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A NEW EXTENSION OF THE MITTAG-LEFFLER FUNCTION

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Abstract The Mittag-Leffler function was introduced by Gosta Mittag-Leffler in 1903. Due to its diverse applications, many of its extensions, generalizations and variants have been presented and investigated. In this paper we introduce a new extension of the Mittag-Leffler function defined by (Shukla and Prajapati) by using extended Beta function. Then we investigate certain useful properties and formulae associated with the extended Mittag-Leffler function such as integral representation, Mellin transform, recurrence relation and derivative formulae.

1 Introduction

Many authors have developed integrals, involving a variety of special functions see [9]-[15]. In 1903, the Swedish Mathematician Gosta Mittag-Leffler [5] introduced the function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad (z \in \mathbb{C}; \Re(\alpha) > 0) \tag{1.1}$$

which is called Mittag-Leffler function. Wiman [2] gave and investigated a generalization of the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad (z, \beta \in \mathbb{C}; \Re(\alpha) > 0) \tag{1.2}$$

Prabhakar [16] introduced the following generalization of $E_{\alpha,\beta}(z)$ as follows

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \qquad (z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0) \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol.
In 2007, Shukla and Prajapati [1] introduced the function $E^{\gamma,q}_{\alpha,\beta}(z)$ which is defined for $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$.

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta) n!} \tag{1.4}$$

In this paper, we extend the Mittag-Leffler function $E^{\gamma,q}_{\alpha,\beta}(z)$ in the following way

$$E^{\gamma,c,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (c)_{qn} z^n}{\Gamma(\alpha n + \beta)(c)_{qn} n!} \tag{1.5}$$

we get,

$$E^{\gamma,c,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + qn, c - \gamma)(c)_{qn} z^n}{B(\gamma, c - \gamma)\Gamma(\alpha n + \beta) n!} \tag{1.6}$$

where $p \geq 0; \Re(c) > \Re(\gamma) > 0; \alpha, \beta, \gamma, c \in \mathbb{C}; \Re(\alpha) > 0.\Re(\beta) > 0; q \in \mathbb{N}$.
Here $B_p(x, y)$ is the extended Euler's Beta function.

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}}\, dt \qquad , [p \geq 0, \Re(x) > 0, \Re(y) > 0] \tag{1.7}$$

and $B_0(x, y) = B(x, y)$ is the familiar Beta function [3]
The generalization of the extended beta function (1.7)

$$B_p^{\lambda,\rho}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left[\lambda; \rho; \frac{-p}{t(1-t)}\right] dt, \quad [p \geq 0, \Re(x) > 0, \Re(y) > 0] \tag{1.8}$$

By the use of (1.8) we further give the extension of the extended Mittag-Leffler function as follows

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma + qn, c - \gamma)(c)_{qn} z^n}{B(\gamma, c-\gamma)\Gamma(\alpha n + \beta) n!} \tag{1.9}$$

where $p \geq 0; \Re(c) > \Re(\gamma) > 0; \alpha, \beta, \gamma, c \in \mathbb{C}; \Re(\alpha) > 0.\Re(\beta) > 0; q \in \mathbb{N}$.
For our purpose, we recall the Fox-Wright function ${}_p\Psi_q$ [3]

$${}_p\Psi_q\left[\begin{array}{cccc} (\alpha_1, A_1), & \cdots & ,(\alpha_p, A_p); & \\ & & & z \\ (\beta_1, B_1), & \cdots & ,(\beta_q, B_q); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n) z^n}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n) n!} \tag{1.10}$$

where the coefficients $A_j \in \mathbb{R}^+ (j = 1, \cdots, p)$ and $B_j \in \mathbb{R}^+ (j = 1, \cdots, q)$ such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$$

A special case of (1.10) reduces to the generalized hypergeometric function ${}_pF_q$

$${}_p\Psi_q\left[\begin{array}{cccc} (\alpha_1, A_1), & \cdots & ,(\alpha_p, A_p); & \\ & & & z \\ (\beta_1, B_1), & \cdots & ,(\beta_q, B_q); & \end{array}\right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} {}_pF_q\left[\begin{array}{cccc} \alpha_1, & \cdots & ,\alpha_p; & \\ & & & z \\ \beta_1, & \cdots & ,\beta_q; & \end{array}\right] \tag{1.11}$$

We also recall the generalized Gamma function $\Gamma_p^{(\alpha,\beta)}$ [4]

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1} {}_1F_1\left(\alpha; \beta; -t - \frac{p}{t}\right) dt, \qquad [min\{\Re(\alpha), \Re(\beta), \Re(x)\} > 0; p \in \mathbb{R}_0^+] \tag{1.12}$$

$$\Gamma_0^{(\alpha,\beta)}(x) =: \Gamma^{(\alpha,\beta)}(x) = \frac{\Gamma(\beta)\Gamma(\alpha - x)\Gamma(x)}{\Gamma(\alpha)\Gamma(\beta - x)} \tag{1.13}$$

The special case of (1.12) when $\alpha = \beta$ and p=0 is seen to yield the familiar Gamma function $\Gamma(x)$.

2 Properties of the extended Mittag-Leffler function

Here we give an integral representation and the Mellin transform of the extended Mittag-Leffler function.

Theorem 2.1. *Let $c, \alpha, \beta, \gamma, \lambda \in C$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0, \Re(\beta) > 0$. Also let $p \in \mathbb{R}^+$ and $\rho \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \frac{1}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} {}_1F_1\left[\lambda; \rho; \frac{-p}{t(1-t)}\right] E_{\alpha,\beta}^{c,q}(t^q z) dt. \tag{2.1}$$

Proof. Using (1.8) in (1.9)

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \int_0^1 t^{\gamma+qn-1}(1-t)^{c-\gamma-1}{}_1F_1\left[\lambda;\rho;\frac{-p}{t(1-t)}\right] dt \frac{(c)_{qn}z^n}{B(\gamma,c-\gamma)\Gamma(\alpha n+\beta)n!}$$

Interchanging the order of summation and integration in the above equation which is verified under the given conditions here, we get

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}{}_1F_1\left[\lambda;\rho;\frac{-p}{t(1-t)}\right] \sum_{n=0}^{\infty} \frac{(c)_{qn}(zt^q)^n}{\Gamma(\alpha n+\beta)n!} dt$$

Using (1.4), we get desired result. □

Corollary 2.2. *Setting $t = \frac{u}{1+u}$ in (2.1) and under the same conditions on parameters we get,*

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^{\infty} \frac{u^{\gamma-1}}{(1+u)^c} {}_1F_1\left[\lambda;\rho;\frac{-p(1+u)^2}{u}\right] E_{\alpha,\beta}^{c,q}\left\{\frac{u^q z}{(1+u)^q}\right\} du \tag{2.2}$$

Corollary 2.3. *Setting $t = sin^2\theta$ in (2.1) and under the same conditions on parameters we get,*

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \frac{2}{B(\gamma,c-\gamma)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\gamma-1}(\cos\theta)^{2c-2\gamma-1}{}_1F_1\left[\lambda;\rho;\frac{-p}{\sin^2\theta\cos^2\theta}\right]$$

$$\times E_{\alpha,\beta}^{c,q}(z\sin^{2q}\theta)d\theta \tag{2.3}$$

Kurulay and Bayram [8] presented the following recurrence relation for the extended Mittag-Leffler function (1.3)

$$E_{\alpha,\beta}^{c}(tz) = \beta E_{\alpha,\beta+1}^{c}(tz) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{c}(tz)$$

We have derived the following recurrence relation for the extended Mittag-Leffler function (1.4)

$$E_{\alpha,\beta}^{c,q}(t^q z) = \beta E_{\alpha,\beta+1}^{c,q}(t^q z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{c,q}(t^q z) \tag{2.4}$$

Corollary 2.4. *Let $c,\alpha,\beta,\gamma,\lambda \in C$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0, \Re(\beta) > 0$. Also let $p \in \mathbb{R}^+$ and $\rho \in \mathbb{C}\setminus\mathbb{Z}^-$, then*

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c,q;\lambda,\rho}(z;p) + \frac{\alpha\gamma(c)_q z}{c}E_{\alpha,\alpha+\beta+1}^{\gamma+1,c+1,q;\lambda,\rho}(z;p) \tag{2.5}$$

Proof. Applying (2.4) to the integrand in (2.1), we obtain

$$E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c,q;\lambda,\rho}(z;p) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,c,q;\lambda,\rho}(z;p)$$

Then we use (3.10) with $n = 1$ in the derivative term in above equation we achieve our result. □

Theorem 2.5. *Let $\gamma,c,\lambda,\beta \in \mathbb{C}$, $\alpha \in (0,\infty)$, $p \in [0,\infty)$, $q \in \mathbb{N}$; $\Re(c) > \Re(\gamma) > 0$, $min\{\Re(\beta),\Re(\lambda),\Re(\rho)\} > 0$. The Mellin transform of the extended Mittag-Leffler function (1.9) is given by*

$$\mathfrak{M}\{E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p);s\} = \frac{\Gamma^{(\lambda,\rho)}(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)}{}_2\Psi_2\left[\begin{matrix}(c,q) & (\gamma+s,q); & \\ & & z\\ (c+2s,q) & (\beta,\alpha); & \end{matrix}\right] \tag{2.6}$$

Proof. Taking Mellin transform of the extended Mittag-Leffler function (1.9), we have

$$\mathfrak{M}\{E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p);s\} = \int_0^\infty p^{s-1} E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p)\,dp \tag{2.7}$$

Applying (1.9) to (2.7), we have

$$\mathfrak{M}\{E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p);s\} = \frac{1}{B(\gamma,c-\gamma)} \int_0^\infty p^{s-1} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} {}_1F_1\left[\lambda;\rho;\frac{-p}{t(1-t)}\right] \times E^{c,q}_{\alpha,\beta}(t^q z)dtdp$$

Interchanging the order of summation and integration, which is gauranteed under the conditions here, we have

$$\mathfrak{M}\{E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p);s\} = \frac{1}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} E^{c,q}_{\alpha,\beta}(t^q z) \times \int_0^\infty p^{s-1} {}_1F_1\left[\lambda;\rho;\frac{-p}{t(1-t)}\right] dpdt \tag{2.8}$$

□

Consider inner integral and take $u = \frac{p}{t(1-t)}$

$$\begin{aligned}\int_0^\infty p^{s-1} {}_1F_1\left[\lambda;\rho;\frac{-p}{t(1-t)}\right] dp &= \int_0^\infty u^{s-1} t^s (1-t)^s {}_1F_1[\lambda;\rho;-u]du \\ &= t^s(1-t)^s \int_0^\infty u^{s-1} {}_1F_1[\lambda;\rho;-u]du \\ &= t^s(1-t)^s \Gamma^{(\lambda,\rho)}(s)\end{aligned}$$

where $\Gamma^{(\lambda,\rho)}(s)$ is the extended gamma function given in (1.10).
Now putting back this value in (2.8), we get

$$\mathfrak{M}\{E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p);s\} = \frac{\Gamma^{(\lambda,\rho)}(s)}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma+s-1}(1-t)^{c-\gamma+s-1} E^{c,q}_{\alpha,\beta}(t^q z)dt$$

Then using (1.4) and interchanging order of summation and integration

$$= \frac{\Gamma^{(\lambda,\rho)}(s)}{B(\gamma,c-\gamma)} \sum_{n=0}^\infty \frac{(c)_{qn}(z)^n}{\Gamma(\alpha n+\beta)n!} \int_0^1 t^{\gamma+s+qn-1}(1-t)^{c-\gamma+s-1}dt$$

Using definition of Beta function

$$= \frac{\Gamma^{(\lambda,\rho)}(s)\Gamma(c)}{\Gamma(\gamma)\Gamma(c-\gamma)} \sum_{n=0}^\infty \frac{(c)_{qn} z^n}{\Gamma(\alpha n+\beta)n!} \frac{\Gamma(\gamma+s+qn)\Gamma(c-\gamma+s)}{\Gamma(c+2s+qn)}$$

$$= \frac{\Gamma^{(\lambda,\rho)}(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)} \sum_{n=0}^\infty \frac{\Gamma(c+qn)\Gamma(\gamma+s+qn)z^n}{\Gamma(c+2s+qn)\Gamma(\alpha n+\beta)n!}$$

By virtue of (1.10), we get our result.

Corollary 2.6. *Setting $s = 1$ in the result of Theorem 2.5 and using (1.11), we obtain an integral formula involving extended Mittag-Leffler function (1.9)*

$$\int_0^\infty E^{\gamma,c,q;\lambda,\rho}_{\alpha,\beta}(z;p)dp = \frac{\rho(c-\gamma)}{\lambda\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (c,q) & (\gamma+1,q); \\ & & z \\ (c+2,q) & (\beta,\alpha); \end{matrix}\right] \tag{2.9}$$

Corollary 2.7. *On setting $\lambda = \rho$ in (2.9), we get an integral formula involving the extended Mittag-Leffler function (1.5)*

$$\int_0^\infty E^{\gamma,c,q}_{\alpha,\beta}(z;p)dp = \frac{(c-\gamma)}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (c,q) & (\gamma+1,q); \\ & & z \\ (c+2,q) & (\beta,\alpha); \end{matrix}\right] \tag{2.10}$$

3 Derivative formula of the extended Mittag-Leffler function

Here, we define a further extension of a known extended Riemann-Liouville fractional derivative and present a derivative formula for the extended Mittag-Leffler function.

The well known Riemann-Liouville fractional derivative of order μ where $\Re(\mu) \geq 0$ is defined by

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dx^m} \int_0^x f(t)(x-t)^{m-\mu-1} dt, \tag{3.1}$$

where $m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x > 0$. In particular,

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x f(t)(x-t)^{-\mu} dt, \ \ [0 < \Re(\mu) < 1;\ x > 0] \tag{3.2}$$

Ozarslan and Yilmaz [7] extended the Riemann-Liouville fractional derivative in (3.1) as follows

$$\mathfrak{D}_x^{\mu,p}\{f(x)\} = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dx^m} \int_0^x f(t)(x-t)^{m-\mu-1} e^{\frac{-px^2}{t(x-t)}} dt \tag{3.3}$$

where $p \in [0,\infty);\ m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x > 0$
In particular,

$$\mathfrak{D}_x^{\mu,p}\{f(x)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{d} \int_0^x f(t)(x-t)^{-\mu} e^{\frac{-px^2}{t(x-t)}} dt \tag{3.4}$$

where $p \in [0,\infty);\ 0 < \Re(\mu) < 1;\ x > 0$ Here, we define a further extension of the extended Riemann-Liouville fractional derivative (3.3) by

$$\mathfrak{D}_{x;\lambda,\rho}^{\mu,p}\{f(x)\} = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dx^m} \int_0^x f(t)(x-t)^{m-\mu-1} {}_1F_1\left[\lambda;\rho;\frac{-px^2}{t(x-t)}\right] dt \tag{3.5}$$

where $p \in [0,\infty);\ m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x > 0;\ \lambda \in \mathbb{C};\ \rho \in \mathbb{C} \setminus \{0,-1,-2,-3\cdots\}$ In particular

$$\mathfrak{D}_{x;\lambda,\rho}^{\mu,p}\{f(x)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{d} \int_0^x f(t)(x-t)^{-\mu} {}_1F_1\left[\lambda;\rho;\frac{-px^2}{t(x-t)}\right] dt \tag{3.6}$$

where $p \in [0,\infty);\ 0 < \Re(\mu) < 1;\ x > 0;\ \lambda \in \mathbb{C};\ \rho \in \mathbb{C} \setminus \{0,-1,-2,-3\cdots\}$
It is obvious that the special cases of (3.5) when $\lambda = \rho$ and $\rho = 0$ reduces to the extended Riemann-Liouville fractional derivative (3.3) and the Riemann-Liouville fractional derivative (3.1), respectively.

We provide some formulas for the $B_p^{\lambda,\rho}(x,y)$ function in (1.6) and the Beta function $B(x,y)$, which are easily derivable and given in the following lemma

Lemma 3.1. *The following formulas holds*

$$B(x,y+1) = \frac{y}{x+y} B(x,y) \tag{3.7}$$

$$B(x+1,y) = \frac{x}{x+y} B(x,y) \tag{3.8}$$

$$B_p^{\lambda,\rho}(x,y) = B_p^{\lambda,\rho}(x+1,y) + B_p^{\lambda,\rho}(x,y+1) \tag{3.9}$$

We give a derivative formula for the extended Mittag-Leffler function (1.7) which is asserted by the following theorem

Theorem 3.2. *Let* $\gamma, c, \alpha, \beta \in \mathbb{C}; n \in \mathbb{N}_0;\ p \in [0,\infty),\ \rho \in \mathbb{C} \setminus \{0,-1,-2,-3\cdots\};\ \Re(\alpha) > 0$ *and* $\Re(c) > \Re(\gamma) > 0$*, then*

$$\frac{d^n}{dz^n} E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p) = \frac{(c)_q(c+q)_q\cdots(c+q(n-1))_q(\gamma)_n}{(c)_n} E_{\alpha,n\alpha+\beta}^{\gamma+qn,c+qn,q;\lambda,\rho}(z;p) \tag{3.10}$$

Proof.
$$\frac{d}{dz}E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p)=\sum_{n=1}^{\infty}\frac{B_p^{\lambda,\rho}(\gamma+qn,c-\gamma)(c)_{qn}z^{n-1}}{B(\gamma,c-\gamma)\Gamma(\alpha n+\beta)(n-1)!}$$
$$=\sum_{n=0}^{\infty}\frac{B_p^{\lambda,\rho}(\gamma+qn+q,c-\gamma)(c)_{qn+q}z^{n}}{B(\gamma,c-\gamma)\Gamma(\alpha n+\alpha+\beta)n!}$$
$$=(c)_q\sum_{n=0}^{\infty}\frac{B_p^{\lambda,\rho}(\gamma+qn+q,c-\gamma)(c+q)_{qn}z^{n}}{B(\gamma,c-\gamma)\Gamma(\alpha n+\alpha+\beta)n!}$$
which upon using (3.8), leads to
$$=\frac{(c)_q\gamma}{c}\sum_{n=0}^{\infty}\frac{B_p^{\lambda,\rho}(\gamma+qn+q,c-\gamma)(c+q)_{qn}z^{n}}{B(\gamma+1,c-\gamma)\Gamma(\alpha n+\alpha+\beta)n!}$$
differentiating again we get
$$\frac{d^2}{dz^2}E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(z;p)=\frac{(c)_q(c+q)_q\gamma(\gamma+1)}{c(c+1)}\sum_{n=0}^{\infty}\frac{B_p^{\lambda,\rho}(\gamma+qn+2q,c-\gamma)(c+2q)_{qn}z^{n}}{B(\gamma+2,c-\gamma)\Gamma(\alpha n+2\alpha+\beta)n!}$$
Differentiating *n* times we get our result. □

Theorem 3.3. *Let* $\gamma,c,\alpha,\beta\in\mathbb{C};n\in\mathbb{N}_0;\ p\in[0,\infty),\ \rho\in\mathbb{C}\setminus\{0,-1,-2,-3\cdots\};\ \Re(\alpha)>0$ *and* $\Re(c)>\Re(\gamma)>0$*, then*
$$\frac{d^n}{dz^n}\{z^{\beta-1}E_{\alpha,\beta}^{\gamma,c,q;\lambda,\rho}(\mu z^{\alpha};p)\}=z^{\beta-n-1}E_{\alpha,\beta-n}^{\gamma,c,q;\lambda,\rho}(\mu z^{\alpha};p)\tag{3.11}$$
Proof. By using (1.9), it is easy to avail the desired result.- □

Theorem 3.4. *Let*$p\in[0,\infty),x\in(0,\infty)$ *and*$\rho\in\mathbb{C}\setminus\{0,-1,-2,-3\cdots\}$. *Also, let* $m-2<\Re(\delta-c)<m-1$ *for* $m\in\mathbb{N}$. *Then*
$$\mathfrak{D}_{x;\lambda,\rho}^{\delta+1-c,p}\{x^{\delta-1}E_{\alpha,\beta}^{c+m-1,q}(x^q)\}=\sum_{l=0}^{m}\binom{m}{l}\frac{\Gamma(\delta+n)}{\Gamma(c+m+l-1)}x^{c+m+l-2}$$
$$\times\frac{(c+m-1)_q(c+m-1+q)_q\cdots(c+m-1+q(n-1))_q}{(c+m-1)_n}E_{\alpha,\alpha n+\beta}^{\delta+qn,c+m-1+qn,q;\lambda,\rho}(x^q;p)\tag{3.12}$$

Proof. Let $\mathfrak{L}$ be the left side of (3.12), by using (3.5), we have
$$\mathfrak{L}=\frac{1}{\Gamma(m+c-\delta-1)}\frac{d^m}{dx^m}\int_0^x t^{\delta-1}(x-t)^{c+m-\delta-2}E_{\alpha,\beta}^{c+m-1,q}(t^q)_1F_1[\lambda;\rho;\frac{-px^2}{t(x-t)}]dt$$
which, upon taking $t=xu$ becomes
$$\mathfrak{L}=\frac{1}{\Gamma(m+c-\delta-1)}$$
$$\times\frac{d^m}{dx^m}\left\{x^{c+m-2}\int_0^1 u^{\delta-1}(1-u)^{c+m-\delta-2}E_{\alpha,\beta}^{c+m-1,q}(x^qu^q)_1F_1[\lambda;\rho;\frac{-p}{u(1-u)}]du\right\}$$
Applying (2.1), we get
$$\mathfrak{L}=\frac{\Gamma(\delta)}{\Gamma(c+m-1)}\frac{d^m}{dx^m}\left\{x^{c+m-2}E_{\alpha,\beta}^{\delta,c+m-1,q;\lambda,\rho}(x^q;p)\right\}$$
By using Leibnitz's generalized rule for differentiation of product of two functions, we have
$$\mathfrak{L}=\frac{\Gamma(\delta)}{\Gamma(c+m-1)}\sum_{l=0}^{m}\binom{m}{l}\left\{\frac{d^{m-l}}{dx^{m-l}}x^{c+m-2}\right\}\left\{\frac{d^l}{dx^l}E_{\alpha,\beta}^{\delta,c+m-1,q;\lambda,\rho}(x^q;p)\right\}$$
Using (3.10) and the following formula we get our result
$$\frac{d^l}{dx^l}x^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-l+1)}x^{\alpha-l}$$
□

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