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Simple Expansions in the Lattice of Čech Closure Operators

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Abstract. Here we discuss simple expansions of Čech closure operators on a fixed non-empty set X. We compare certain properties of closure operators with their simple expansions.

1 Introduction

Larson R. E. and Throne W. J. introduced covering relations in the lattice of T_1 Topologies[5]. Then Agashe P. and Levine N. discussed sum and product of covers in the lattice of topologies[1]. Čech E. introduced a generalized notion of Kuratowski closure operator which in tern makes a space which is a generalization of topological spaces[2]. Many mathematicians contributed in this area. P.T. Ramachandran studied the lattice of Čech Closure operators [7, 8, 9, 10]. Analogous to the concept of immediate successors Kunheenkutty M. defined upper neighbours in the lattice of Čech closure operators[3]. Kunheenkutty M.et.al. discussed adjacency of generalization of Co-finite closure operators in [4].

Simple Extensions of topologies were introduced by Norman Levine in [6]. The simple extension of a topology τ on X by $A \subseteq X$ is the smallest topology on X containing A and τ [6]. The concept of simple expansions of Čech closure operators is introduced in the same[4, 3]. We note that some properties of a closure operator V need not be shared with the simple expansion of V. In this paper we investigate under what conditions does certain properties of closure operators hold for its simple expansion.

2 Preliminaries

Let X be a set and $P(X)$ denotes the power set of X. A Čech closure operator on a set X is a function $V : P(X) \to P(X)$ satisfying $V(\phi) = \phi,\ A \subseteq V(A)$, and $V(A \cup B) = V(A) \cup V(B)$ for every $A,\ B \in P(X)$. For brevity we call V a closure operator on X and the pair (X, V) a closure space.

A subset S of a closure space (X, V) is said to be closed if $V(S) = S$, and is said to be open if its complement is closed. The collection of all open sets in a closure space (X, V) is a topology on X, called the topology associated with V. A closure operator V is said to be topological if V is idempotent.

The closure operator $I : P(X) \to P(X)$ defined by $I(\phi) = \phi$ and $I(A) = X$, for every subset $A \subseteq X,\ A \neq \phi$ is called the indiscrete closure operator. This closure operator is the topological closure operator associated with the indiscrete topology on X. The closure operator D on X given by $D(A) = A$ for all $A \in P(X)$, is the topological closure operator associated with the discrete topology on X, called the discrete closure operator.

Define C_α on $P(X)$, where α is any infinite cardinal number such that $\alpha \leq |X|$, by,

$$C_\alpha(A) = \begin{cases} A & ; \text{ if } |A| < \alpha \\ X & ; \text{ otherwise} \end{cases}$$

The closure operator $C_0 : P(X) \to P(X)$ given by

$$C_0(A) = \begin{cases} A & \text{if } A \text{is finite,} \\ X & \text{otherwise.} \end{cases}$$

When $\alpha = \aleph_0$, we get that $C_\alpha = C_{\aleph_0} = C_0$. We have $C_0(C_0(A)) = C_0(A)$ for every $A \subseteq X$. Thus C_0 is topological closure operator and is associated with co-finite topology.

Let V_1, V_2 be two closure operators on set X. Then V_1 is said to be coarser than V_2 if $V_2(A) \subseteq V_1(A)$ for every $A \in P(X)$ and is denoted by $V_1 \leq V_2$. This relation in the set of all closure operators on X is a partial order. The set of all closure operators on X forms a lattice under this partial order and is denoted by L(X). The smallest element of this lattice is the indiscrete closure operator I and the largest is the discrete closure operator D. Let $\{V_a; a \in A\}$ where A is some indexing set, be a non empty collection of closure operators on $L(X)$. A closure operator V which is the infimum of $\{V_a\}$ in $L(X)$ is given by $V(A) = \underset{a \in A}{\cup} V_a(A)$ for $A \in P(X)$[2]. If U is the supremum of a non empty collection $\{V_a\}$ in $L(X)$ and $A \in P(X)$, then $x \in U(A)$ if and only if for each finite cover $\{A_1, A_2, \ldots, A_n\}$ of A, there exists an A_i such that $x \in V_a(A_i)$ for each $a \in A$[2]. Let U and V be two closure operators on X such that $U < V$. Then V is called an upper neighbour of U [4] if W is any closure operator on X such that $U \leq W \leq V$, then either $W = U$ or $W = V$. Then U and V are said to be adjacent.

Simple expansions of a closure operator is defined and discussed in [4]. First of all we need closure operators of the form $V_{A,x}$. Then we look at some properties of this closure operators and give the definition of simple expansion. In this section we investigate some properties of simple expansions also.

Definition 2.1. [3] Let A be a non empty proper subset of X and $x \in A$. Define, $V_{(A,x)} : P(X) \to P(X)$ by,

$$V_{(A,x)}(S) = \begin{cases} \phi & ; \text{ if } S = \phi, \\ X - \{x\} & ; \text{ if } S \neq \phi \text{ and } S \subseteq X - A, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{(A,x)}$ is a closure operator on X.

Lemma 2.2. *Let V be a closure operator on X. Then $V_{(A,x)} \leq V$ if and only if $x \notin V(X - A)$.*

Proof. Suppose $V_{(A,x)} \leq V$. Then $V(X-A) \subseteq V_{(A,x)}(X-A) = X - \{x\}$. Thus $x \notin V(X-A)$. Conversely let $x \notin V(X-A)$. Suppose $S \subseteq X - A$, then $V(S) \subseteq V(X-A)$. $x \notin V(X-A) \Rightarrow x \notin V(S) \Rightarrow V(S) \subseteq X - \{x\} = V_{(A,x)}(S)$. If $S \nsubseteq X - A$, then $V(S) \subseteq V_{(A,x)}(S) = X$. Hence $V_{(A,x)} \leq V$. □

Theorem 2.3. *Let A and B be subsets of X such that $B \in A$ and let $x \in X$ such that $x \in B$. Let $y \in X$, $x \neq y$. Then $V_{(B,x)}$ is an upper neighbour of $V_{(A,x)}$ if and only if $A \setminus B$ is a singleton subset of X.*

Proof. Suppose $A = B \cup \{y\}$. Since $B \subseteq A$, then $x \notin V_{(A,x)}(S) \Rightarrow S \subseteq X \setminus A \subseteq X \setminus B \Rightarrow S \subseteq X \setminus B \Rightarrow x \notin V_{(B,x)}(S)$. That is $V_{(A,x)} \leq V_{(B,x)}$. Let W be a closure operator on X such that $V_{(A,x)} \leq W \leq V_{(B,x)}$. Then $V_{(B,x)}(X \setminus A) \subseteq W(X \setminus A) \subseteq V_{(A,x)}(X \setminus A)$. Since $B \subseteq A$, $X \setminus A \subseteq X \setminus B$. Thus $W(X \setminus A) \subseteq W(X \setminus B)$ and this implies that $V_{(A,x)}(X \setminus A) = X \setminus \{x\}$. Also $x \notin V_{(B,x)}(X \setminus A) = X \setminus \{x\}$. Thus $x \notin W(X \setminus B)$. Then by Lemma 2.2, $V_{(B,x)} \leq W$. Hence $W = V_{(B,x)}$.

Now suppose $V_{(B,x)}$ is an upper neighbour of $V_{(A,x)}$. Then $V_{(A,x)} < V_{(B,x)}$. Then $V_{(B,x)}(X \setminus A) \subseteq V_{(A,x)}(X \setminus A) = X \setminus \{x\}$. Thus $x \notin V_{(B,x)}(X \setminus A)$. This implies that $(X \setminus A) \subseteq (X \setminus B)$. That is $B \subseteq A$. Suppose $A \setminus B$ is a non-empty set such that its cardinality is greater than or equal to 2. Then we can find a $C \subseteq X$ such that $B \subset C \subset A$. Then $V_{(A,x)} \leq V_{(C,x)} \leq V_{(B,x)}$. Then either $V_{(C,x)} = V_{(A,x)}$ or $V_{(C,x)} \leq V_{(B,x)}$. This is a contradiction. Hence $A \setminus B$ is a singleton set. □

Theorem 2.4. *Suppose that A and B are two infinite subsets of X such that $x \in A \cap B$, then $V_{(A,x)} \bigvee V_{(B,x)} = V_{(A \cap B,x)}$.*

Proof. We have $V_{(A,x)} \leq V_{(A \cap B,x)}$ and $V_{(B,x)} \leq V_{(A \cap B,x)}$. Thus $V_{(A,x)} \bigvee V_{(B,x)} \leq V_{(A \cap B,x)}$. In order to prove $V_{(A \cap B,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$, it is enough to prove that $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X - (A \cap B))$ by Lemma 2.2.

Since $V_{(A,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$ and $V_{(B,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$, $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X - A)$ and $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X - A)$. Hence $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X - (A \cap B))$. □

Definition 2.5. [3] Let V be any closure operator on X and A be a subset of X such that $x \in A$. The closure operator $V_A^x = V \bigvee V_{(A,x)}$ is called a simple expansion of V by A at x. Then it it is easy to see that,

$$V_A^x(S) = \begin{cases} V(S) - \{x\} & ; \quad \text{if } S \cap (X - A) \neq \phi \text{ and } x \notin V(S \cap A) \\ V(S) & ; \quad \text{otherwise.} \end{cases}$$

Remark 2.6. If A is closed in (X, V), then A is closed in (X, V_A^x).

Lemma 2.7. *Let (X, V) be a closure space. Suppose that A and B are two subsets of X such that $x \in A \cap B$. Then $V_A^x \leq V_{A\cap B}^x$ and $V_B^x \leq V_{A\cap B}^x$.*

Proof. Let $S \subseteq X$. Then

$$\begin{aligned} x \in V_{A\cap B}^x(S) \quad &\Rightarrow \quad S \subseteq A \cap B \text{ or } x \in V(S \cap A \cap B) \\ &\Rightarrow \quad S \subseteq A \text{ or } x \in V(S \cap A) \\ &\Rightarrow \quad x \in V_A^x(S). \end{aligned}$$

Then $V_{A\cap B}^x(S) \subseteq V_A^x(S)$. Hence $V_A^x \leq V_{A\cap B}^x$. Similarly we can prove that $V_B^x \leq V_{A\cap B}^x$. □

Lemma 2.8. *Let (X, V) be a closure space. Suppose that A and B are two subsets of X such that $x \in A \cup B$. Then $V_{A\cup B}^x \leq V_A^x$ and $V_{A\cup B}^x \leq V_B^x$.*

Proof. Let S be a subset of X. Then

$$\begin{aligned} x \in V_A^x(S) \quad &\Rightarrow \quad S \subseteq A \text{ or } x \in V(S \cap A) \\ &\Rightarrow \quad S \subseteq A \cap B \text{ or } x \in V(S \cap (A \cup B)) \text{ since } S \cap A \subseteq S \cap (A \cup B) \\ &\Rightarrow \quad x \in V_{A\cup B}^x(S). \end{aligned}$$

Thus $V_{A\cup B}^x \leq V_A^x$. Similarly we can prove that $V_{A\cup B}^x \leq V_B^x$. □

Remark 2.9. We have $V \leq V_{A\cup B}^x \leq V_A^x \leq V_{A\cap B}^x$ by Lemma 2.7 and Lemma 2.8.

Lemma 2.10. *Let A and B are subsets of X such that $A \cup B \neq X$. Let $x \in A \cap B$. Suppose V is a closure operator on X such that $x \notin V((X \setminus B) \cap A)$. Then $V_A^x \vee V_B^x = V_{A\cap B}^x$.*

Proof. We have by Lemma 2.7, $V_A^x \leq V_{A\cap B}^x$ and $V_B^x \leq V_{A\cap B}^x$. Therefore we get $V_A^x \bigvee V_B^x \leq V_{A\cap B}^x$. To prove the converse inequality it is enough to prove that $x \notin V_A^x \bigvee V_B^x(X \setminus (A \cap B))$. We have $x \notin V_A^x(X \setminus A)$ and $x \notin V_B^x(X \setminus B)$. This implies that $x \notin V_A^x \bigvee V_B^x(X \setminus A)$ and $x \notin V_A^x \bigvee V_B^x(X \setminus B)$. Hence $x \notin V_A^x \bigvee V_B^x(X \setminus (A \cap B)$. Thus $V_{A\cap B}^x \leq V_A^x \bigvee V_B^x$. □

Next we find out the relation between simple expansion of closure operators and upper neighbours of closure operators.

Theorem 2.11. *Let U and V be two closure operators on a set X such that U is an upper neighbour of V. Then U is a simple expansion of V at some point $x \in X$.*

Proof. Suppose that U is an upper neighbour of V. Then there exists a subset A of X such that $U(A) \subset V(A)$. Let $x \in A$. Consider the simple expansion of V by A at x. We have $V \leq V_A^x$. We prove that $V_A^x \leq U$. That is to prove that $U(S) \subseteq V_A^x(S)$ for each $S \subseteq X$.
Case (i) $S \subseteq X$ or $x \in V(S \cap A)$.
In this case $V_A^x(S) = V(S)$. Therefore $U(S) \subseteq V_A^x(S)$.
Case (ii) $S \cap (X \setminus A) \neq \phi$ and $x \notin V(S \cap A)$.
In this case $V_A^x(S) = V(S) \setminus \{x\}$. In order to prove $U(S) \subseteq V_A^x(S)$, it is enough to prove that $x \notin U(S)$. Since $x \notin V(S \cap A)$ and $V < U$, we have

$$x \notin U(S \cap A) \tag{2.1}$$

Since A is a V-neighbourhood of x, A is a U-neighbourhood of x. Therefore $A \cup (X \setminus S)$ is a U-neighbourhood of x. This implies that $x \notin U(X \setminus (A \cup (X \setminus S)))$. That is

$$x \notin U(S \cap (X \setminus A)) \tag{2.2}$$

Hence from 2.1 and 2.2, $x \notin U(S)$. Therefore we get $U(S) \subseteq V_A^x(S)$ for each $S \subseteq X$. Thus $V \leq V_A^x \leq U$. Now by the definition of upper neighbours either $V = V_A^x$ or $U = V_A^x$. Since $V \neq V_A^x$, we have $U = V_A^x$. This completes the proof. $\square$

Example 2.12. Converse of the Lemma 2.11 is not true. Let $X = \{a, b, c, d\}$. Define $V : P(X) \to P(X)$ as $V(\phi) = \phi$, $V(\{a\}) = \{a, b\}$, $V(\{b\}) = \{a, b, c\}$, $V(\{c\}) = \{c\}$, $V(\{d\}) = \{a, d\}$ and $V(B) = \underset{x\in B}{\cup} V(\{x\})$. Now let $A = \{a\}$ and consider V_A^a. Then $V_A^a(\{a\}) = \{a, b\}$, $V_A^a(\{b\}) = \{b, c\}$, $V_A^a(\{c\}) = \{c\}$ and $V_A^a(\{d\}) = \{d\}$. Thus V_A^a is not an upper neighbour of V, since $V \leqslant U \leqslant V_A^x$ where U is given by $U(\phi) = \phi$. $U(\{a\}) = \{a, b\}$, $U(\{b\}) = \{a, b, c\}$, $U(\{c\}) = \{c\}$, $U(\{d\}) = \{d\}$ and $U(B) = \underset{x\in B}{\cup} V(\{x\})$.

If V is a topological closure operator, then simple expansion of V by A at a point x need not be a topological closure operator.

Example 2.13. Let $X = \mathbb{R}$, $A = \mathbb{Z}$ and $x = 1$. Then $C_0(\mathbb{Z}) = \mathbb{R}$, where C_0 is the co-finite closure operator. Now $C_{0A}^x(\mathbb{R} - \mathbb{Z}) = \mathbb{R} - \{1\}$. But $C_{0A}^x(C_{0A}^x(\mathbb{R} - \mathbb{Z})) = C_{0A}^x(\mathbb{R} - \{1\}) = C_0(\mathbb{R} - \{1\}) = \mathbb{R}$. Here C_0 is a topological closure operator, but C_{0A}^x is not a topological closure operator.

Now we check when simple expansion of a topological closure operator becomes topological.

Theorem 2.14. *Let V be a closure operator on X and let $A \subseteq X$ with $x \in A$ such that $V(A) = A$ and $V(X \setminus \{x\}) = X \setminus \{x\}$. Then if V is a topological closure operator, then V_A^x is a topological closure operator.*

Proof. Suppose V is a topological closure operator. Then $V(V(A)) = V(A)$ for every $A \subseteq X$. Now suppose that $S \subseteq X$.
Case (i):$S \cap (X \setminus A) = \emptyset$.
That is $S \subseteq A$. In this case $V_A^x(S) = V(S)$. Since $S \subseteq A$, we have $V(S) \subseteq V(A) = A$. Thus $V_A^x(V(S)) = V(S)$ since $V(S) \subseteq A$. Now $V_A^x(V_A^x(S)) = V_A^x(V(S)) = V(V(S)) = V(S) = V_A^x(S)$.
Case (ii): $x \in V(S \cap A)$.
In this case $V_A^x(S) = V(S)$. We have $S \subseteq V(S) \Rightarrow S \cap A \subseteq V(S) \cap A \Rightarrow V(S \cap A) \subseteq V(V(S) \cap A)$. Hence $x \in V(V(S) \cap A)$ and therefore $V_A^x(V_A^x(S)) = V_A^x(S)$.
Case (iii): $S \cap (X \setminus A) \neq \emptyset$ and $x \notin V(S \cap A)$.
Then $V_A^x(S) = V(S) \setminus \{x\}$. Since $S \subseteq V_A^x(S)$, $S \cap (X \setminus A) \neq \emptyset \Rightarrow V_A^x(S) \cap (X \setminus A) \neq \emptyset$ and $x \notin V(V_A^x(S) \cap A)$. Hence $x \notin V_A^x(S) \Rightarrow x \notin V_A^x(V_A^x(S))$. Thus $V_A^x(V_A^x(S)) \subseteq V_A^x(S)$. If V is a topological closure operator, V_A^x is a topological closure operator. $\square$

Remark 2.15. Converse of the above Theorem 2.14 is not true. That is there exists a closure operator V which is not a topological closure operator but it has a simple expansion V_A^x where $V(A) = A$ which is topological.

Example 2.16. Let $X = \{a, b, c\}$. Let $V : P(X) \to P(X)$ be defined as $V(\{a\}) = \{a\}$, $V(\{b\}) = \{b, c\}$, $V(\{c\}) = \{c, a\}$ and $V(A) = \underset{s\in S}{\cup} V(\{s\})$. Then V is a closure operator on X which is not a topological closure operator. Now consider the simple expansion of V by $A = \{a\}$ at a. We have $V_A^a(\{a\}) = V(\{a\}) = \{a\}$, $V_A^a(\{b\}) = \{b, c\}$, $V_A^a(\{c\}) = V(\{c\}) - \{a\} = \{c\}$ and $V_A^a(S) = \underset{s\in S}{\cup} V(\{s\})$. Then V_A^a is a topological closure operator.

3 Properties of Simple Expansions of Closure Operators

In this section we check under what conditions does various properties of closure operators like regularity, normality and separation axioms hold for its simple expansions. Recall the definition of a regular closure space.

Definition 3.1. [2] A closure space (X, V) is said to be regular if for each point $x \in X$ and a subset S of X, such that $x \notin V(S)$, there exists neighbourhoods U_1 of x and U_2 of S such that $U_1 \cap U_2 = \phi$.

Now we analyze regularity property of the simple expansion of a closure operator.

Theorem 3.2. *Suppose that (X,V) is a regular closure space and let $A \subseteq X$ such that $V(A) = A$. Let $x \in A$. Then (X,V_A^x) is a regular closure space.*

Proof. Suppose V is regular and $A \subseteq X$ such that $V(A) = A$. Let $S \subseteq X$. If $S \subseteq A$, then $V_A^x(S) = V(S)$. Since $V \leq V_A^x$, every V- neighbourhood is a V_A^x-neighbourhood of S therefore V_A^x is regular.

If $y \neq x$, then $y \notin V_A^x(S)$ implies that $y \notin V(S)$. Since V is regular, there exists neighbourhoods U_1 of x and U_2 of S such that $U_1 \cap U_2 = \phi$. Again, since every V-neighbourhood is a V_A^x neighbourhood, V_A^x is regular. Now suppose $x \notin V_A^x(S)$, by definition of V_A^x, $S \cap (X-A) \neq \phi$ and $x \notin V(S \cap A)$. Now $x \notin V(S \cap A)$ implies that there exists a neighbourhood U of x and W of S such that $U \cap W = \phi$.

We have U and A are V_A^x-neighbourhood of x implies that $U \cap A$ is a neighbourhood of x. Now $U \cap A \subseteq A$ and therefore $V_A^x(U \cap A) \subseteq V_A^x(A) = V(A) = A$. Also $U \cap A \subseteq U$ implies that $V_A^x(U \cap A) \subseteq V(U)$. Thus $V_A^x(U \cap A) \subseteq V(U) \cap A$. We have $U \cap W = \phi$ implies that $U \subseteq X - W$. Then $V(U) \subseteq V(X-W)$. Now $X - V(X-W) \subseteq X - V(U)$.
$V(U) \cap A \subseteq V(U)$ then $X - V(U) \subseteq X - (V(U) \cap A)$. Thus we have

$$\begin{aligned} S &\subseteq X - V(X-W) \\ &\subseteq X - V(U) \\ &\subseteq X - (V(U) \cap A). \end{aligned}$$

As mentioned above we have $V_A^x(U \cap A) \subseteq V(U) \cap A$. Thus $X \setminus V(U) \cap A \subseteq X \setminus V_A^x(U \cap A)$. Therefore $S \cap A \subseteq X \setminus V_A^x(U \cap A)$. Now $V_A^x(U \cap A) = V(U \cap A) \subseteq V(A) = A$. Then $S \cap (X \setminus A) \subseteq (X \setminus A) \subseteq X \setminus (V_A^x(U \cap A))$. Hence $X \setminus (U \cap A)$ is a neighbourhood of S. Thus $U \cap A$ is a neighbourhood of x and $X \setminus (U \cap A)$ is a neighbourhood of S. Hence (X, V_A^x) is a regular closure space. □

Definition 3.3. [2] A closure space (X,V) is said to be T_0 if $x \in V(\{y\})$, $y \in V(\{x\})$ implies that $x = y$, and is said to be T_1 if every singleton subset of X is closed in X.

Definition 3.4. Let (X,V) be a closure space. Two subsets S_1 and S_2 of X are said to be separated if there exists neighbourhoods U_1 of S_1 and U_2 of S_2 such that $U_1 \cap U_2 = \phi$. A closure space (X,V) is said to be separated if any two distinct points of X are said to be separated.

Note that any expansion of a T_0 (respectively T_1, T_2) topological space has the same separation property.

Theorem 3.5. *Let (X,V) be a closure space which is T_0, T_1 or separated and $x \in X$ and $A \subseteq X$ such that $x \in A$. Then (X, V_A^x) is T_0, T_1 and separated.*

Proof. Suppose (X,V) is T_0. Suppose $y \in V_A^x(\{z\})$ and $z \in V_A^x(\{y\})$. Then $y \in V(\{z\})$ and $z \in V(\{y\})$. Since V is T_0, $y = z$.
Now suppose V is a T_1 closure operator. Then since $V_A^x(S) \subseteq V(S)$ for every subset S of X, V_A^x is T_1. Now we have to prove that if any two subsets in (X,V) is separated in (X,V), then any two sets in (X,V_A^x) is also separated. For that suppose that S_1 and S_2 are subsets of X. Then there exists neighbourhoods U_1 of S_1 and U_2 of S_2 such that $U_1 \cap U_2 = \phi$. That is $S_1 \subseteq X - V(X - U_1)$ and $S_2 \subseteq X - V(X - U_2)$. We have $V_A^x(S) \subseteq V(S)$ for every subset S of X. Then $S_1 \subseteq X - V_A^x(X - U_1)$ and $S_2 \subseteq X - V_A^x(X - U_2)$. Hence S_1 and S_2 are separated in (X, V_A^x). □

There is a characterization theorem for normal closure spaces in [2] as follows,

Theorem 3.6. *[2] A closure space (X,V) is normal if and only if S_1, S_2 subsets of X such that $V(S_1) \cap V(S_2) = \phi$, then S_1 and S_2 are separated and if $x \in X$ and $S \subseteq X$ are such that $V(\{x\}) \cap V(S) \neq \phi$, then $x \in V(S)$.*

Theorem 3.7. *Suppose (X,V) be a normal closure space. Let $A \subseteq X$ such that $x \in A$ and $V(\{x\}) = \{x\}$. Consider the simple expansion V_A^x. Then (X, V_A^x) is a normal closure space.*

Proof. Suppose (X,V) is a normal closure space. Then we have $V(S_1)\cap V(S_2)=\phi$, implies that S_1 and S_2 are separated. Now suppose $V_A^x(S_1)\cap V_A^x(S_2)=\phi$. Then $V_A^x(S_1)\subseteq X-V_A^x(S_2)$ and $V_A^x(S_2)\subseteq X-V_A^x(S_1)$. Then $X-S_1$ is a neighbourhood of S_2 and $X-S_2$ is a neighbourhood of S_1. Hence S_1 and S_2 are separated. Now we have to prove that $V_A^x(\{y\})\cap V_A^x(S)\neq\phi$ implies that $y\in V_A^x(S)$. Suppose $y\neq x$, $V_A^x(\{y\})\cap V_A^x(S)\neq\phi$, $S\subseteq X$. We have $V_A^x(\{y\})=V(\{y\})$ or $V_A^x(\{y\})=V(\{y\})-\{x\}$. Since V is normal, this implies that $y\in V(S)$. Next consider the case when $y=x$. Then $V_A^x(\{x\})=V(\{x\})$ by definition and by assumption $V_A^x(\{x\})=\{x\}$. If $V_A^x(\{x\})\cap V_A^x(S)\neq\phi$ implies that $\{x\}\cap V_A^x(S)\neq\phi$. Then $x\in V_A^x(S)$. Hence (X,V_A^x) is normal. □

Theorem 3.8. *[6] Let (X,τ) be separable and $A\notin\tau$. Then $(X,\tau(A))$ is separable if and only if $(A,\tau\cap A)$ is separable.*

Theorem 3.9. *Let (X,V) be a closure space. Let $x\in X$ and $A\subseteq X$ such that $x\in A$. Then (X,V) is separable if and only if (X,V_A^x) is separable.*

Proof. Suppose (X,V_A^x) is separable. Then there exists a countable set S of X such that $V_A^x(S)=X$. Since $V_A^x(S)\subseteq V(S)$, we have $V(S)=X$. Hence (X,V) is separable.

Conversely let (X,V) be separable. Let $S\subseteq X$ such that S is countable and $V(S)=X$. If $S\subseteq A$, then $V_A^x(S)=V(S)=X$ and if $x\in V(S\cap A)$, then $V_A^x(S)=V(S)=X$. Now suppose $S\cap X-A\neq\phi$ and $x\notin V(S\cap A)$, then $V_A^x(S)=V(S)-\{x\}=X-\{x\}$. Now consider the set $S\cup\{x\}$. We have $V_A^x(S\cup\{x\})=V_A^x(S)\cup V_A^x(\{x\})=X-\{x\}\cup V(\{x\})=X$. Thus (X,V_A^x) is separable. □

Next we check simple expansion of a connected closure space. First of all let us define semi separated subsets of a closure space.

Definition 3.10. [2] Let (X,V) be a closure space. Two subsets S_1 and S_2 of X are said to be semi separated if there exist neighbourhoods U_1 of S_1 and U_2 of S_2 such that $U_1\cap S_2=\emptyset=U_2\cap S_2$.

Definition 3.11. [2] A subset S of a closure space (X,V) is said to be connected if S is not the union of two non empty semi separated subsets of (X,V). That is $S=S_1\cup S_2$, $(V(S_1)\cap S_2)\cup(S_1\cap V(S_2))=\phi$ implies that $S_1=\phi$ or $S_2=\phi$.

Example 3.12. Let $X=\mathbb{Z}$, $A=\{1,2,\dots,n\}$ and $x=2$. Then $C_{0A}^2(\mathbb{Z}-\{2\})=\mathbb{Z}-\{2\}$ since $\mathbb{Z}-\{2\}\cap\mathbb{Z}-\{1,2,\dots,n\}\neq\phi$ and $2\notin C_0(\mathbb{Z}-\{2\}\cap A)$. Then $\mathbb{Z}=\{2\}\cap\mathbb{Z}-\{2\}$. And $\{2\}\cap C_{0A}^2(\mathbb{Z}-\{2\})=\phi$ and $C_0(\{2\})\cap(\mathbb{Z}-\{2\})=\phi$. That is (X,C_0) is connected but (X,C_{0A}^2) is not connected.

Now we characterize simple expansion of connected sets.

Theorem 3.13. *Let (X,V) be a closure space. Let $A\subseteq X$ be such that $x\in X$. If $(X\setminus A,V|_{X\setminus A})$ is connected, then (X,V_A^x) is connected.*

Proof. Suppose that $(X\setminus A,V|_{X\setminus A})$ is connected. Assume that $X=X_1\cup X_2$ such that $V_A^x(X_1)\cap X_2=\emptyset$ and $X_1\cap V_A^x(X_2)=\emptyset$. We have $V_A^x(S)=V(S)$ or $V_A^x(S)=V(S)\setminus\{x\}$ for every $S\subseteq X$.
Case (i): $V_A^x(X_1)=V(X_1)$ and $V_A^x(X_2)=V(X_2)$.
In this case $V(X_1)\cap X_2=\emptyset$ and $X_1\cap V(X_2)=\emptyset$. This implies that $V(X_1)\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\cap X\setminus A=\emptyset$. Thus by assumption $X_1=\emptyset$ or $X_2=\emptyset$.
Case (ii): $V_A^x(X_1)=V(X_1)\setminus\{x\}$ and $V_A^x(X_2)=V(X_2)$.
In this case $V(X_1)\setminus\{x\}\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\cap X\setminus A=\emptyset$. This implies that $V(X_1)\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\cap X\setminus A=\emptyset$, since $x\in A$. Hence by assumption $X_1=\emptyset$ or $X_2=\emptyset$.
Case (iii): $V_A^x(X_1)=V(X_1)$ and $V_A^x(X_2)=V(X_2)\setminus\{x\}$.
Then $V(X_1)\cap X_2=\emptyset$ and $X_1\cap V(X_2)\setminus\{x\}=\emptyset$, which implies that $V(X_1)\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\setminus\{x\}\cap X\setminus A=\emptyset$. Since $x\in A$, we have $V(X_1)\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\cap X\setminus A=\emptyset$. Thus $X_1=\emptyset$ or $X_2=\emptyset$.
Case (iv): $V_A^x(X_1)=V(X_1)\setminus\{x\}$ and $V_A^x(X_2)=V(X_2)\setminus\{x\}$.
In this case also since $x\in A$, $V(X_1)\cap X\setminus A\cap X_2=\emptyset$ and $X_1\cap V(X_2)\cap X\setminus A=\emptyset$. So $X_1=\emptyset$ or $X_2=\emptyset$ by assumption. □

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