u^m(1-u)^m}\right)dt, \tag{1.8}$$

where $\Re(p)>0,m>0$. Clearly when $m=1$, equation(1.7)reduce to Chaudhary et al.[2]extended beta function (EBF)and $p=0$, it reduces to Euler's beta function [9].

The Gauss hypergeometric function, denoted by $F(a,b;c;z)$ and confluent hypergeometric function of the first kind, denoted by $\Phi(b;c;z)$, for $\Re(c)>\Re(b)>0$, are defined as follows (see [9], see also [14]):

$$F(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{1.9}$$

$$|arg(1-z)|<\pi$$

and

$$\Phi(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp(zt)dt \tag{1.10}$$

$$F(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{1.11}$$

$$|arg(1-z)|<\pi$$

and

$$\Phi(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp(zt)dt.$$

By using the series expansion of $(1-zt)^{-a}$ and $\exp(zt)$ in (1.9) and (1.10) respectively, we obtain

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_nB(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.12}$$

$$(|z| < 1, \Re(c) > \Re(b) > 0).$$

$$\Phi(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}. \tag{1.13}$$

$$(\Re(c) > \Re(b) > 0)$$

In (2004), Chaudhary et al. [5] (see also [11]) used beta function $B_\sigma(a,b)$ to extended the hypergeometric and confluent hypergeometric function as follows:

$$F_\sigma(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n B_\sigma(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.14}$$

$$(\sigma \geq 0; |z| < 1, \Re(c) > \Re(b) > 0),$$

$$\Phi_\sigma(b;c;z) = \sum_{n=0}^{\infty} \frac{B_\sigma(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.15}$$

$$(\sigma \geq 0; \Re(c) > \Re(b) > 0)$$

and gave their Euler's type integral representation:

$$F_\sigma(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{\sigma}{t(1-t)}\right] dt, \tag{1.16}$$

$$(\sigma > 0; \sigma = 0 \quad and \quad |arg(1-z)| < \pi; \Re(c) > \Re(b) > 0),$$

$$\Phi_\sigma(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{\sigma}{t(1-t)}\right] dt, \tag{1.17}$$

$$(\sigma > 0; \sigma = 0 \quad and \quad \Re(c) > \Re(b) > 0).$$

In this paper, we obtained some theorems on Euler type integral involving generalized Mittag-Leffler function and discussed some special cases.

2 Euler type integral operator involving generalized Mittag-Leffler function

Theorem 2.1. If $\alpha, \beta, \gamma, \delta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(A) > 0$, $p, q \geq 0$ and $q < \Re(\alpha) + p$, then

$$\int_0^1 t^{\rho-1}(1-t)^{\delta-1} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(zt^\alpha) dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta) n!} B(n\alpha + \rho, \delta; A; m). \tag{2.1}$$

Proof. In order to derive (2.1),we denote L.H.S. of (2.1) by I_1 and then expanding $E_{\alpha,\beta}^{\gamma,q}(zt^\alpha)$ by using (1.5), we get:

$$I_1 = \int_0^1 t^{\rho-1}(1-t)^{\delta-1} \exp\left(\frac{-A}{t^m(1-t)^m}\right) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n t^{n\alpha}}{\Gamma(\alpha n + \beta) n!} dt.$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta) n!} \int_0^1 t^{n\alpha+\rho-1}(1-t)^{\delta-1} \exp\left(\frac{-A}{t^m(1-t)^m}\right) dt.$$

Using beta definition of generalization of beta function (1.6),we get

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(n\alpha+\rho, \delta; A; m).$$

Remark 2.1. Setting $m = 1$ in Theorem 2.1, the result reduces to known result of Ahmed and Khan [1,p.481].

Theorem 2.2. If $\alpha, \beta, \gamma, \delta, \rho, \mu, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\rho), \Re(\mu), \Re(\lambda) > 0, \Re(A) > 0$, $p, q \geq 0$ and $q < \Re(\alpha)+p$; $\mid arg(\frac{bc+d}{ac+d}) \mid < \pi$, then

$$\int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{(t-a)^m(b-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(z(b-t)^f)dt.$$

$$= (ac+d)^{\lambda} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n! r!} B(\rho-mr, \delta+fn-mr)(b-a)^{\rho+\delta+fn-2mr-1}$$

$$\times {}_2F_1\left[\rho-mr, -\lambda; \rho+\delta+fn-2mr; \frac{-(b-a)c}{ca+d}\right]. \quad (2.2)$$

Proof. On the L.H.S. of (2.2), expending the generalized Mittag-Leffler function in their respective series, to get

$$\int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{(t-a)^m(b-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(z(b-t)^f)dt.$$

$$= \int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda} \sum_{r=0}^{\infty} \frac{(-A)^r}{(t-a)^{mr}(b-t)^{rm} r!} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n (b-t)^{fn}}{\Gamma(\alpha n+\beta) n!} dt.$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) r!} \int_a^b (t-a)^{\rho-mr-1}(b-t)^{\delta+fn-mr-1}(ct+d)^{\lambda} dt,$$

which further on using the integral [13, p.263], yields the required result (2.2).

$$= (ac+d)^{\lambda} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n! r!} B(\rho-mr, \delta+fn-mr)(b-a)^{\rho+\delta+fn-2mr-1}$$

$$\times {}_2F_1\left[\rho-mr, -\lambda; \rho+\delta+fn-2mr; \frac{-(b-a)c}{ca+d}\right].$$

Corollary 2.2. Putting $a = 0$, $b = 1$ in Theorem 2.2, the result reduce the following as

$$\int_0^1 (t)^{\rho-1}(1-t)^{\delta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(z(1-t)^f)dt$$

$$= d^{\lambda} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(\rho-mr, \delta+fn-mr)$$

$$\times {}_2F_1\left[\rho-mr, -\lambda; \rho+\delta+fn-2mr; \frac{-c}{d}\right]. \quad (2.3)$$

Remark 2.2. Setting $m = 1$ in Theorem 2.2, the result reduces to known result of Ahmed and Khan [1, p.482].

Theorem 2.3. If $\alpha, \beta, \gamma, \delta, \rho, \mu, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\rho), \Re(\mu), \Re(\lambda) > 0, \Re(A) > 0$, $p, q \geq 0$ and $q < \Re(\alpha) + p$, then

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1}(1-ut^{\rho}(1-t)^{\sigma})^{-a} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(zt^{\alpha})dt$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_r(\gamma)_{qn}u^r z^n}{\Gamma(\alpha n+\beta)n!} B(\lambda+\alpha n+\rho r, \mu-\lambda+\sigma r; A; m). \tag{2.4}$$

Proof. On Taking L.H.S. of Theorem 2.3, using the definition of generalized Mittag-Leffler function (1.6) , we get

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1}(1-ut^{\rho}(1-t)^{\sigma})^{-a} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(zt^{\alpha})dt.$$

$$= \int_0^1 t^{\lambda+\alpha n-1}(1-t)^{\mu-\lambda-1} \sum_{r=0}^{\infty} \frac{u^r t^{\rho r}(1-t)^{\sigma r}(a)_r}{r!} \exp\left(\frac{-A}{t^m(1-t)^m}\right) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n t^{n\alpha}}{\Gamma(\alpha n+\beta)n!} dt$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\gamma)_{qn} z^n u^r (a)_r}{\Gamma(\alpha n+\beta)n!r!} \int_0^1 t^{\lambda+\rho r+\alpha n-1}(1-t)^{\mu+\sigma r-\lambda-1}dt.$$

Using the definition of generalized beta function,(1.6),we get

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_r(\gamma)_{qn}u^r z^n}{\Gamma(\alpha n+\beta)n!} B(\lambda+\alpha n+\rho r, \mu-\lambda+\sigma r; A; m).$$

Corollary 2.4. For $a = 0$ in Theorem 2.3 reduces to the following result as

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}^{\gamma,q}(zt^{\alpha})dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)n!} B(\lambda+\alpha n, \mu-\lambda; A; m). \tag{2.5}$$

Remark 2.3. For $m = 1$ in Theorem 2.3, the result reduces to known result of Ahmed and Khan [1, p.483].

3 Special Cases

1. On setting $\delta = p = q = 1$ in Theorem 2.1, we get

$$\int_0^1 t^{\rho-1}(1-t)^{\delta-1} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}(zt^{\alpha})dt = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} B(\alpha n+\rho, \delta; A; m), \tag{3.1}$$

where $E_{\alpha,\beta}(z)$ is a Wiman function (1.2).

2. For $q=1$ in Theorem 2.1, we get

$$\int_0^1 t^{rho-1}(1-t)^{\delta-1}\exp\left(\frac{-A}{t^m(1-t)^m}\right)E^{\gamma}_{\alpha,\beta}(zt^{\alpha})dt$$

$$=\sum_{n=0}^{\infty}\frac{z^n(\gamma)_n}{\Gamma(\alpha n+\beta)n!}B(\alpha n+\rho,\delta;A;m), \tag{3.2}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is a Prabhakar function (1.3).

3. Setting $\beta=\gamma=q=1$ in Theorem 2.1, we get

$$\int_0^1 t^{\rho-1}(1-t)^{\delta-1}\exp\left(\frac{-A}{t^m(1-t)^m}\right)E_{\alpha}(zt^{\alpha})dt$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}B(n\alpha+\rho,\delta;A;m). \tag{3.3}$$

4. Letting $\gamma=q=1$ in Theorem 2.2, we get

$$\int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda}\exp\left(\frac{-A}{(t-a)^m(b-t)^m}\right)E_{\alpha,\beta}(z(b-t)^f)dt.$$

$$=(ac+d)^{\lambda}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^r z^n}{r!}\Gamma(\alpha n+\beta)B(\rho-mr,\delta+fn-mr)(b-a)^{\rho+\delta+fn-2mr-1}$$
$$\times{}_2F_1\left[\rho-mr,-\lambda,\rho+\delta+fn-2mr;\frac{(b-a)c}{ca+d}\right], \tag{3.4}$$

where $E_{\alpha,\beta}(z)$ is a Wiman function (1.2).

5. On putting $=q=1$ in Theorem 2.2, we get

$$\int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda}\exp\left(\frac{-A}{(t-a)^m(b-t)^m}\right)E^{\gamma}_{\alpha,\beta}(z(b-t)^f)dt.$$

$$=(ac+d)^{\lambda}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^r z^n(\gamma)_n}{r!n!}\Gamma(\alpha n+\beta)B(\rho-mr,\delta+fn-mr)(b-a)^{\rho+\delta+fn-2mr-1}$$
$$\times{}_2F_1\left[\rho-mr,-\lambda,\rho+\delta+fn-2mr;\frac{(b-a)c}{ca+d}\right], \tag{3.5}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is a Prabhakar function (1.3).

6. On setting $\alpha=\beta=\gamma=q=1$ in Theorem 2.2, we get

$$\int_a^b (t-a)^{\rho-1}(b-t)^{\delta-1}(ct+d)^{\lambda}\exp\left(\frac{-A}{(t-a)^m(b-t)^m+z(b-t)^f}\right)dt.$$

$$=(ac+d)^{\lambda}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^r}{r!n!}B(\rho-mr,\delta+fn-mr)(b-a)^{\rho+\delta+fn-2mr-1}$$

$$\times {}_2F_1\left[\rho - mr, -\lambda, \rho+\delta+fn-2mr; \frac{(b-a)c}{ca+d}\right], \tag{3.6}$$

where exp(z) is the exponential function and $\Re(a) > 0, \Re(\lambda) > 0; |arg(\frac{bc+d}{ac+d})| < \pi$.

7. Setting $\gamma = q = 1$ in Theorem 2.3, we get

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1}\{1-ut^{\rho}(1-t)^{\sigma}\}^{-a} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E_{\alpha,\beta}(zt^{\alpha})dt$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_r u^r z^n}{\Gamma(\alpha n+\beta) r!} B(\lambda+\rho r+n\alpha, \mu-\lambda+\sigma r; A; m), \tag{3.7}$$

where $E_{\alpha,\beta}(z)$ is a Wiman function (1.2).

8. Setting $q = 1$ in Theorem 2.3, we get

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1}\{1-ut^{\rho}(1-t)^{\sigma}\}^{-a} \exp\left(\frac{-A}{t^m(1-t)^m}\right) E^{\gamma}_{\alpha,\beta}(zt^{\alpha})dt.$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\gamma)_n (a)_r u^r z^n}{\Gamma(\alpha n+\beta) r! n!} B(\lambda+\rho r+n\alpha, \mu-\lambda+\sigma r; m; A), \tag{3.8}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is a Prabhakar function (1.3).

9. Setting $\alpha = \beta = \gamma = q = 1$ in theorem 2.3, we get

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1}\{1-ut^{\rho}(1-t)^{\sigma}\}^{-a} \exp\left(\frac{-A}{t^m(1-t)^m+zt}\right) dt.$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{((a)_r u^r z^n}{r! n!} B(\lambda+\rho r+n, \mu-\lambda+\sigma r; m; A). \tag{3.9}$$

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