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$*$-RICCI SOLITONS ON $(\epsilon)-$KENMOTSU MANIFOLDS

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Abstract In the present paper we study $*$-Ricci solitons and provide the condition for a $*$-Ricci soliton in an $(\epsilon)-$Kenmotsu 3-manifold M with constant scalar curvature to be steady. Beside these, we study gradient $*$-Ricci solitons on $(\epsilon)-$Kenmotsu 3-manifolds.

1 Introduction

In modern mathematics, the methods of contact geometry play an important role. Contact geometry has evolved from the mathematical formalism of classical mechanics. The roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has multiple connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optic, phase space of dynamical systems, thermodynamics and control theory.

The properties of a manifold depend on the nature of the metric defined on it. With the help of *indefinite metric*, A. Bejancu and K. L. Duggal [2] introduced $(\epsilon)-$*Sasakian manifolds*. Also Xufeng and Xiaoli [27] showed that *every* $(\epsilon)-$*Sasakian manifold must be a real hypersurface of some indefinite Kähler manifold.* In 2009, De and Sarkar [12] introduced the notion of $(\epsilon)-$*Kenmotsu manifolds*. Since Sasakian manifolds with indefinite metric play significant role in physics [15], our natural trend is to study various contact manifolds with indefinite metric. In 1972, K. Kenmotsu [20] introduced a new class of almost contact Riemannian manifolds which are known as *Kenmotsu manifolds*. Kenmotsu manifolds were studied by many authors such as G. Pitis [22], De and De ([9],[10],[11]), Binh, Tamassy, De and Tarafdar [3], Özgür ([23],[24]) and many others. In this paper, we introduce a new type of Ricci solitons, called $*$-*Ricci solitons in* $(\epsilon)-$*Kenmotsu manifolds* with indefinite metric which also enclose the usual case of Kenmotsu manifolds.

In 1959, Tachibana [26] introduced the notion of $*$-*Ricci tensor* on almost Hermitian manifolds. Later, in [17] Hamada studied $*$-Ricci flat real hypersurfaces in non-flat complex space forms and Blair [4] defined $*$-Ricci tensor in contact metric manifolds by

$$S^*(X,Y) = g(Q^*X,Y) = Trace\{\phi \circ R(X,\phi Y)\}, \tag{1.1}$$

where Q^* is called the $*$-*Ricci operator*.

In 1982, Hamilton [18] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial_t} g_{ij} = -2R_{ij}. \tag{1.2}$$

Ricci solitons are special solutions of the Ricci flow equation (1.2) of the form $g_{ij} = \sigma(t)\psi_t^* g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [6]. On a manifold M, a Ricci soliton is a triple (g,V,λ) with g a Riemannian metric, V a vector field (called potential vector field) and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.3) are interesting and useful in physics and are often referred as *quasi-Einstein* ([7],[8]).

The Ricci soliton is said to be *shrinking*, *steady* and *expanding* according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([13],[14],[18],[16],[19]) and many others. We refer to ([1],[5],[25]) and references therein for a survey and further references on the geometry of Ricci solitons on semi-Riemannian manifolds.

Definition 1.1. [21] A Riemannian (or semi-Riemannian) metric g on M is called $*$*-Ricci soliton* if

$$\pounds_V g + 2S^* + 2\lambda g = 0, \tag{1.4}$$

where λ is a constant.

Definition 1.2. [21] A Riemannian (or semi-Riemannian) metric g on M is called *gradient* $*$*-Ricci soliton* if

$$\nabla\nabla f + S^* + \lambda g = 0, \tag{1.5}$$

where $\nabla\nabla f$ denotes the hessian of the smooth function f on M with respect to g and λ is a constant.

Definition 1.3. A contact metric manifold of dimension $n > 2$ is called $*$*-Einstein* if the $*$-Ricci tensor S^* of type $(0,2)$ satisfies the relation

$$S^* = \lambda g,$$

where λ is a constant.

If an $(\epsilon)-$Kenmotsu manifold M satisfies the relation (1.4), then we say that M admits a $*$-Ricci soliton.

The present paper focuses on the study of $(\epsilon)-$Kenmotsu 3-manifolds admitting a $*$-Ricci soliton. More precisely, the following theorems are proved.

Theorem 1.4. *If an $(\epsilon)-$Kenmotsu 3-manifold $(M, \phi, \xi, \eta, g, \epsilon)$ with constant scalar curvature admits a $*$-Ricci soliton with potential vector field V, then the $*$-Ricci soliton is steady if and only if $\pounds_V\xi$ is g-orthogonal to ξ.*

Theorem 1.5. *A gradient $*$-Ricci soliton with potential vector field of gradient type, $V = Df$, satisfying $\pounds_\xi f = 0$ on an $(\epsilon)-$Kenmotsu 3-manifold $(M, \phi, \xi, \eta, g, \epsilon)$ is $*$-Einstein.*

2 Preliminaries

In 1990, Duggal [15] introduced a larger class of contact manifolds as follows.

A $(2n+1)-$dimensional smooth manifold M together with a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a semi-Riemannian metric g is called an $(\epsilon)-$*almost contact metric manifold* if

$$\phi^2 X = -X + \eta(X)\xi, \;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi, \xi) = \epsilon, \;\; \eta(X) = \epsilon g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.3}$$

where ϵ is 1 or -1 according as ξ is spacelike or timelike, and the rank of ϕ is $2n$. It is important to mention that in the above definition, ξ is never a lightlike vector field. If $d\eta(X,Y) = g(\phi X, Y)$, for any $X, Y \in \chi(M)$, then we say that M is an $(\epsilon)-$*contact metric manifold*. It follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and $g(X, \phi Y) = -g(\phi X, Y)$, for any $X, Y \in \chi(M)$.

If moreover, the manifold satisfies

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g, then we shall call the manifold an $(\epsilon)-$*Kenmotsu manifold* [12].

On an $(\epsilon)-$Kenmotsu manifold $(M, \phi, \xi, \eta, g, \epsilon)$, the following relations hold:

$$\nabla_X \xi = \epsilon[X - \eta(X)\xi], \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$\begin{aligned} R(X,Y)\phi Z &= \phi R(X,Y)Z + \epsilon[g(Y,Z)\phi X \\ &- g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X], \end{aligned} \tag{2.8}$$

$$\eta(R(X,Y)Z) = \epsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{2.9}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.10}$$

$$Q\xi = -\epsilon 2n\xi, \tag{2.11}$$

where ∇, R, S and Q denote respectively, the Riemannian connection, the curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$ and the Ricci operator of type $(1,1)$.

Lemma 2.1. *In an $(\epsilon)-$Kenmotsu manifold $(M,\phi,\xi,\eta,g,\epsilon)$, we have*

$$\begin{aligned} \tilde{R}(X,Y,\phi Z,\phi W) &= \tilde{R}(X,Y,Z,W) + \epsilon g(X,\phi Z)g(Y,\phi W) \\ &-\epsilon g(Y,\phi Z)g(X,\phi W) + \epsilon g(X,W)g(Y,Z) \\ &-\epsilon g(Y,W)g(X,Z), \end{aligned} \tag{2.12}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, for $X,Y,Z,W \in \chi(M)$.

Proof. To prove the above Lemma we shall use the equation (2.8).

Now

$$\begin{aligned} & \tilde{R}(X,Y,\phi Z,\phi W) \\ = \ & g(\phi R(X,Y)Z,\phi W) + \epsilon[g(Y,Z)g(\phi X,\phi W) \\ - \ & g(X,Z)g(\phi Y,\phi W) + g(X,\phi Z)g(Y,\phi W) - g(Y,\phi Z)g(X,\phi W) \\ = \ & g(R(X,Y)Z,W) - \epsilon\eta(R(X,Y)Z)\eta(W) + \epsilon[g(Y,Z)g(X,W) \\ - \ & \epsilon g(Y,Z)\eta(X)\eta(W) - g(X,Z)g(Y,W) + \epsilon g(X,Z)\eta(Y)\eta(W) \\ + \ & g(X,\phi Z)g(Y,\phi W) - g(Y,\phi Z)g(X,\phi W)] \\ = \ & g(R(X,Y)Z,W) + \epsilon g(Y,Z)g(X,W) - \epsilon g(X,Z)g(Y,W) \\ + \ & \epsilon g(X,\phi Z)g(Y,\phi W) - \epsilon g(Y,\phi Z)g(X,\phi W)]. \end{aligned}$$

This completes the proof. □

For a 3-dimensional $(\epsilon)-$Kenmotsu manifold $(M,\phi,\xi,\eta,g,\epsilon)$, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.13}$$

for any $X,Y,Z \in \chi(M)$, where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Putting $Z = \xi$ in (2.13) and using (2.7) we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} + \epsilon\right)[\eta(Y)X - \eta(X)Y]. \tag{2.14}$$

Again replacing Y by ξ in the foregoing equation and using (2.10), we get

$$QX = \left(\frac{r}{2} + \epsilon\right)X - \left(\frac{r}{2} + 3\epsilon\right)\eta(X)\xi, \tag{2.15}$$

which implies

$$S(X,Y) = \left(\frac{r}{2}+\epsilon\right) g(X,Y) - \left(\frac{r}{2}+3\epsilon\right) \epsilon\eta(X)\eta(Y). \tag{2.16}$$

Now we prove the following Lemma which will be used later.

Lemma 2.2. *In an* $(\epsilon)-$*Kenmotsu* 3*-manifold* $(M,\phi,\xi,\eta,g,\epsilon)$*, the* $*$*-Ricci tensor is given by*

$$S^*(X,Y) = -S(X,Y) - \epsilon g(X,Y) - \eta(X)\eta(Y), \tag{2.17}$$

where S *and* S^* *are the Ricci tensor and the* $*$*-Ricci tensor of type* $(0,2)$*, respectively.*

Proof. Let $\{e_i\}$, $i=1,2,3$ be an orthonormal basis of the tangent space at each point of the manifold. From (1.1) and using (2.12), we infer

$$\begin{aligned} S^*(Y,Z) &= -\sum_{i=1}^{3} \tilde{R}(e_i,Y,\phi Z,\phi e_i) \\ &= \sum_{i=1}^{3}[-\tilde{R}(e_i,Y,Z,e_i) + \epsilon g(e_i,\phi Z)g(Y,\phi e_i) \\ &\quad -\epsilon g(Y,\phi Z)g(e_i,\phi e_i) + \epsilon g(e_i,e_i)g(Y,Z) \\ &\quad -\epsilon g(Y,e_i)g(e_i,Z)] \\ &= -S(Y,Z) - \epsilon g(Y,Z) - \eta(Y)\eta(Z). \end{aligned}$$

Hence, the $*$-Ricci tensor is

$$S^*(Y,Z) = -S(Y,Z) - \epsilon g(Y,Z) - \eta(Y)\eta(Z),$$

for any $Y,Z \in \chi(M)$. This completes the proof. □

From the above Lemma, the $(1,1)$ $*$-Ricci operator Q^* and the $*$-scalar curvature r^* are given by

$$Q^*X = -QX - \epsilon X - \epsilon\eta(X)\xi, \tag{2.18}$$

$$r^* = -r - 4\epsilon. \tag{2.19}$$

3 Proof of the main theorems

In view of the equation (2.16), the $*$-Ricci tensor is given by

$$S^*(X,Y) = -\left(\frac{r}{2}+2\epsilon\right) g(X,Y) + \left(\frac{r}{2}+2\epsilon\right) \epsilon\eta(X)\eta(Y). \tag{3.1}$$

Again from the equation of $*$-Ricci soliton we have

$$\begin{aligned} (\pounds_V g)(X,Y) &= -2S^*(X,Y) - 2\lambda g(X,Y) \\ &= (r+4\epsilon-2\lambda)g(X,Y) - (r+4\epsilon)\epsilon\eta(X)\eta(Y). \end{aligned} \tag{3.2}$$

Taking the covariant derivative with respect to Z, we get

$$\begin{aligned} (\nabla_Z \pounds_V g)(X,Y) =& (Zr)[g(X,Y) - \epsilon\eta(X)\eta(Y)] \\ &- (r+4\epsilon)\epsilon[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) \\ &- 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \end{aligned} \tag{3.3}$$

Following Yano ([28], p. 23), the following formula holds

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y),$$

for any $X, Y, Z \in \chi(M)$. As g is parallel with respect to the Levi-Civita connection ∇, the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y), \tag{3.4}$$

for any $X, Y, Z \in \chi(M)$. Since $\pounds_V \nabla$ is a symmetric tensor of type $(1,2)$, that is, $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)$, then it follows from (3.4) that

$$\begin{aligned} &g((\pounds_V \nabla)(X,Y),Z) \\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) - \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{3.5}$$

Using (3.3) in (3.5) yields

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) = {} & (Xr)[g(Y,Z) - \epsilon\eta(Y)\eta(Z)] \\ & - (r+4\epsilon)\epsilon[g(X,Z)\eta(Y) + g(X,Y)\eta(Z) \\ & - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \\ & + (Yr)[g(X,Z) - \epsilon\eta(X)\eta(Z)] \\ & - (r+4\epsilon)\epsilon[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) \\ & - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \\ & - (Zr)[g(X,Y) - \epsilon\eta(X)\eta(Y)] \\ & + (r+4\epsilon)\epsilon[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) \\ & - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \end{aligned} \tag{3.6}$$

Removing Z from (3.6), it follows that

$$\begin{aligned} 2(\pounds_V \nabla)(X,Y) = {} & (Xr)[Y - \eta(Y)\xi] + (Yr)[X - \eta(X)\xi] \\ & - [g(X,Y) - \epsilon\eta(X)\eta(Y)][(Dr) + 2(r+4\epsilon)\xi], \end{aligned} \tag{3.7}$$

where $(X\alpha) = g(D\alpha, X)$, for D the gradient operator with respect to g. Substituting $Y = \xi$ in the foregoing equation and using $r =$ constant (hence, $(Dr) = 0$ and (ξr)=0), we have

$$(\pounds_V \nabla)(X,\xi) = 0. \tag{3.8}$$

Taking the covariant derivative of (3.8) with respect to Y, we infer

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = -(\pounds_V \nabla)(X,Y). \tag{3.9}$$

Again from [28]

$$(\pounds_V R)(X,Y,Z) = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z). \tag{3.10}$$

Therefore (3.9) and (3.10) yields

$$(\pounds_V R)(X,Y,\xi) = 0, \tag{3.11}$$

for any $X, Y \in \chi(M)$. Setting $Y = \xi$ in (3.2) it follows that $(\pounds_V g)(X,\xi) = -2\lambda\epsilon\eta(X)$. Lie-differentiating the equation (2.2) along V and by virtue of the last equation we have

$$(\pounds_V \eta)(X) - \epsilon g(\pounds_V \xi, X) + 2\lambda\eta(X) = 0. \tag{3.12}$$

Putting $X = \xi$ in (3.12) we get

$$\lambda = \eta(\pounds_V \xi). \tag{3.13}$$

Thus we can say that the $*$-Ricci soliton is steady if and only if $\pounds_V \xi$ is g-orthogonal to ξ. This completes the proof of the Theorem 1.4. □

Remark 3.1. If (1.4) defines a steady $*$-Ricci soliton with potential vector field V on the $(\epsilon)-$Kenmotsu 3-manifold $(M, \phi, \xi, \eta, g, \epsilon)$ with vanishing $*$-scalar curvature, then V is a Killing vector field and g is an η-Ricci soliton defined by $(V, -\epsilon, -1)$ which can be either shrinking or expanding according as ξ is spacelike or timelike.

Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an $(\epsilon)-$Kenmotsu 3-manifold with g as a gradient $*$-Ricci soliton. Then the equation (1.5) can be written as

$$\nabla_X Df + Q^*X + \lambda X = 0, \tag{3.14}$$

for any $X \in \chi(M)$, where D denotes the gradient operator with respect to g. From (3.14) it follows that

$$R(X,Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y, \quad X, Y \in \chi(M). \tag{3.15}$$

Using (2.7), we have

$$g(R(\xi, X)Df, \xi) = \eta(X)(\xi f) - (Xf). \tag{3.16}$$

With the help of (3.1), we have

$$\begin{aligned}(\nabla_X Q^*)Y = &-\frac{(Xr)}{2}[Y - \eta(Y)\xi] \\ &+ \left(\frac{r}{2} + 2\epsilon\right)[g(X,Y)\xi + \eta(Y)X \\ &- (\epsilon + 1)\eta(X)\eta(Y)\xi].\end{aligned} \tag{3.17}$$

Interchanging X and Y, we have

$$\begin{aligned}(\nabla_Y Q^*)X = &-\frac{(Yr)}{2}[X - \eta(X)\xi] \\ &+ \left(\frac{r}{2} + 2\epsilon\right)[g(X,Y)\xi + \eta(X)Y \\ &- (\epsilon + 1)\eta(X)\eta(Y)\xi].\end{aligned} \tag{3.18}$$

Making use of (3.17) and (3.18) we get

$$\begin{aligned}(\nabla_Y Q^*)X - (\nabla_X Q^*)Y = &\frac{(Xr)}{2}[Y - \eta(Y)\xi] - \frac{(Yr)}{2}[X - \eta(X)\xi] \\ &- \left(\frac{r}{2} + 2\epsilon\right)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{3.19}$$

Putting $X = \xi$ in (3.19) and taking inner product with ξ, we infer that

$$g((\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y, \xi) = 0, \tag{3.20}$$

for any $Y \in \chi(M)$. From (3.16) and (3.20) we get

$$(Xf) = \eta(X)(\xi f), \tag{3.21}$$

for any $X \in \chi(M)$. Therefore, $Df = \epsilon(\xi f)\xi$. Taking the covariant derivative with respect to X and using (3.14) it follows that

$$S^*(X,Y) = -[\lambda + \epsilon(\xi f)]g(X,Y) - \eta(Y)[X(\xi f) - \eta(X)(\xi f)], \tag{3.22}$$

for any $X, Y \in \chi(M)$. This completes the proof of the Theorem 1.5. □

Also remark that if we assume $\pounds_\xi f = 0$, from (3.21) we obtain that f is a constant function.

Remark 3.2. If (3.14) defines a gradient $*$-Ricci soliton on the $(\epsilon)-$Kenmotsu 3-manifold $(M, \phi, \xi, \eta, g, \epsilon)$ with $\pounds_\xi f = 0$, then M has constant scalar curvature and g is an η-Ricci soliton defined by $(V, -\epsilon, -1)$, for any Killing vector field V, which can be either shrinking or expanding according as ξ is spacelike or timelike.

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