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On the estimates for solutions of a nonlinear neutral differential system with periodic coefficients and time-varying lag

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Abstract. In this paper, we consider a nonlinear neutral differential system with periodic coefficients and time-varying lag. We obtain new estimates to characterize the exponential decay of solutions of that system at $+\infty$ and depending on the norms of the powers of a constant matrix D. To reach the desirable results, we use a Lyapunov-Krasovskii functional and give an example to show applicability of the constructed assumptions. We also use MATLAB-Simulink to show the behaviors of the paths of the solutions of the system considered in the special case.

1 Introduction

In 2015, Demidenko and Matveeva [10] consider the following n- dimensional constant delay differential equation (DDE) of neutral type

$$\frac{d}{dt}(y(t)+Dy(t-\tau))=Ay(t)+By(t-\tau)+F(t,y(t),y(t-\tau)). \tag{1.1}$$

The authors use their previous results on neutral linear DDEs to establish sufficient conditions for the exponential asymptotic stability of the zero solution of equation (1.1) with explicit estimates for the exponential rate decay of solutions at $+\infty$. The method used in [10] does not require finding the position of the zeros of the characteristic equation for the linearized equation. It is based on suitable Lyapunov-Krasovskii functionals and imposes some restrictions on the spectrum of the matrix D in equation (1.1), the rate decays obtained depend on the norms $\|D^j\|$ of powers of D.

In addition, qualitative behaviors of solutions of equation (1.1) were discussed in [5, 6, 7, 8, 9]. In that papers, conditions for exponential stability of the zero solution, estimates for the exponential decay of solutions at infinity and estimates of attraction sets of the zero solution were obtained. Note that in [5, 6] some estimates for exponential decay of solutions to (1.1) were established when $\|D\| < 1$. Further, in [7], for the linear case when $F(t,u,v)\equiv 0$ in equation (1.1), analogous estimates were given when the spectrum of the matrix D belongs to the unit disk $\{\lambda \in C : |\lambda| < 1\}$. However, in the case of $\|D\| < 1$ the estimates in [7] are weaker in comparison with the estimates obtained in [5]. More precise exponential estimates for the linear systems were obtained in [8, 9]. Moreover, in [9] the authors established estimates of exponential decay of solutions of the linear time-delay systems of neutral type with periodic coefficients.

In this paper, motivated by Demidenko and Matveeva [10], we consider the following nonlinear neutral differential system with periodic coefficients and time-varying lag

$$\frac{d}{dt}(y(t)+Dy(t-\tau(t)))=A(t)y(t)+B(t)y(t-\tau(t))+F(t,y(t),y(t-\tau(t))), \tag{1.2}$$

where $t\in\Re$, $t>0$, $y\in\Re^n$, D is a constant $n\times n$- matrix, $A(t)$ and $B(t)$ are $n\times n$- continuous T- periodic matrices, that is,

$$A(t+T)\equiv A(t), B(t+T)\equiv B(t), T>0,$$

and $F(t,u,v)$ is a real-vector valued continuous function satisfying the Lipschitz condition in u and the inequality

$$\|F(t,u,v\| \leq q_1\|u\| + q_2\|v\|, q_1, q_2 \in \Re, q_1, q_2 \geq 0, \tag{1.3}$$

and $\tau(t) \in C^1([0,\infty)), \tau(t)$ is differentiable and T- periodic variable delay, and it also satisfies that

$$\tau(t+T) = \tau(t), 0 < \tau_1 \leq \tau(t) \leq \tau_2 < \infty, \tau'(t) \leq \alpha < 1, \alpha \in (0,1), \tag{1.4}$$

where τ_1, τ_2 and α are some constants. In this paper, we consider the system (1.2) when the spectrum of the matrix D belongs to the unit disk. Our aim is to obtain estimates characterizing exponential decay of solutions at infinity depending on the norms $\|D^j\|$.

Through the paper, we use the following dot product and vector norm

$$\langle x, z\rangle = \sum_{j=1}^{n} x_j \bar{z}_j, \|x\| = \sqrt{\langle x, x\rangle}.$$

The symbol $\|D\|$ means the spectral norm of the matrix D and H^* is the conjugate transpose of H.

The aim of this paper is to obtain new estimates on the exponential decay of solutions of system (1.2). In recent years, in particular, for constant coefficients, constant delay and $F(t,y,y(t-\tau(t))) = 0$, there are a lot of nice works in the literature for linear differential equations including equations of neutral type or various kinds of the other differential equations (see [1-22]). However, we would only like to summarize a few related results on the topic.

For the following constant delay differential system with periodic coefficients

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau) + F(t, y(t), y(t-\tau)),$$

some constructive estimates of attraction sets are obtained in [4] by Lyapunov -Krasovskii functionals associated with the exponentially stable linear system

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau). \tag{1.5}$$

Further, in [3], to discuss the asymptotic stability of solutions of system (1.5), the authors have usage of the Lyapunov-Krasovskii functional

$$\langle H(t)y(t), y(t)\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle ds. \tag{1.6}$$

In this paper, in the particular case when $F(t,y,y(t-\tau(t))) \equiv 0$ in system (1.2), to investigate the exponential stability of solutions of the following linear differential system of neutral type with periodic coefficients

$$\frac{d}{dt}(y(t) + Dy(t-\tau(t))) = A(t)y(t) + B(t)y(t-\tau(t)), \tag{1.7}$$

we introduce the following Lyapunov-Krasovskii functional

$$\begin{aligned} W(\vartheta) =& \langle H(0)(\vartheta(0) + D\vartheta(-\tau(0))), (\vartheta(0) + D\vartheta(-\tau(0)))\rangle \\ &+ \int_{-\tau(0)}^{0} \langle K(-s)\vartheta(s), \vartheta(s)\rangle ds, \vartheta(s) \in C[-\tau_2, 0], \end{aligned} \tag{1.8}$$

where the matrix valued functions $H \in C(\bar{\Re}_+) \cap C^1([lT,(l+1)T]), l = 0,1,..., K(s) \in C^1([0,\tau_2])$ are such that

$$H(t) = H^*(t), H(t) = H(t+T) > 0, t \geq 0, \tag{1.9}$$

$$K(s) = K^*(s) > 0, \frac{d}{ds}K(s) < 0, s \in [0, \tau_2]. \tag{1.10}$$

Here and later, the matrix inequality $Q > 0$ (or $Q < 0$) means that the Hermitian matrix Q is positive (or negative) definite. In case of the T- periodic matrix $A(t)$ such that the zero solution to the system of ordinary differential equations $\frac{dx}{dt} = A(t)x$ is asymptotically stable, it is not difficult to construct the functional (1.6) by the usage of the asymptotic stability criterion of Demidenko and Matveeva [2].

Indeed, in accord with this criterion, the following boundary value problem (BVP) for the Lyapunov differential equation

$$\begin{cases} \frac{d}{dt}H + HA(t) + A^*(t)H = -Q(t), t \in [0,T], \\ H(0) = H(T) > 0, \end{cases} \tag{1.11}$$

is uniquely solvable for every continuous matrix $Q(t)$, moreover, if $Q(t) = Q^*(t) > 0$, then $H(t) = H^*(t) > 0$ on $[0,T]$. Extend T- periodically the matrix $H(t)$ to the whole half-axis $\{t > 0\}$, and use it in (1.6), since (1.9) is fulfilled. In view of the results of [3, 4], it follows that solutions of equation (1.5) are asymptotically stable if there exists a matrix $K(s)$ satisfying (1.10) and such that the matrix

$$\begin{pmatrix} Q(t) - K(0) & -H(t)B(t) \\ -B^*(t)H(t) & K(\tau) \end{pmatrix}, t \in [0,T],$$

is positive definite. Note that this condition is equivalent to the matrix inequality

$$K(0) + H(t)B(t)(K(\tau))^{-1}B^*(t)H(t) < Q(t), t \in [0,T].$$

In fact, for a wide class of T- periodic matrices $B(t)$, the matrix $K(s)$ can be found in the form

$$K(s) = \alpha(s)K_0, K_0 = K_0^* > 0,$$

where

$$\alpha(s) > 0, \alpha'(s) < 0, s \in [0,\tau].$$

The usage of the functional (1.6) allows us to obtain estimates of exponential decay of solutions of system (1.5).

2 Main results

We introduce the main results of this paper.

Theorem 1. Let $H(t)$ and $K(s)$ be matrices satisfying the relations (1.9) and (1.10) such that the compound matrix

$$C(t) = -\begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix} \tag{2.1}$$

is positive definite for all $t \in [0,T]$, where

$$C_{11}(t) = \frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) + K(0),$$

$$C_{12}(t) = \frac{d}{dt}H(t)D + H(t)B(t) + A^*(t)H(t)D,$$

$$C_{21}(t) = D^*\frac{d}{dt}H(t) + D^*H(t)A(t) + B^*(t)H(t)$$

and

$$C_{22}(t) = D^*\frac{d}{dt}H(t)D + D^*H(t)B(t) + B^*(t)H(t)D - (1-\alpha)K(\tau_2).$$

Then, the zero solution of system (1.7) is exponentially stable.

Proof. The proof of this theorem can be done easily. Therefore, we omit the details of the proof.

We now consider the initial value problem (IVP)

$$\begin{aligned} \frac{d}{dt}(y(t)+Dy(t-\tau(t))) =& A(t)y(t)+B(t)y(t-\tau(t)) \\ &+F(t,y(t),y(t-\tau(t))), \\ y(t)=\vartheta(t), t \in& [-\tau_2,0], y(+0)=\vartheta(0), \end{aligned} \tag{2.2}$$

where $\vartheta(t) \in C^1([-\tau_2,0])$ is a given vector-valued function.

If the matrix $H(t)$ satisfies the conditions of Theorem 1, then we have

$$\frac{d}{dt}H(t)+H(t)A(t)+A^*(t)H(t) < -K(0),$$

that is, $H(t)$ is a solution of BVP (1.11) with $Q(t) = Q^*(t) > 0$. In this case, $H(t) > 0$, $t \in [0,T]$ (see Demidenko and Matveeva [2]). Extend T- periodically this matrix to the whole half-axis $\{t \geq 0\}$, keeping the same notation. Using the matrices $H(t)$ and $K(s)$ indicated in Theorem 1, we now consider the Lyapunov-Krasovskii functional (1.8). Let

$$y_t : \theta \to y(t+\theta), \theta \in [-\tau_2, 0].$$

Then, we have

$$\begin{aligned} V(y_t) =& \langle H(0)(y_t(0)+Dy_t(-\tau(0))),(y_t(0)+Dy_t(-\tau(0)))\rangle \\ &+\int_{-\tau(0)}^{0} \langle K(-\theta)y_t(\theta),y_t(\theta)\rangle d\theta \\ =& \langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle \\ &+\int_{t-\tau(t)}^{t} \langle K(t-s)y(s),y(s)\rangle ds. \end{aligned} \tag{2.3}$$

We assume that the conditions of Theorem 1 hold. Using the matrices $H(t)$ and $K(s)$, we introduce the following matrix function

$$S(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} S_{11}(t) =& -\frac{d}{dt}H(t)-H(t)A(t)-A^*(t)H(t)-K(0), \\ S_{12}(t) =& H(t)A(t)D+K(0)D-H(t)B(t), \\ S_{21}(t) =& D^*A^*(t)H(t)+D^*K(0)-B^*(t)H(t), \\ S_{22}(t) =& (1-\alpha)K(\tau_2)-D^*K(0)D, \end{aligned}$$

$$q(t) = (q_1+\sqrt{q_1^2+(q_1\|D\|+q_2)^2})\|H(t)\|, \tag{2.5}$$

$$\begin{aligned} R(t) =& -\frac{d}{dt}H(t)-H(t)A(t)-A^*(t)H(t)-K(0)-qI \\ &-(H(t)A(t)D+K(0)D-H(t)B(t))[(1-\alpha)K(\tau_2)-D^*K(0)D-qI]^{-1} \\ &\times (H(t)A(t)D+K(0)D-H(t)B(t))^*, \end{aligned} \tag{2.6}$$

where I is the unit matrix.

It is not hard to verify that the matrix $C(t)$ in (2.1) is positive definite for all $t \in [0,T]$ if and only if the matrix $S(t)$ is positive definite for all $t \in [0,T]$. In addition, note that $R(t)$ is positive definite for $t \in [0,T]$ if the matrix $S(t)-q(t)I$ is positive definite for $t \in [0,T]$.

Lemma 1. (Skvortsova [20]). If the matrix $S(t)$ in (2.4) is positive definite, then the spectrum of the matrix D is contained in the disc $\{\lambda \in C : |\lambda| < \sqrt{1-\alpha}\}$.

It should be noted that if the assumption of Lemma 1 holds, then there exists a solution $\tilde{H}$ to the discrete Lyapunov matrix equation

$$\tilde{H} - \frac{1}{1-\alpha} D^* \tilde{H} D = I, \tag{2.7}$$

where I is the identity matrix. Furthermore, we have the counterpart of the Krein inequality

$$\|D^j\| \le \sqrt{\|\tilde{H}\| \|\tilde{H}^{-1}\|} \rho^j, \ \ j \in N, \tag{2.8}$$

where

$$\rho = \sqrt{(1-\alpha)(1 - 1/\|\tilde{H}\|)} \tag{2.9}$$

(see Godunov [12]).

The above estimate will be used in the proof of the main result.

Theorem 2. Let the conditions of Theorem 1 hold. Suppose that the parameters q_1, q_2 are such that the matrix $S(t) - q(t)I$ is positive definite for $t \in [0, T]$. Let $k > 0$ be the maximal number such that

$$\frac{d}{ds} K(s) + kK(s) \le 0, \ \ s \in [0, \tau_2]. \tag{2.10}$$

Let $r(t) > 0$ be the minimal eigenvalue of the matrix $R(t)$. Then, each solution of the IVP (2.2) satisfies

$$\|y(t) + Dy(t-\tau(t))\| \le \sqrt{\frac{W(\vartheta)}{h(t)}} exp(-\int_0^t \frac{\gamma(s)}{2\|H(s)\|} ds), t > 0, \tag{2.11}$$

where $W(\vartheta)$ is defined by (1.8), $h(t) > 0$ is the minimal eigenvalue of the matrix $H(t)$, and

$$\gamma(t) = \min\{r(t), k\|H(t)\|\} > 0. \tag{2.12}$$

Proof. We follow the strategy in [3]. Let $y(t)$ be a solution of the IVP (2.2). Differentiating of the functional $V(y_t)$ along solutions of the IVP (2.2), we find

$$\begin{aligned}
\frac{d}{dt} V(y_t) =& \langle \frac{d}{dt} H(t)(y(t) + Dy(t-\tau(t))), (y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle H(t) \frac{d}{dt}(y(t) + Dy(t-\tau(t))), (y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), \frac{d}{dt}(y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle K(0)y(t), y(t) \rangle - (1-\tau'(t)) \langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t)) \rangle \\
&+ \int_{t-\tau(t)}^t \langle \frac{d}{dt} K(t-s)y(s), y(s) \rangle ds \\
=& \langle \frac{d}{dt} H(t)(y(t) + Dy(t-\tau(t))), (y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle H(t)(A(t)y(t) + B(t)y(t-\tau(t))), (y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), (A(t)y(t) + B(t)y(t-\tau(t))) \rangle \\
&+ \langle H(t)F(t, y(t), y(t-\tau(t))), (y(t) + Dy(t-\tau(t))) \rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), F(t, y(t), y(t-\tau(t))) \rangle \\
&+ \langle K(0)y(t), y(t) \rangle - (1-\tau'(t)) \langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t)) \rangle \\
&+ \int_{t-\tau(t)}^t \langle \frac{d}{dt} K(t-s)y(s), y(s) \rangle ds
\end{aligned}$$

$$
\begin{aligned}
=&\langle \frac{d}{dt}H(t)y(t), y(t)\rangle + \langle \frac{d}{dt}H(t)y(t), Dy(t-\tau(t)))\rangle \\
&+ \langle \frac{d}{dt}H(t)Dy(t-\tau(t), y(t)\rangle + \langle \frac{d}{dt}H(t)Dy(t-\tau(t)), Dy(t-\tau(t))\rangle \\
&+ \langle H(t)A(t)y(t), y(t)\rangle, \langle H(t)A(t)y(t), Dy(t-\tau(t))\rangle \\
&+ \langle H(t)B(t)y(t-\tau(t)), y(t)\rangle + \langle H(t)B(t)y(t-\tau(t)), Dy(t-\tau(t))\rangle \\
&+ \langle H(t)y(t), A(t)y(t)\rangle + \langle H(t)y(t), B(t)y(t-\tau(t)))\rangle \\
&+ \langle H(t)Dy(t-\tau(t)), A(t)y(t)\rangle + \langle H(t)Dy(t-\tau(t)), B(t)y(t-\tau(t))\rangle \\
&+ \langle H(t)F(t, y(t), y(t-\tau(t))), (y(t) + Dy(t-\tau(t)))\rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), F(t, y(t), y(t-\tau(t)))\rangle \\
&+ \langle K(0)y(t), y(t)\rangle - (1-\tau'(t))\langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle \\
&+ \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds. \qquad (2.13)
\end{aligned}
$$

Consider the expression

$$(1-\tau'(t))\langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle.$$

By (1.4) and the condition $K(s) = K^*(s) > 0$, $s \in [0, \tau_2]$, we obtain

$$
\begin{aligned}
&(1-\tau'(t))\langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle \\
&\qquad \geq (1-\alpha)\langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle.
\end{aligned}
$$

Using the conditions $\frac{d}{ds}K(s) < 0$ and $\tau(t) \leq \tau_2$, we have $K(\tau(t)) \geq K(\tau_2)$. Hence,

$$
\begin{aligned}
&(1-\tau'(t))\langle K(\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle \\
&\geq (1-\alpha)\langle K(\tau_2)y(t-\tau(t)), y(t-\tau(t))\rangle.
\end{aligned}
$$

Therefore, it follows from (2.13) that

$$
\begin{aligned}
\frac{d}{dt}V(y_t) \leq&\langle \frac{d}{dt}H(t)y(t), y(t)\rangle + \langle D^*\frac{d}{dt}H(t)y(t), y(t-\tau(t))\rangle \\
&+ \langle \frac{d}{dt}H(t)Dy(t-\tau), y(t)\rangle + \langle D^*\frac{d}{dt}H(t)Dy(t-\tau(t)), y(t-\tau(t))\rangle \\
&+ \langle H(t)A(t)y(t), y(t)\rangle + \langle D^*H(t)A(t)y(t), y(t-\tau(t))\rangle \\
&+ \langle H(t)B(t)y(t-\tau(t)), y(t)\rangle + \langle D^*H(t)B(t)y(t-\tau(t)), y(t-\tau(t))\rangle \\
&+ \langle A^*(t)H(t)y(t), y(t)\rangle + \langle B^*(t)H(t)y(t), y(t-\tau(t))\rangle \\
&+ \langle A^*(t)H(t)Dy(t-\tau(t)), y(t)\rangle + \langle B^*(t)H(t)Dy(t-\tau(t)), y(t-\tau(t))\rangle \\
&+ \langle H(t)F(t, y(t), y(t-\tau(t))), (y(t) + Dy(t-\tau(t)))\rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), F(t, y(t), y(t-\tau(t)))\rangle \\
&+ \langle K(0)y(t), y(t)\rangle - (1-\alpha)\langle K(\tau_2)y(t-\tau(t)), y(t-\tau(t))\rangle \\
&+ \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds.
\end{aligned}
$$

Using the matrix $C(t)$ defined by (2.1), we obtain

$$
\begin{aligned}
\frac{d}{dv}V(y_t) \leq& -\left\langle C(t)\begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix}\right\rangle \\
&+ \langle H(t)F(t, y(t), y(t-\tau(t))), (y(t) + Dy(t-\tau(t)))\rangle \\
&+ \langle H(t)(y(t) + Dy(t-\tau(t))), F(t, y(t), y(t-\tau(t)))\rangle \\
&+ \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds. \qquad (2.14)
\end{aligned}
$$

Consider the first summand in the right-hand side of (2.14). Since

$$\begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix} = \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix},$$

then

$$\begin{aligned} &\left\langle C(t) \begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix} \right\rangle \\ &\equiv \left\langle S(t) \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix} \right\rangle, \end{aligned}$$

where

$$S(t) = \begin{pmatrix} I & 0 \\ -D^* & I \end{pmatrix} C(t) \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix},$$

which is defined in (2.4).

We now consider the second and the third summands in the right-hand side of (2.14). In view of (1.3), we have

$$\begin{aligned} &\langle H(t)F(t,y(t),y(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle \\ &\quad + \langle H(t)(y(t)+Dy(t-\tau(t))),F(t,y(t),y(t-\tau(t)))\rangle \\ &\le 2\|H(t)\|(q_1\|y(t)\|+q_2\|y(t-\tau(t))\|)\|y(t)+Dy(t-\tau(t))\| \\ &\le 2q_1\|H(t)\|\|y(t)+Dy(t-\tau(t))\|^2 \\ &\quad + 2(q_1\|D\|+q_2)\|H(t)\|\|y(t-\tau(t)\|\|y(t)+Dy(t-\tau(t))\| \\ &\le q(t)(\|y(t)+Dy(t-\tau(t))\|^2+\|y(t-\tau(t))\|^2), \end{aligned}$$

where $q(t)$ is given in (2.5). Hence,

$$\begin{aligned} &-\left\langle C(t) \begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \end{pmatrix} \right\rangle \\ &+\langle H(t)F(t,y(t),y(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle \\ &+\langle H(t)(y(t)+Dy(t-\tau(t))),F(t,y(t),y(t-\tau(t)))\rangle \\ &\le -\left\langle (S(t)-q(t)I) \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix} \right\rangle. \end{aligned} \tag{2.15}$$

By the conditions of Theorem 2, the matrix $S(t)-q(t)I$ is positive definite for $t \in [0,T]$. Using the representation

$$\begin{aligned} S(t)-q(t)I = &\begin{pmatrix} I & S_{12}(t)(S_{22}-qI)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_{11}(t)-qI-S_{12}(t)(S_{22}-qI)^{-1}S_{12}^*(t) & 0 \\ 0 & S_{22}-qI \end{pmatrix} \\ &\times \begin{pmatrix} I & 0 \\ (S_{22}-qI)^{-1}S_{12}^*(t) & I \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} &\left\langle (S(t)-q(t)I) \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix} \right\rangle \\ &\ge \langle [S_{11}(t)-qI-S_{12}(t)(S_{22}-q(t)I)^{-1}S_{12}^*(t)](y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle. \end{aligned}$$

Since the matrix $S(t)-q(t)I$ is positive definite for $t\in[0,T]$, then the matrix

$$R(t)=S_{11}(t)-q(t)I-S_{12}(t)(S_{22}-q(t)I)^{-1}S_{12}^{*}(t)$$

is positive definite for $t\in[0,T]$. Taking into account (2.4), it is seen that the matrix $R(t)$ has the form (2.6). Consequently, from (2.15), we obtain

$$\begin{aligned}
&-\left\langle C(t)\begin{pmatrix} y(t)\\ y(t-\tau(t))\end{pmatrix},\begin{pmatrix} y(t)\\ y(t-\tau(t))\end{pmatrix}\right\rangle\\
&+\langle H(t)F(t,y(t),y(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle\\
&+\langle H(t)(y(t)+Dy(t-\tau(t))),F(t,y(t),y(t-\tau(t)))\rangle\\
&\quad\le-\langle R(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle\\
&\quad\le-r(t)\|y(t)+Dy(t-\tau(t))\|^{2}, \qquad (2.16)
\end{aligned}$$

where $r(t)>0$ is the minimal eigenvalue of $R(t)$. Using the matrix $H(t)$, we have

$$\|y(t)+Dy(t-\tau(t))\|^{2}\ge\frac{1}{\|H(t)\|}\langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle.$$

In view of (2.10), (2.14) and (2.16), we obtain

$$\begin{aligned}
\frac{d}{dt}V(y_t)\le&-\frac{r(t)}{\|H(t)\|}\langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle\\
&+\int_{t-\tau(t)}^{t}\langle\frac{d}{dt}K(t-s)y(s),y(s)\rangle ds\\
\le&-\frac{r(t)}{\|H(t)\|}\langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle\\
&-k\int_{t-\tau(t)}^{t}\langle K(t-s)y(s),y(s)\rangle ds.
\end{aligned}$$

Taking into account the definition of the functional W given by (1.8), we obtain

$$\frac{d}{dt}V(y_t)\le-\frac{\gamma(t)}{\|H(t)\|}V(y_t)\le W(\vartheta)\exp(-\int_0^t\frac{\gamma(s)}{\|H(s)\|}ds),$$

where $\gamma(t)=\min\{r(t),k\|H(t)\|\}>0$.

It is also clear that

$$\|y(t)+Dy(t-\tau(t))\|^{2}\le\frac{1}{h(t)}\langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle,$$

where $h(t)$ is the minimal eigenvalue of $H(t)$. At the end, in view of (1.8), we can conclude that

$$\|y(t)+Dy(t-\tau(t))\|\le\sqrt{\frac{V(y_t)}{h(t)}}\le\sqrt{\frac{W(\vartheta)}{h(t)}}\exp(-\int_0^t\frac{\gamma(s)}{2\|H(s)\|}ds).$$

The last inequality ends the proof of the theorem.

In the next theorem, Theorem 3, based on the inequality (2.11), we prove an estimate for the solutions of the IVP (2.2). The estimate obtained will be used for proving our main results.

Let

$$\begin{aligned}
&\mu=\max_{t\in[0,T]}\sqrt{\frac{W(\vartheta)}{h(t)}},\qquad \Phi=\max_{t\in[-\tau_2,0]}\|\vartheta(t)\|,\\
&\beta(t)=\frac{\gamma(t)}{2\|H(t)\|},\quad\beta^{+}=\max_{t\in[0,T]}\beta(t),\quad\beta^{-}=\min_{t\in[0,T]}\beta(t). \qquad (2.17)
\end{aligned}$$

Theorem 3. If the conditions of Theorem 2, then, on each segment $t \in [k\tau_1, (k+1)\tau_1), k = 0, 1, ...$, the solution $y(t)$ of IVP (2.2) satisfies

$$\|y(t)\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu \sum_{j=0}^{k} (\rho e^{\beta^+ \tau_2})^j e^{-\int_0^t \beta(s)ds} + \rho^{k+1}\Phi), \tag{2.18}$$

where $\tilde{H}, \rho, \mu, \Phi, \beta^+$ and $\beta(t)$ are defined in (2.7), (2.9) and (2.17), respectively.
Proof. Clearly, if we take into account the estimates (2.8), (2.9) and (2.17), then, by (2.11) for $t \in [0, \tau_1)$, it follows that

$$\begin{aligned}
||y(t)|| \leq& \mu e^{-\int_0^t \beta(s)ds} + \|Dy(t-\tau(t))\| \\
\leq& \mu e^{-\int_0^t \beta(s)ds} + \|D\|\Phi \\
\leq& \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu e^{-\int_0^t \beta(s)ds} + \rho\Phi),
\end{aligned}$$

which implies (2.18) for $k = 0$.

Let $t \in [k\tau_1, (k+1)\tau_1)$, $k = 1, 2, ...$ It is not difficulty to show the following sequence of the inequalities

$$\begin{aligned}
||y(t)|| \leq& \mu e^{-\int_0^t \beta(s)ds} + \|Dy(t-\tau(t))\| \\
\leq& \mu e^{-\int_0^t \beta(s)ds} + \|Dy(t-\tau(t)) + D^2 y(t-2\tau(t))\| \\
&+ \|D^2 y(t-2\tau(t)) + D^3 y(t-3\tau(t))\| + ... \\
&+ \|D^k y(t-k\tau(t)) + D^{k+1} y(t-(k+1)\tau(t))\| \\
&+ \|D^{k+1} y(t-(k+1)\tau(t))\| \\
\leq& \mu e^{-\int_0^t \beta(s)ds} + \|D\|\|y(t-\tau(t)) + Dy(t-2\tau(t))\| \\
&+ \|D^2\|\|y(t-2\tau(t)) + Dy(t-3\tau(t))\| + ... \\
&+ \|D^k\|\|y(t-k\tau(t)) + Dy(t-(k+1)\tau(t))\| \\
&+ \|D^{k+1}\|\|y(t-(k+1)\tau(t))\|.
\end{aligned}$$

In view of (2.11), we can derive the estimate

$$\begin{aligned}
||y(t)|| \leq& \mu e^{-\int_0^t \beta(s)ds} + \mu||D||e^{-\int_0^{t-\tau(t)} \beta(s)ds} + \mu||D^2||e^{-\int_0^{t-2\tau(t)} \beta(s)ds} \\
&+ ... + \mu\|D^k\|e^{-\int_0^{t-k\tau(t)} \beta(s)ds} + \|D^{k+1}\|\Phi \\
=& \mu \sum_{j=0}^{k} \|D^j\| e^{-\int_0^{t-j\tau(t)} \beta(s)ds} + \|D^{k+1}\|\Phi \\
=& \mu \sum_{j=0}^{k} \|D^j\| e^{\int_{t-j\tau(t)}^{t} \beta(s)ds} e^{-\int_0^t \beta(s)ds} + \|D^{k+1}\|\Phi.
\end{aligned}$$

By the estimates (1.4), (2.8) and (2.17), we can obtain

$$\begin{aligned}
\|y(t)\| \leq& \mu \sum_{j=0}^{k} \|D^j\| e^{\beta^+ j\tau_2} e^{-\int_0^t \beta(s)ds} + \|D^{k+1}\|\Phi \\
\leq& \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu \sum_{j=0}^{k} \rho^j e^{\beta^+ j\tau_2} e^{-\int_0^t \beta(s)ds} + \rho^{k+1}\Phi),
\end{aligned}$$

which is required.

Next, we obtain some estimates for solutions of the IVP (2.2) on the whole half-line $\{t > 0\}$. Like in [5], we distinguish three cases allowing us to obtain more precise estimates. Since the spectrum of the matrix D belongs to the unit disk $\{\lambda \in C : |\lambda| < \sqrt{1-\alpha}\}$, then it follows that $\|D^j\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\rho^j \to 0$ as $j \to \infty$. Let $l > 0$ be the minimal integer such that $\|D^l\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\rho^l < 1$. In the following theorems, Theorems 4-6, we establish estimates when

$$\rho^1 < e^{-l\beta^+\tau_2}, \rho^l = e^{-l\beta^+\tau_2}, e^{-l\beta^+\tau_2} < \rho^l < e^{-(l\beta^+\tau_2 - l\beta^-\tau_1)},$$

where ρ and β^+ are defined by (2.9) and (2.17), respectively.

Theorem 4. We assume that

$$\rho^l < e^{-l\beta^+\tau_2}. \tag{2.19}$$

Then, the solution of the IVP (2.2) satisfies

$$\begin{aligned} y(t) \leq & \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(1-\rho^l e^{l\beta^+\tau_2})^{-1}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j \\ & + \max\{\rho e^{\beta^+\tau_1}, ..., \rho^l e^{l\beta^+\tau_1}\}\Phi]e^{-\int_0^t \beta(s)ds} \end{aligned} \tag{2.20}$$

for $t > 0$, where $\tilde{H}, \rho, \mu, \Phi, \beta^+$ and $\beta(t)$ are defined in (2.7), (2.9) and (2.17), respectively.

Proof. Using the inequality (2.18), on each segment $t \in [k\tau_1, (k+1)\tau_1)$, $k = 0, 1, ...,$ one can write the inequality

$$\begin{aligned} ||y(t)|| \leq & \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j e^{-\int_0^t \beta(s)ds} + \rho^{k+1}\Phi) \\ = & \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j + \rho^{k+1}e^{\int_0^t \beta(s)ds}\Phi]e^{-\int_0^t \beta(s)ds} \\ \leq & \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j + \rho^{k+1}e^{(k+1)\beta^+\tau_1}\Phi]e^{-\int_0^t \beta(s)ds}. \end{aligned}$$

In view the condition on ρ^l, we obtain the estimate on the whole half-line $\{t > 0\}$,

$$y(t) \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu\sum_{j=0}^{\infty}(\rho e^{\beta^+\tau_2})^j + \max\{\rho e^{\beta^+\tau_1}, ..., \rho^l e^{l\beta^+\tau_1}\}\Phi]e^{-\int_0^t \beta(s)ds}. \tag{2.21}$$

Consider the series $\sum_{j=0}^{\infty}(\rho e^{\beta^+\tau_2})^j$. Then, it is clear that

$$\begin{aligned} \sum_{j=0}^{\infty}(\rho e^{\beta^+\tau_2})^j &= \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=l}^{2l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=2l}^{3l-1}(\rho e^{\beta^+\tau_2})^j + ... \\ &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \rho^l e^{l\beta^+\tau_2}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + (\rho^l e^{l\beta^+\tau_2})^2\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + ... \\ &= (1 + \rho^l e^{l\beta^+\tau_2} + (\rho^l e^{l\beta^+\tau_2})^2 + ...)\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j. \end{aligned}$$

Since $\rho^l e^{l\beta^+\tau_2} < 1$, then, by (2.19), we have

$$\sum_{j=0}^{\infty}(\rho e^{\beta^+\tau_2})^j \leq (1-\rho^l e^{l\beta^+\tau_2})^{-1}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j.$$

Using this inequality, from (2.21) we derive the required estimate (2.20).

Theorem 5. We assume that

$$\rho^l = e^{-l\beta^+\tau_2}.$$

Then, the solution to the initial value problem (2.2) satisfies

$$y(t) \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(1+\frac{t}{l\tau_1})\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \max\{e^{-\beta^+(\tau_2-\tau_1)},...,e^{-l\beta^+(\tau_2-\tau_1)}\}\Phi]e^{-\int_0^t \beta(s)ds} \tag{2.22}$$

for $t > 0$, where $\tilde{H}, \rho, \mu, \Phi, \beta^+$ and $\beta(t)$ are defined in (2.7), (2.9) and (2.17), respectively.
Proof. By Theorem 3, the solution to (2.2) satisfies (2.18) on each segment $t \in [k\tau_1, (k+1)\tau_1)$, $k = 0, 1, ...$ Hence

$$y(t) \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j + \rho^{k+1}e^{(k+1)\beta^+\tau_1}\Phi]e^{-\int_0^t \beta(s)ds}. \tag{2.23}$$

Taking into account condition ρ^l in Theorem 5, we obtain

$$y(t) \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j + \max\{e^{-\beta^+(\tau_2-\tau_1)},...,e^{-l\beta^+(\tau_2-\tau_1)}\}\Phi]e^{-\int_0^t \beta(s)ds}. \tag{2.24}$$

If $k \leq l-1$, then (2.23) follows from (2.24) for $t \in [0, l\tau_1)$.

Let $l \leq k \leq 2l-1$, that is, $1 \leq \frac{t}{l\tau_1} < 2$. Consider the sum $\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j$. Clearly, we see

$$\begin{aligned}\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &= \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=l}^{k}(\rho e^{\beta^+\tau_2})^j\\ &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \rho^l e^{l\beta^+\tau_2}\sum_{j=0}^{k-l}(\rho e^{\beta^+\tau_2})^j.\end{aligned}$$

Since $\rho^l = e^{-l\beta^+\tau_2}$, we have

$$\begin{aligned}\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=0}^{k-l}(\rho e^{\beta^+\tau_2})^j\\ &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \frac{t}{l\tau_1}\sum_{j=0}^{k-l}(\rho e^{\beta^+\tau_2})^j.\end{aligned}$$

By this inequality, (2.23) follows from (2.24) for $t \in [l\tau_1, 2l\tau_1)$.

Let $ml \leq k \leq (m+1)l-1, m = 2, 3, ...$, i.e., $m \leq \frac{t}{l\tau_1} < m+1$. Consider the sum $\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j$. It is not difficult to see that

$$\begin{aligned}\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &= \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=l}^{2l-1}(\rho e^{\beta^+\tau_2})^j + ... + \sum_{j=ml}^{k}(\rho e^{\beta^+\tau_2})^j\\ &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \rho^l e^{l\beta^+\tau_2}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + ... + \rho^{ml}e^{ml\beta^+\tau_2}\sum_{j=0}^{k-ml}(\rho e^{\beta^+\tau_2})^j.\end{aligned}$$

Since $\rho^l = e^{-l\beta^+\tau_2}$, then we have

$$\begin{aligned}\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &\leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + ... + \sum_{j=0}^{k-ml}(\rho e^{\beta^+\tau_2})^j\\ &\leq (1+m)\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j.\end{aligned}$$

In view of the above estimates, it follows that

$$\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j \leq (1+\frac{t}{l\tau_1})\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j$$

so that (2.23) follows from (2.24) for $t\in[ml\tau_1,(m+1)l\tau_1)$. Owing to arbitrariness of m, (2.23) is valid for all $t>0$.

Theorem 6. We suppose that

$$e^{-l\beta^+\tau_2} < \rho < e^{-(l\beta^+\tau_2 - l\beta^-\tau_1)}. \tag{2.25}$$

Then, the solution to the IVP (2.2) satisfies

$$\begin{aligned} y(t) \leq & \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(e^{l\beta^+\tau_2-l\beta^-\tau_1})^{\frac{t}{l\tau_1}}[1-(\rho^l e^{l\beta^+\tau_2})^{-1}]^{-1} \\ & \times\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + (\rho^l)^{\frac{1}{l}-1}\max\{1,\rho,...,\rho^{l-1}\}\Phi]\exp(\frac{t}{l\tau_1}\ln\rho^l) \end{aligned} \tag{2.26}$$

for $t>0$, where $\tilde{H},\rho,\mu,\Phi,\beta^+$ and $\beta(t)$ are defined in (2.7), (2.9) and (2.17), respectively.

Proof. In view of Theorem 3, a solution of the IVP (2.2) satisfies (2.18) on each segment $t\in[k\tau_1,(k+1)\tau_1)$, $k=0,1,....$

At first we consider the first summand in the right-hand side of (2.18). For $k\leq l-1$, we have

$$\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j \leq \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j.$$

Let $ml\leq k\leq(m+1)l-1$, $m=1,2,3,....$ Then,

$$\begin{aligned} \sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &= \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \sum_{j=l}^{2l-1}(\rho e^{\beta^+\tau_2})^j + ... + \sum_{j=ml}^{k}(\rho e^{\beta^+\tau_2})^j \\ &= \sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + \rho^l e^{l\beta^+\tau_2}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j + ... + \rho^{ml}e^{ml\beta^+\tau_2}\sum_{j=0}^{k-ml}(\rho e^{\beta^+\tau_2})^j \\ &\leq (1+\rho^l e^{l\beta^+\tau_2}+...+\rho^{ml}e^{ml\beta^+\tau_2})\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j &\leq \rho^{ml}e^{ml\beta^+\tau_2}[1+(\rho^l e^{l\beta^+\tau_2})^{-1}+...+(\rho^l e^{l\beta^+\tau_2})^{-m}]\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j \\ &\leq \rho^{ml}e^{ml\beta^+\tau_2}[1+(\rho^l e^{l\beta^+\tau_2})^{-1}+...+(\rho^l e^{l\beta^+\tau_2})^{-m}+...]\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j. \end{aligned}$$

Since $\rho^l e^{l\beta^+\tau_2}>1$, then, owing to (2.25), it follows that

$$\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j \leq \rho^{ml}e^{ml\beta^+\tau_2}[1-(\rho^l e^{l\beta^+\tau_2})^{-1}]^{-1}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j.$$

Taking into account the inequality $ml\tau_1\leq t\leq(m+1)l\tau_1$, we can derive the estimate for the first summand in (2.18) for every k such below

$$\begin{aligned}
&\sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j e^{-\int_0^t \beta(s)ds})\\
&\quad\le\sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}(\mu\sum_{j=0}^{k}(\rho e^{\beta^+\tau_2})^j e^{-\beta^- t})\\
&\quad\le\sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\mu\rho^{ml}e^{ml\beta^+\tau_2}e^{-ml\beta^-\tau_1}[1-(\rho^l e^{l\beta^+\tau_2})^{-1}]^{-1}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j\\
&\quad\le\sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\mu(\rho^l e^{l\beta^+\tau_2-l\beta^-\tau_1})^{t/l\tau_1}[1-(\rho^l e^{l\beta^+\tau_2})^{-1}]^{-1}\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j\\
&\quad=\sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\mu(e^{l\beta^+\tau_2-l\beta^-\tau_1})^{t/l\tau_1}[1-(\rho^l e^{l\beta^+\tau_2})^{-1}]^{-1}\\
&\qquad\times\sum_{j=0}^{l-1}(\rho e^{\beta^+\tau_2})^j\exp(\frac{t}{l\tau_1}\ln\rho^l). \qquad (2.27)
\end{aligned}$$

We now consider the second summand in the right-hand side of (2.18). Obviously, for $0\le k\le l-2$, we have

$$\rho^{k+1}\le\max\{\rho,...,\rho^{l-1}\}.$$

Let $ml-1\le k\le(m+1)l-2, m=1,2,....$ Hence,

$$\rho^{k+1}=\rho^{ml}\rho^{k+1-ml}\le\rho^{ml}\max\{1,\rho,...,\rho^{l-1}\}.$$

Since $\rho^l<1$ and $t<((m+1)l-1)\tau_1$, then

$$\rho^{ml}\le(\rho^l)^{\frac{t-(l-1)\tau_1}{l\tau_1}}=(\rho^l)^{\frac{1}{l}-1}\exp(\frac{t}{l\tau_1}\ln\rho^l).$$

Owing to arbitrariness of m, we infer that

$$\rho^{k+1}\le(\rho^l)^{\frac{1}{l}-1}\max\{1,\rho,...,\rho^{l-1}\}\exp(\frac{t}{l\tau_1}\ln\rho^l)$$

for every k. Taking into account the estimate (2.27) for the first summand in the right-hand side of (2.18), we derive (2.26). The proof is complete.

Example 1. For the case $n=2$ as a special case of equation (1.2), we consider the following a time-varying delay system of nonlinear neutral differential equations with periodic coefficients

$$\begin{aligned}
&\frac{d}{dt}\left(\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}+\begin{bmatrix}0.11 & 0.12\\ -0.04 & 0.16\end{bmatrix}\times\begin{bmatrix}y_1(t-\tau(t))\\ y_2(t-\tau(t))\end{bmatrix}\right)\\
&=\begin{bmatrix}-8+0.1\cos t & 1-0.5\cos t\\ 1+0.2\cos t & -14-0.2\cos t\end{bmatrix}\times\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}\\
&\quad+\begin{bmatrix}0.1\sin t & 0.2\cos t\\ 0 & -1\end{bmatrix}\times\begin{bmatrix}y_1(t-\tau(t))\\ y_2(t-\tau(t))\end{bmatrix}\\
&\quad+\begin{bmatrix}q_1y_1(t)e^{-y_1^2(t)}+q_2y_1(t-\tau(t))e^{-y_1^2(t-\tau(t))}\\ q_1y_2(t)e^{-y_2^2(t)}+q_2y_2(t-\tau(t))e^{-y_2^2(t-\tau(t))}\end{bmatrix}, t>0. \qquad (2.28)
\end{aligned}$$

When we compare equation (2.28) with equation (1.2), it can be seen that

$$D=\begin{bmatrix}0.11 & 0.12\\ -0.04 & 0.16\end{bmatrix},$$

$$A(t) = \begin{bmatrix} -8+0.1\cos t & 1-0.5\cos t \\ 1+0.2\cos t & -14-0.2\cos t \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0.1\sin t & 0.2\cos t \\ 0 & -1 \end{bmatrix},$$

$$F(t,y,y(t-\tau(t))) = \begin{bmatrix} q_1 y_1(t)e^{-y_1^2(t)} + q_2 y_1(t-\tau(t))e^{-y_1^2(t-\tau(t))} \\ q_1 y_2(t)e^{-y_2^2(t)} + q_2 y_2(t-\tau(t))e^{-y_2^2(t-\tau(t))} \end{bmatrix},$$

and

$$\tau_1 = \frac{1}{20} \le \tau(t) = \frac{1+\sin^2 t}{20} \le \frac{1}{10} = \tau_2,$$

where $F(t,u,v)$ is a real-valued vector function satisfying the Lipschitz condition with respect to u and the inequality

$$\|F(t,u,v)\| \le q_1\|u\| + q_2\|v\|$$

for some constants $q_1, q_2 \ge 0$.

In addition, it can be followed that

$$H(t) = \begin{bmatrix} 5-2.4\sin t & 1-1.3\sin t \\ 1-1.3\sin t & 6+2.4\sin t \end{bmatrix}$$

and

$$K(s) = e^{-ks}K_0, \;\; k = 0.05, \;\; K(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let $h_{\min}(t) > 0$ be the minimal eigenvalue of the matrix $H(t)$. Hence, it is notable that

$$2.52 \le h_{\min}(t) \le 4.38, \;\; 6.62 \le \|H(t)\| \le 8.48.$$

Therefore, for $\alpha = 0.05, q_1 = 0, q_2 = 0.02, q = 0.16$ and the former particular choices, one can easily check that the matrix $C(t), S(t)$ and $R(t)$ are positive definite for all $t \in [0, 2\pi]$ and the minimal eigenvalue $c_{\min}(t), s_{\min}(t)$ and $r_{\min}(t)$ of the matrix $C(t), S(t)$ and $R(t)$, respectively, satisfies $c_{\min}(t) \ge 0.5365, s_{\min}(t) \ge 0.5379$ and $r_{\min}(t) \ge 0.3779$ by MATLAB-Simulink. Finally, we have

$$\gamma(t) = \min\{r(t), k\|H(t)\|\} = k\|H(t)\| > 0,$$

$$\beta(t) = \frac{\gamma(t)}{2\|H(t)\|} = \beta^+ = \beta^- = \frac{k}{2} = 0.025$$

and $\rho < e^{-\beta^+\tau_2}$ so that

$$\|y(t)\| \le r \max_{-\tau_2 \le s \le 0} \|y(s)\| e^{-0.025t}, \;\; t \ge 0$$

for a proper positive constant r.

As a result, it is seen that all the assumptions of Theorem 1 can be held.

Let

$$\tau(t) = \frac{1+\sin^2(t)}{20}, \; t > 0.$$

Benefited from by MATLAB-Simulink, the desired result for the behaviors of the orbits of solutions of the considered differential system is shown by the following graph.

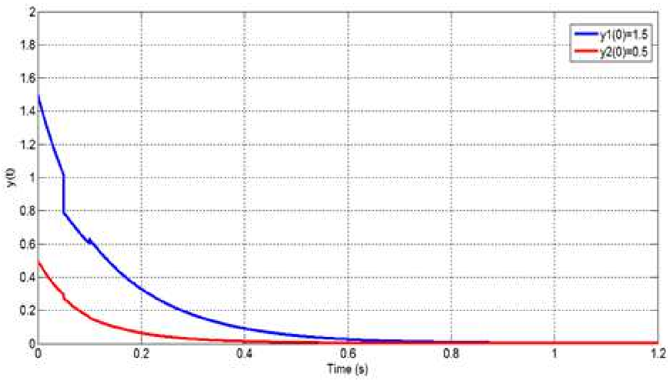

Figure 1. Trajectories of solutions $y(t)$ of system (2.27) when $\tau(t) = \frac{1+\sin^2(t)}{20}, t > 0$

References

[1] Altun, Y., Tunç, C. On the global stability of a neutral differential equation with variable time-lags. Bull. Math. Anal. Appl. 9 (2017), no. 4, 31-41.

[2] Demidenko, G. V., Matveeva, I. I., On the stability of solutions of linear systems with periodic coefficients. (Russian. Russian summary) Sibirsk. Mat. Zh. 42 (2001), no. 2, 332-348, ii; translation in Siberian Math. J., 42 (2001), no. 2, 282-296.

[3] Demidenko, G. V., Matveeva, I. I., Asymptotic properties of solutions to delay differential equations. Vestn. Novosib. Gos. Univ., Ser. Mat. Mekh. Inform., 5:3 (2005), 20-28.

[4] Demidenko, G. V., Matveeva, I. I., Stability of solutions of differential equations with retarded argument and periodic coefficients in the linear terms. . Sibirsk. Mat. Zh. 48 (2007), no. 5, 1025-1040; translation in Siberian Math. J. 48 (2007), no. 5, 824-836.

[5] Demidenko, G. V., Stability of solutions to linear differential equations of neutral type, J. Anal. Appl. 7 (2009), no. 3, 119-130.

[6] Demidenko, G. V., Kotova, T. V., Skvortsova, M. A., Stability of solutions to differential equations of neutral type. J. Math. Sci. (N. Y.) 186 (2012), no. 3, 394-406.

[7] Demidenko, G. V., Vodop'yanov, E. S., Skvortsova, M. A., Estimates for the solutions of linear differential equations of neutral type with several deviations of the argument. Sib. Zh. Ind. Mat. 16 (2013), no. 3, 53-60. ; translation in J. Appl. Ind. Math. 7 (2013), no. 4, 472-479.

[8] Demidenko, G. V., Matveeva, I. I., On the exponential stability of solutions of a class of systems of differential equations of neutral type, J. Appl. Indust. Math., 8 (2014), 510-520.

[9] Demidenko, G. V., Matveeva, I. I., On estimates for solutions of systems of differential equations of neutral type with periodic coefficients, Siberian Math. J., 55 (2014), 866-881.

[10] Demidenko, G. V., Matveeva, I. I., Estimates for solutions to a class of nonlinear time-delay systems of neutral type. Electron. J. Differential Equations 2015, No. 34, 14 pp.

[11] El'sgol'ts, L. E., Norkin, S. B., Introduction to the theory and application of differential equations with deviating arguments. Translated from the Russian by John L. Casti. Mathematics in Science and Engineering, Vol. 105. Academic Press, New York-London, (1973).

[12] Godunov, S. K., Modern aspects of linear algebra. Translated from the 1997 Russian original by Tamara Rozhkovskaya. Translations of Mathematical Monographs, 175. American Mathematical Society, Providence, RI, (1998).

[13] Gu, K., Kharitonov, V. L., Chen, J., Stability of time-delay systems, Control Engineering, Birkhäuser Boston, Inc., Boston, MA, 2003.

[14] Kharitonov, V. L., Time-delay systems. Lyapunov functionals and matrices. Control Engineering. Birkhäuser/Springer, New York, 2013.

[15] Khusainov, D. Ya., Shatyrko, A. V., The Method of Lyapunov functions in the investigation of the stability of functional differential systems, Izdatelstvo Kievskogo Universiteta, Kiev, 1997. 236 pp.

[16] Matveeva, I. I., Shcheglova, A. A., Estimates for the solutions of differential equations with retarded argument and with parameters. (Russian. Russian summary) Sib. Zh. Ind. Mat. 14 (2011), no. 1, 83-92; translation J. Appl. Ind. Math. 5 (2011), no. 3, 391-399.

[17] Matveeva, I. I., Estimates for the solutions of a class of systems of nonlinear delay differential equations. (Russian) Sib. Zh. Ind. Mat. 16 (2013), no. 3, 122-132; translation in J. Appl. Ind. Math. 7 (2013), no. 4, 557-566.

[18] Melchor-Aguilar, D., Niculescu, S.-I., Estimates of the attraction region for a class of nonlinear time-delay systems. IMA J. Math. Control Inform. 24 (2007), no. 4, 523-550.

[19] Michiels, W., Niculescu, S.I., Stability and stabilization of time-delay systems. An eigenvalue-based approach. , Advances in Design and Control, 12. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

[20] Skvortsova, M. A., Asymptotic properties of solutions to systems of neutral type differential equations with variable delay. Journal of mathematical sciences 205 (2015), no. 3, 455-463.

[21] Tunç, C., Altun, Y., Asymptotic stability in neutral differential equations with multiple delays. J. Math. Anal. 7 (2016), no. 5, 40-53.

[22] Tunç, C., Altun, Y., On the nature of solutions of neutral differential equations with periodic coefficients. Appl. Math. Inf. Sci. (AMIS), 11, (2017), no.2, 393-399.

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