Palestine Journal of Mathematics

Vol. 8(1)(2019) , 148–159

Strength of Cartesian product of certain strong fuzzy graphs

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Communicated by Ayman Badawi

MSC 2010 Classifications: 05C72.

Keywords and phrases:Strength of fuzzy graphs, Cartesian product of fuzzy graphs, fuzzy cycle, fuzzy butterfly graph, fuzzy star graph, domino graph, fuzzy book.

Abstract. In this paper we prove the Cartesian product of two fuzzy paths is again a strong fuzzy path. Also we find the strength of Cartesian product of two strong fuzzy graphs with underlying crisp graphs are the paths P_m and P_n, for all values of m and n and that of P_2 and C_n for all n. The strength of a strong fuzzy butterfly graph, Cartesian product of two strong fuzzy graphs with its underlying crisp graphs are P_2 and a star graph S_n are also determined.

1 Introduction

In this paper we find the strength of Cartesian product of various fuzzy graphs. The notion of a fuzzy subset was introduced for the first time in 1965 by Lofti A. Zadeh [15]. Azriel Rosenfeld [11], in 1975, defined the fuzzy graph based on definitions of fuzzy sets and relations. He was the one who developed the theory of fuzzy graphs. J. N. Mordeson [6] together with Premchand S. Nair [4] studied different operations on fuzzy graphs and their properties. The concept of strength of connectivity between two vertices of a fuzzy graph was introduced by M. S. Sunitha [12] and extended by Sheeba M. B. [13], [14] to arbitrary fuzzy graphs. Sheeba called it, strength of the fuzzy graph and determined it, in two different ways, of which one is by introducing weight matrix of a fuzzy graph and other by introducing the concept of extra strong path between its vertices.

Throughout this paper only undirected fuzzy graphs are considered.

2 Preliminaries

A fuzzy graph $G = G(V, \mu, \sigma)$ [4] is a nonempty set V together with a pair of functions $\mu : V \longrightarrow [0, 1]$ and $\sigma : V \times V \longrightarrow [0, 1]$ such that for all $u, v \in V$, $\sigma(u, v) = \sigma(uv) \leq \mu(u) \wedge \mu(v)$. We call μ the fuzzy vertex set of G and σ the fuzzy edge set of G.

Given any fuzzy graph there is a crisp graph associated with it called the underlying crisp graph. The vertex set of the crisp graph of a given fuzzy graph G is that of G and its edge set is $E = \{uv : u, v \in V \text{ such that } \sigma(uv) > 0\}$. If $uv \in E$ we say that u and v are adjacent in the associated crisp of G and also in G for convenience.

A fuzzy graph G is complete [4] if $\sigma(uv) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. A fuzzy graph G is a strong fuzzy graph [4] if $\sigma(uv) = \mu(u) \wedge \mu(v)$, $\forall uv \in E$. The strength of a strong fuzzy complete graph is one [13]. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with the underlying crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ respectively. If $V_1 \cap V_2 = \phi$ then their join [4] is the fuzzy graph $G = G_1 \vee G_2(V_1 \cup V_2, \mu_1 \vee \mu_2, \sigma_1 \vee \sigma_2)$ with the underlying crisp graph $G(V_1 \cup V_2, E_1 \cup E_2 \cup E')$ where E' is the set of all edges joining the vertices of V_1 and V_2 and

$$(\mu_1 \vee \mu_2)(u) = \begin{cases} \mu_1(u) & \text{if } u \in V_1 \\ \mu_2(u) & \text{if } u \in V2; \end{cases}$$

$$(\sigma_1 \vee \sigma_2)(uv) = \begin{cases} \sigma_1(uv) & \text{if } uv \in E_1, \\ \sigma_2(uv) & \text{if } uv \in E_2, \\ \mu_1(u) \wedge \mu_2(v) & \text{if } u \in E_1 \text{ and } v \in E_2. \end{cases}$$

A strong fuzzy complete bipartite graph is a strong fuzzy graph with its underlying crisp graph is a complete bipartite graph [10]. A fuzzy graph G is called a path if its underlying crisp graph is a path. A path P of length $n-1$ on n vertices in a fuzzy graph G [4] is a sequence of distinct vertices $v_1, v_2, v_3, \ldots, v_n$, such that $\sigma(v_i, v_{i+1}) > 0$, $i = 1, 2, 3, \ldots, n-1$, also we denote this by P_n. A path P on the vertices $v_1, v_2, \ldots, v_n$, $n \geq 3$ is called a fuzzy cycle if $\sigma(v_1 v_n) \geq 0$ and there exists at least two edges e_1 and e_2 in P such that $\sigma(e_1) = \sigma(e_2)$ and is denoted by C_n.

The vertices v_1 and v_n are called the end vertices of P. The strength of a path is defined as the weight of the weakest edge of the path [4]. A path in a fuzzy graph G is a partial fuzzy graph which itself is a path. A path P in a fuzzy graph is said to connect the vertices u and v of G strongly if its strength is maximum among all paths between u and v. Such paths are called strong paths [15]. Any strong path between two distinct vertices u and v in G with minimum length is called an extra strong path between them [13]. There may exists more than one extra strong paths between two vertices in a fuzzy graph G. But, by the definition of extra strong path, each such path between two vertices has the same length. The maximum length of extra strong paths between every pair of distinct vertices in G is called the strength of the graph G [13]. For a fuzzy graph G, with the underlying crisp graph is a path $P = v_1 v_2 \ldots v_n$ on n vertices then the strength of the graph G is its length $(n-1)$ [13]. The strength of a strong fuzzy complete graph is one [13].

Here after for a fuzzy graph G, we use $\mathscr{S}(G)$ to denote its strength. The following theorems determine the strength of a fuzzy cycle.

Theorem 2.1. [14] *In a fuzzy cycle G of length n, suppose there are l weakest edges where $l \leq [\frac{n+1}{2}]$. If these weakest edges altogether form a subpath then $\mathscr{S}(G)$ is $n-l$.*

Theorem 2.2. [14] *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which altogether form a subpath. If $l > [\frac{n+1}{2}]$, then $\mathscr{S}(G)$ is $[\frac{n}{2}]$.*

Theorem 2.3. [14] *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which do not altogether form a subpath. If $l > [\frac{n}{2}] - 1$ then the strength of the graph is $[\frac{n}{2}]$ and if $l = [\frac{n}{2}] - 1$ then $\mathscr{S}(G)$ is $[\frac{n+1}{2}]$.*

Theorem 2.4. [14] *In a fuzzy cycle of length n suppose there are $l < [\frac{n}{2}] - 1$ weakest edges which do not altogether form a subpath. Let s denote the maximum length of a subpath which does not contain any weakest edge. If $s \leq [\frac{n}{2}]$ then the strength of the graph is $[\frac{n}{2}]$ and if $s > [\frac{n}{2}]$ then the strength of the graph is s.*

Lemma 2.5. *The diameter of the Cartesian product of the graphs P_2 with vertex set $\{u_1, u_2\}$ and a butterfly graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ is* 3.

Lemma 2.6. *[8] In a strong fuzzy graph G if any two vertices are adjacent the the strength of the $u - v$ path in G is* 1.

Theorem 2.7. *[8] Let G be a strong fuzzy graph with its underlying crisp graph a butterfly graph. Then the strength of G is* 2.

Definition 2.8. [16] For $i = 1, 2$, let $G_i(V_i, \mu_i, \sigma_i)$ be two fuzzy graphs with underlying crisp graphs $G_i(V_i, E_i)$. Their Cartesian product G, denoted by $G_1 \Box G_2$ is the fuzzy graph $G(V, \mu, \sigma)$ with the underlying crisp graph $G(V, E)$, the Cartesian product of the crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with vertex set $V = V_1 \times V_2$ and edge set $E = \{(u, u_2)(u, v_2) | u \in V_1, u_2 v_2 \in E_2\} \cup \{(u_1, w)(v_1, w) | w \in V_2, u_1 v_1 \in E_1\}$ and whose membership functions μ and σ are defined as

$$\mu(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2); (u_1, u_2) \in V,$$

$$\sigma((u_1, u_2)(v_1, v_2)) = \begin{cases} \mu_1(u_1) \wedge \sigma_2(u_2 v_2) & \text{if } u_1 = v_1 \text{ and } u_2 v_2 \in E_2, \\ \mu_2(u_2) \wedge \sigma_1(u_1 v_1) & \text{if } u_2 = v_2 \text{ and } u_1 v_1 \in E_1. \end{cases}$$

Notation 2.9. Unless otherwise specified for $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ the notation w_{ij} is used to denote the vertex $(u_i, v_j) \in V_1 \times V_2$.

Lemma 2.10. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy paths, each has P_2 as its underlying crisp graph. Then the Cartesian product $G_1 \Box G_2$ of G_1 and G_2 is a fuzzy cycle.*

Proof. Let G_1 and G_2 be two fuzzy graphs with P_2 as their underlying crisp graph. The fuzzy graph $G_1 \Box G_2$ is depicted in Figure 1.

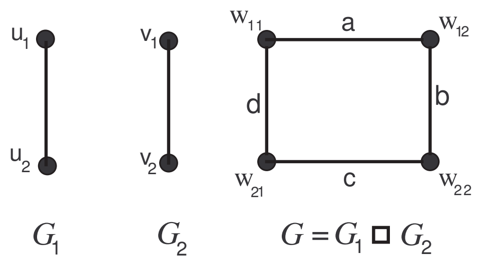

Figure 1. The fuzzy paths G_1, G_2 and their Cartesian product $G_1 \Box G_2$

Suppose that $\sigma_1(u_1u_2) \leq \sigma_2(v_1v_2)$. Then $\sigma(w_{11}w_{12}) = \sigma(w_{21}w_{22}) = \sigma(v_1v_2)$ and $\sigma(w_{11}w_{21}) = \sigma(w_{12}w_{22}) = \sigma(u_1u_2)$. Thus there are at least two weakest edges in $G_1 \Box G_2$. Hence $G_1 \Box G_2$ is a fuzzy cycle since the underlying graph of $G_1 \Box G_2$ is a cycle.

Note 2.11. If G_1 and G_2 are two strong fuzzy graphs then $\sigma(u_1u_2) = \mu_1(u_1) \wedge \mu_1(u_2)$ and $\sigma_2(v_1v_2) = \mu_2(v_1) \wedge \mu_2(v_2)$. If let us suppose that $\mu_1(u_1) = min\{\mu_1(u_1), \mu_1(u_2), \mu_2(v_1), \mu_2(v_2)\}$. Then $\sigma(w_{11}w_{12}) = \sigma(w_{11}w_{21}) = \sigma(w_{12}w_{22}) = a$ say and $\sigma(w_{21}w_{22})$ is greater than or equal to this common value a. Thus if G_1 and G_2 are strong fuzzy graphs then at least three edges of $G_1 \Box G_2$ are weakest edges.

□

Lemma 2.12. *Let G_1 and G_2 be two strong fuzzy graphs. Suppose both the graphs have underlying crisp graphs P_2 on two vertices. Then the strength of the Cartesian product of G_1 and G_2 is two.*

Lemma 2.13. *Let G_1 and G_2 be two fuzzy graphs with crisp graphs P_2 and P_3 respectively. Then the strength of $G_1 \Box G_2$ is* 3.

Proof. Let the fuzzy graphs G_1, G_2, and their Cartesian product $G_1 \Box G_2$ be as depicted in Figure 2.

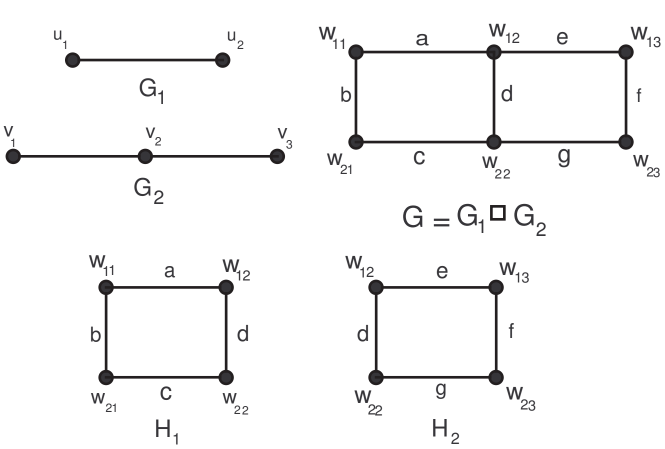

Figure 2. The fuzzy subgraphs G_1 and G_2, their Cartesian product $G_1 \Box G_2$ and two partial fuzzy subgraphs H_1 and H_2 of $G_1 \Box G_2$

The two partial fuzzy subgraphs H_1 and H_2 of $G_1 \square G_2$ shown in Figure 2 are fuzzy cycles by Lemma 2.10. Theorem 2.12 shows that both H_1 and H_2 have strength 2. Suppose the weakest edge of H_1 has weight α and those of H_2 have weight β.

Case 1. $\alpha \geq \beta$.
In this case $d \geq \alpha$.

Subcase 1. $d > \beta$. Then $e = g = f = \beta \longrightarrow (1)$. Let u and v be two vertices of G. If u and v are in $V(H_1)$, then the length of the extra strong path joining u and v is $\leq$ the strength of H_1, ie 2. Because, if a $u - v$ path P passes through a vertex in $V(G) \setminus V(H_1)$ then it has strength $\leq$ any $u - v$ path in H_1 and its length must be greater than or equal to any $u - v$ path in H_1, then either its strength is α or β, according as the path is a subpath of H_1 or it contains at least one edge of $G \backslash H_1$.

If u and v are in $V(G \backslash H_1)$ then $u, v \in \{w_{13}, w_{23}\}$ and hence adjacent. Therefore, the extra strong path joining u and v is $w_{13}w_{23}$, which is of length one.

If u is in $V(G \backslash H_2)$ and v is in$V(G \backslash H_1)$. Then all the paths joining u and v must pass through an edge having weight β. Therefore, all the paths joining u and v have same strength. So, length of the extra strong path joining u and v is ≤ 3.

In particular if $u = w_{11}$ and $v = w_{23}$ or $u = w_{21}$ and $v = w_{13}$ then the length of extra strong path is equal to 3.

Subcase 2. $d = \beta$.

Then $\mu_1(u_1) = \beta$ or $\mu_1(u_2) = \beta$ or $\mu_2(v_2) = \beta$. In the first case $d = f = e = \beta$. In the second case $d = e = g = \beta$. In these cases also as in Subcase 1 we can prove that the strength of G is 3.

Case 2. $\alpha < \beta$.
The proof follows by interchanging the roles of H_1 and H_2.

□

Theorem 2.14. *Let G_1 and G_2 be two strong fuzzy graphs with respective underlying crisp graphs P_2 and P_n. Then the strength of Cartesian product $G_1 \square G_2$ of G_1 and G_2 is n.*

Proof. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graphs P_2 with vertex set $\{u_1, u_2\}$ and P_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ respectively.

Let $G(V, \mu, \sigma)$ be the Cartesian product $G_1 \square G_2$ of G_1 and G_2 with underlying crisp graph $G(V, E)$ where the vertex set $V = \{(u_i, v_j) = w_{ij} : u_i \in V_1, v_j \in V_2, i = 1, 2, j = 1, 2, \ldots, n\}$ and edge set $E = \{w_{ij}w_{ij+1} : 1 \leq j \leq n - 1, i = 1, 2\} \cup \{w_{1j}w_{2j} : j = 1, 2, \ldots, n\}$.

We prove the theorem by induction on n. The result is trivial when $n = 1$ and the result is true for $n = 2$, and $n = 3$ by Lemmas 2.12 and 2.13. When $n = 2$, ie, when G_1 and G_2 are two fuzzy graphs with respective crisp graphs P_2, we proved that, the strength of the graph is 2, by showing that if $u = w_{11}$ and $v = w_{22}$ (or $u = w_{21}$ and $v = w_{12}$) then length of the extra strong $u - v$ path is 2 and for any other u and v, it is 1. Also in the case, G_1 is a fuzzy graph with the underlying crisp graph P_2 and G_2 a fuzzy graph with underlying crisp graph P_3, we proved that the length of any extra strong $u - v$ path is 3, when $u = w_{11}$ and $v = w_{23}$ or $u = w_{21}$ and $v = w_{13}$. For all other choices of u and v the length of the extra strong $u - v$ path is < 3 and the extra strong $w_{11} - w_{13}$ path is $w_{11}w_{12}w_{13}$.

We assume that the result is true for $n = m$, where $m \geq 3$. That is if G_1 is the fuzzy path P_2 with vertex set $\{u_1, u_2\}$ and G_2 is a fuzzy path P_m with vertex set $\{v_1, v_2, \ldots, v_m\}$ then assume that length of the extra strong path joining the vertices w_{11} and $w_{2\,m}$ or the vertices w_{21} and $w_{1\,m}$ in $G_1 \square G_2$ is m and if $u = w_{11}$ and $v = w_{1\,m}$ or if $u = w_{21}$ and $v = w_{2m}$ then the length of the extra strong $u - v$ path is $m - 1$, and in fact $w_{11}w_{12} \ldots w_{1m}$ is the extra strong $w_{11} - w_{1m}$ path. u and v are any other vertices of $G_1 \square G_2$ then the length of the extra strong $u - v$ path is $< m - 1$

Let us suppose that G_1 be the fuzzy path on the vertex set $\{u_1, u_2\}$ and G_2 be the fuzzy path on the vertex set $\{v_1, v_2, \ldots, v_{m+1}\}$. For $1 \leq p < q \leq m + 1$, H_{pq} denotes the maximal partial fuzzy subgraph of G with vertex set $\{w_{ij}; i = 1, 2, p \leq j \leq q\}$. (See Figure 3).

Clearly, $H_{1\,n+1} = G_1 \Box G_2$.

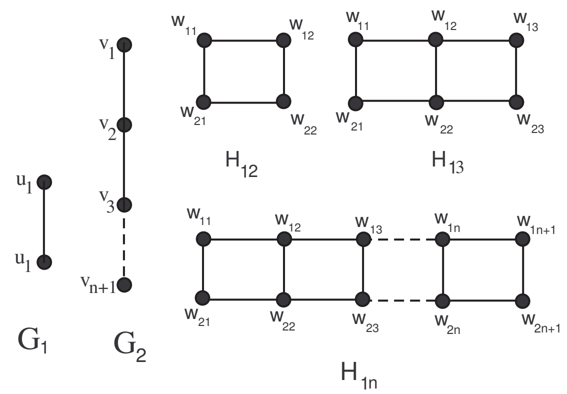

Figure 3. Partial fuzzy subgraphs H_{12} , H_{13} and $H_{1\,n+1}$ of $G = G_1 \Box G_2$

Let u and v be two non -adjacent vertices of $G_1 \Box G_2$. We assert that if $u = w_{ij}$ and $v = w_{kl} \in H_{2\,m+1}$ then any extra strong $u - v$ path of G lie in H_{2m+1} and the length of any extra strong $u - v$ path in $G_1 \Box G_2$ is $\leq m + 1$, by the induction hypothesis when $u = w_{21}$ and when $v = w_{1\,m+1}$ then the length of the extra strong $u - v$ path is $m + 1$.

Case 1. Suppose that u and v are in $\{w_{ij} : i = 1, 2; j = 2, 3, \ldots, m\}$.

Then any path joining u and v in G can be viewed either as a path in the maximal partial fuzzy graph H_{1n} with vertex set $\{w_{ij} : i = 1, 2, 1 \leq j \leq m\}$ or as a path in the maximal partial fuzzy graph H_{2m+1} with vertex set $\{w_{ij} : i = 1, 2; 2 \leq j \leq (m + 1)\}$. Note that both these graphs have $P_2 \Box P_m$ as their underlying crisp graphs. Therefore by induction hypothesis the length of the extra strong $u - v$ path is $\leq m < (m + 1)$

Case 2. $u, v \in \{w_{11}, w_{21}, w_{1\,m+1}, w_{2\,m+1}\}$.

Suppose $u \in \{w_{11}, w_{21}\}$ and $v \in \{w_{1\ m+1}, w_{2\ m+1}\}$. Then we can prove the result in two steps.

(i) If $u = w_{11}$ and $v = w_{1m+1}$ (or $u = w_{21}$ and $v = w_{2m+1}$). Any path P_m in $H_{1\,m+1}$ joining w_{11} and w_{1m+1} can be considered as sum of two paths P^1 and P^2 where P^1 is a path in H_{1m} joining w_{11} and w_{1m} or it is a path joining w_{11} and w_{2m} in H_{1m} and P^2 is $P \cap H_{m\ m+1}$. Note that the strength of the path P is minimum of strength of the paths $P^i : i = 1, 2$. By induction hypothesis if P^1 is a path joining w_{11} and w_{1m} then it has maximum strength if $P^1 = w_{11}w_{12 \ldots w_{1m}}$. Since w_{1m} and $w_{1\ m+1}$ are adjacent, the path $w_{1m}w_{1m+1}$ is the extra strong path joining w_{1m} and $w_{1\ m+1}$. In the second case, that is P' is a path from w_{11} to w_{2m} in H_{1m} and $P^2 = P \cap H_{m\ m+1}$ then by induction hypothesis P^1 has length m when P^1 is an extra strong path. Therefore in this length of the path P is $m+2$ and it has strength $\leq$ the strength of the path $w_{11}w_{12} \ldots w_{1\ m+1}$. Therefore, we can conclude that the path P has maximum strength if $P^1 = w_{11}w_{12} \ldots w_{1m}$ and $P^2 = w_{1m}w_{1\ m+1}$. Also the length of P' is minimum among all paths in H_{1m} between w_{11} and $w_{1\ m}$.

(ii) If $u = w_{11}$ and $v = w_{2m+1}$ (or $u = w_{21}$ and $v = w_{1m+1}$).

In this case as in the proof of (i) we can prove that the strength of $u - v$ path is $m + 1$ in $H_{1\ m+1}$.

Hence the theorem. □

Theorem 2.15. *Let G_1 and G_2 be two strong fuzzy graphs with the underlying crisp graphs the path P_m and the path P_n on m and n vertices respectively. Then the strength of the Cartesian product $G = G_1 \Box G_2$ of G_1 and G_2 is $m + n - 2$.*

Proof. For a fixed n, we prove this theorem by induction on m. If $m = 1$ then G_1 is a fuzzy trivial graph. Thus when $m = 1$, $G = G_1 \Box G_2$ is a copy of P_n, a fuzzy path on n vertices. If $n = 1$, its strength is zero. If $n > 1$ then its strength is $n - 1$. In either case we have the strength is $m + n - 2$. Assume that the result is true for $m = k > 1$. To prove the result for $m = k + 1$, let G_1 and G_2 be strong fuzzy graphs with underlying crisp graphs P_{k+1} and P_n respectively and let G be their Cartesian product. If $n = 1$ then G is a copy of G_1. Therefore strength of G is $k = m + n - 2$ thus in this case the theorem holds. So assume that $n > 1$. Also let $u, v \in V(G)$.

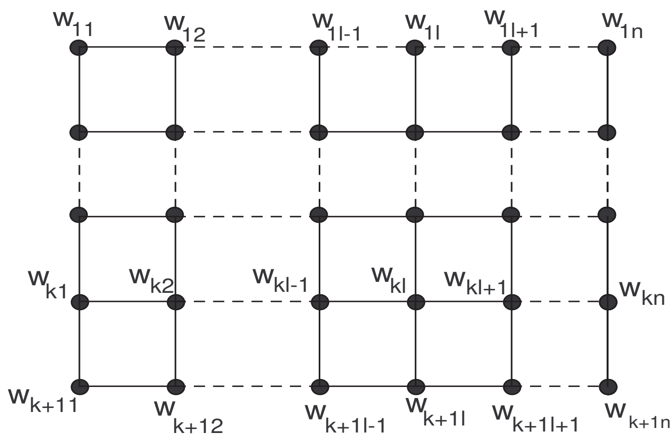

Figure 4. Cartesian product of two fuzzy graphs with underlying graphs P_{k+1} and P_n.

Case 1. $u, v \in \{w_{ij} : 1 \le i \le k, 1 \le j \le n\}$ or $u, v \in \{w_{ij} : 2 \le i \le k+1, 1 \le j \le n\}$. Let H_1 and H_2 be the two maximal partial fuzzy subgraphs of G with vertex set $\{w_{ij} : 1 \le i \le k, 1 \le j \le n\}$, $\{w_{ij} : 2 \le i \le k+1, 1 \le j \le n\}$ respectively. Then any extra strong path joining u and v in G can be either a path in H_1 or in H_2 of G.

To prove this assertion we proceed as follows. Let us suppose that $u, v \in V(H_1)$. Suppose P is an extra strong $u - v$ path in G, which passes through at least one of the vertices $w_{11}, w_{12}, \ldots, w_{1n}$. Then, we claim that P does not pass through any of the vertices $w_{k+1\,1}, w_{k+1\,2}, \ldots, w_{k+1\,n}$. If so, it contains a subpath $w_{kl}w_{k+1\,l}w_{k+1\,l+1} \ldots w_{k+1\,j}w_{kj}$ of G, which can be viewed as a path of the maximal partial fuzzy subgraph with vertex set $\{w_{k1}w_{k2} \ldots w_{kn}w_{k+1\,1} \ldots w_{k+1\,n-1}w_{k+1\,n}\}$ of G which is of the form $P_2 \Box P_n$. Therefore the extra strong path joining w_{kl} and w_{kj} is $w_{kl}w_{k\,l+1} \ldots w_{kj}$ by the proof of Theorem 2.14. Therefore we can conclude that every path like P is contained in H_1. Hence its length by induction $\le k + n - 2$. Similar is the case when $u, v \in V(H_2)$.

Case 2. $u \in \{w_{1l} : l = 1, 2, \ldots, n\}$ and $v \in \{w_{k+1l} : l = 1, 2, \ldots, n\}$.

Let us suppose that $u = w_{1j}$ and $v = w_{k+1l}$. For $l = 1, 2, \ldots, k+1$ we denote the path $w_{i1}w_{i2} \ldots w_{in}$ with vertices $w_{i1}, w_{i2}, \ldots, w_{in}$ in G by L_i. We claim that for a fixed $l, l = 1, 2, \ldots, n$ the edge $w_{k+1l}w_{kl}$ has strength greater than or equal to the strength of any path from v to any vertex w of L_k. Suppose a path P_1 from v to a vertex of L_k contains a subpath $Q_1 = w_{k+1\,j}w_{k+1\,j-1} \ldots w_{k+1\,l}$ of L_{k+1}, then the path P_1 has strength less than or equal to that of the edge $w_{k+1l}w_{kl}$. For if the edge $w_{k+1\,l}w_{k\,l}$ is not a weakest edge of the cycle $C : w_{k\,l+1}w_{k+1\,l+1}w_{k+1\,l}w_{k\,l}w_{k\,l+1}$ then weight of $w_{k+1\,l}w_{k+1\,l+1} <$ weight of $w_{k+1\,l}w_{k\,l}$. Therefore the strength of $P_1 <$ strength of $w_{k+1\,l}w_{k\,l}$.

If $w_{k+1l}w_{kl}$ is a weakest edge of C then the subpath Q_1 of P_1 which belongs to L_{k+1} has strength $\ge$ strength of $w_{k+1l}w_{kl}$. If Q_1 has strength greater than that of $w_{k+1\,l}w_{k\,l}$ then all the edges $w_{k+1\,l}w_{k\,l}, \ldots, w_{k+1\,j}w_{kj}$ have weight equal to that of $w_{k+1l}w_{kl}$. Therefore we can conclude that in this case the path P_1 has strength $\le$ that of $w_{k+1l}w_{kl}$. If P_1 contains no subpath of L_{k+1} then any path from v to a vertex of L_k pass through the edge vw_{kl}. Hence its strength must be less than or equal to the strength of the edge vw_{kl}. Hence the path having minimum length and with maximum strength from w_{k+1l} to a vertex of L_k is just the edge $w_{k+1l}w_{kl}$.

By the same argument, the edge $w_{kl}w_{k-1l}$ has the maximum strength and minimum length from w_{kl} to any vertex in L_{k-1}. Therefore the path $w_{k+1l}w_{kl}w_{k-1l}$ is the path from w_{k+1l} to L_{k-1}. Proceeding similarly we get the path $w_{k+1l}\dots w_{1l}$ is the path with maximum strength and minimum length from w_{k+1l} to any vertex of L_1. Proceeding similarly $w_{1l}\dots w_{1j}$ is the path with maximum strength and minimum length path joining w_{1j} and w_{1l}. Therefore the strength of the $u-v$ path is $\leq (n-1)+k = k+n-1$.

When $u = w_{11}$ and $v = w_{k+1n}$, the strength of the $u-v$ path is equal to $k+n-1$. Thus the theorem is true for $m = k+1$. Therefore the theorem follows by induction. □

Next we consider the Cartesian product of the fuzzy graphs P_2 and a fuzzy cycle C_n. Suppose $V_1 = \{u_1, u_2\}$, and $V_2 = \{v_1, v_2, \dots, v_n\}$ are the vertex set of G_1 and G_2 respectively. Then the Cartesian product of G_1 and G_2 is the fuzzy graph $G(V, \mu, \sigma)$ where the underlying crisp graph is $G(V, E)$ with vertex set $V = \{w_{ij}, i = 1, 2, j = 1, 2, \dots, n\}$ and edge set $E = \{w_{ij}w_{ij+1}, 1 \leq j < n, i = 1, 2\} \cup \{w_{1j}w_{2j}, 1 \leq j < n\} \cup \{w_{i1}w_{in}, i = 1, 2\}$ where $\mu(w_{ij}) = \mu_1(u_i) \wedge \mu_2(v_j), \forall w_{ij} \in V$
$\sigma(w_{ij}w_{ij+1}) = \mu_1(u_i) \wedge \sigma_2(v_jv_{j+1}), u_i \in V_1, (v_j, v_{j+1}) \in E_2$;
$\sigma(w_{1j}w_{2j}) = \sigma_1(u_1u_2) \wedge \mu_2(v_j)$; $\sigma(w_{i1}w_{in}) = \mu_1(u_i) \wedge \sigma_2(v_1v_n)$.

For example

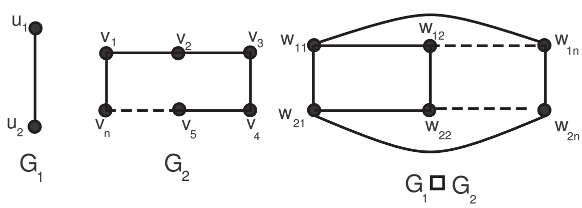

Figure 5. Cartesian product of the fuzzy graphs G_1 with underlying crisp graph P_2 and G_2 with underlying crisp graph C_n.

Theorem 2.16. *Suppose G_1 and G_2 are two strong fuzzy graphs with underlying crisp graphs the path P_2 with vertex set $V_1 = \{u_1, u_2\}$ and the cycle C_n with vertex set $V_2 = \{v_1, v_2, \dots, v_n\}$ respectively and the weight of the weakest vertices of G_1 is greater than the weight of the weakest vertices of G_2. If the weakest vertices of G_2 altogether form a subpath of length l in G_2 then the strength of the Cartesian product of G_1 and G_2 is $(n-l+1)$ if $l < [\frac{n+1}{2}]$ and $[\frac{n}{2}]$ if $l \geq [\frac{n+1}{2}]$.*

Proof. Let u and v be two non-adjacent vertices of G. Without loss of generality assume that $v_1, v_2, \dots, v_{l-1}$ are the weakest vertices of G_2. Also assume that the weight of each $v_i, i = 1, 2, \dots, l-1$ is w and these vertices altogether form a subpath in G_2. Then in G, the vertices $w_{11}, w_{12}, \dots, w_{1l-1}$ and $w_{21}, w_{22}, \dots, w_{2l-1}$ have the same weight w (See Figure 6).

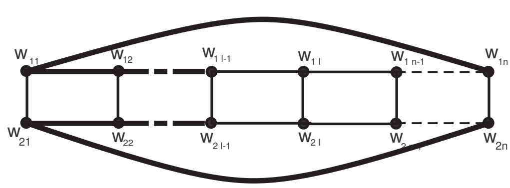

Figure 6. The Cartesian product of G_1 and $G_2 - \{v_1, \dots, v_{l-1}\}$.

Case 1. $l < [\frac{n+1}{2}]$.

If $u, v \in V(G) - \{w_{11}, \dots, w_{1l-1}, w_{21}, \dots w_{2l-1}\}$ then the strength of the $u-v$ path in G is $\leq n-l+1$, since the extra strong paths joining u and v lie completely in the maximal partial subgraph $G_1\Box(G_2 - \{v_1, v_2, \ldots, v_{l-1}\})$ of G with underlying crisp graph $P_2\Box P_{n-(l-1)}$. Therefore by Theorem 2.15 the length of the extra strong $u - v$ path in $G \le n - l + 1$.

If $u, v \in \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2\,l-1}\}$ then all the $u - v$ paths have same strength in G. So all the extra strong paths joining u and v lie in the maximal partial subgraph $G_1\Box G_2'$ of G, where G_2' is the maximal partial fuzzy graph of G_2 with vertex set $\{v_1, v_2, \ldots, v_{l-1}\}$ as shown in Figure 6. Also since $l \le [\frac{n+1}{2}]$ the length of the extra strong $u - v$ path is $\le l - 1 \le n - l + 1$.

If $u \in \{w_{11}, \ldots, w_{1\,l-1}, w_{2\,1}, \ldots, w_{2\,l-1}\}$ and $v \in V(G) - \{w_{11}, \ldots, w_{1\,l-2}, w_{2\,1}, \ldots, w_{2\,l-2}\}$ or vice versa then all the paths joining u and v have same strength. So the length of the extra strong $u - v$ path is the minimum distance between u and v in the underlying crisp graph of G, $P_2\Box C_n$ which is $\le (n - l + 1)$.

If $u = w_{2\,l}$ and $v = w_{1\,n}$ then the length of the extra strong $u - v$ path is equal to $n - l + 1$.

Case 2. $l \ge [\frac{n+1}{2}]$.

If $u, v \in V(G) \setminus \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2l-1}\}$ then as in Case 1 strength of $u - v$ path in G is $n - l + 1 \le [\frac{n}{2}]$. If $u, v \in \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2l-1}\}$ or $u \in G - \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2\,l-1}\}$ and $v \in \{w_{11}, \ldots, w_{1\,l-1}, w_{2\,1}, \ldots, w_{2\,l-1}\}$ then all the $u - v$ paths must have same strength in G, and therefore the length of the extra strong path joining u and v is $\le [\frac{n}{2}]$, since $l > [\frac{n+1}{2}]$. When $u = w_{11}$ and $v = w_{1k}$ where $k = [\frac{n}{2}]$ then strength of the $u - v$ path in G is exactly equal to $[\frac{n}{2}]$. Hence the Theorem. □

Theorem 2.17. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs $K_1 =< u >$ and the cycle $C_n = v_1, v_2, \ldots, v_n, v_1$ respectively. Let $G(V, \mu, \sigma)$ be the Cartesian product of G_1 and G_2. If v be a weakest vertex of G_2 then*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } \mu_1(u) \le \mu_2(v) \\ \mathscr{S}(G_2) & \text{otherwise} \end{cases}$$

Proof. If $\mu_1(u) \le \mu_2(v)$ then all the vertices of $G_1\Box G_2$ have the same weight $\mu_1(u)$. Therefore it is a regular fuzzy cycle. Hence by Theorem 2.2, strength of $G_1\Box G_2$ is $[\frac{n}{2}]$.

If $\mu_1(u) > \mu_2(v)$, then,

$$\mu(u, v_i) = \begin{cases} \mu_2(v_i) & \text{if } \mu_2(v_i) \le \mu_1(u) \\ \mu_1(u) & \text{otherwise} \end{cases}$$

Thus a vertex (u, v_i) of G is a weakest vertex of G if and only if v_i is a weakest vertex of G_2. Therefore, the strength $\mathscr{S}(G)$ of G is that of G_2. □

Theorem 2.18. *Suppose $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ are two strong fuzzy graphs with underlying crisp graphs the path $P_2 = u_1u_2$ and $C_n = v_1v_2 \ldots v_nv_1$ respectively. Suppose that $\mu_1(u_1) \le \mu_1(u_2) \wedge \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_n)$. Let $G = G_1\Box G_2$ be the Cartesian product of G_1 and G_2. Then the strength $\mathscr{S}(G)$ of the Cartesian product G of G_1 and G_2 is,*

$$\mathscr{S}(G) = max\left\{\mathscr{S}(G_2\Box G_3), \left\lceil\frac{n+1}{2}\right\rceil\right\};$$

where G_3 is the null graph with vertex set $\{u_2\}$.

Proof. Let u and v be two distinct vertices of G.

Case 1. $\mu_1(u_2) > \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$

Subcase 1. Let $u, v \in \{w_{1j}, 1 \le j \le n\}$. Since $\mu(w_{1j}) = \mu_1(u_1); 1 \le j \le n$, all the edges having w_{1j} as one of the end vertices, $1 \le j \le n$ have weight equal to $\mu_1(u_1)$. Therefore, the length of the extra strong path joining u and v is the minimum length of the path joining u and v in G. That is less than or equal to $\left[\frac{n}{2}\right]$.

Subcase 2. Let $u, v \in \{w_{2j}, 1 \leq j \leq n\}$
Since $\mu(w_{1j}) \leq \mu(w_{2j})$, the extra strong path joining u and v lies in the maximal partial fuzzy subgraph $G_3 \square G_2$ of G. So we have by Theorem 2.17, the length of the extra strong $u - v$ path is the strength of G_2.

Subcase 3. Let $u \in \{w_{1j} : 1 \leq j \leq n\}$ and $v \in \{w_{2j} : 1 \leq j \leq n\}$.

Since $\mu_1(u_1) \leq \mu_1(u_2) \wedge \mu_2(v_1) \wedge \ldots \wedge \mu_2(v_n)$, all the $u - v$ paths in G have the strength $\mu_1(u_1)$. So length of the extra strong $u - v$ path in G is the length of the shortest $u - v$ path in G which is $\leq \lceil \frac{n+1}{2} \rceil$.

Case 2. $\mu_1(u_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$.

Subcase 1. $\mu_1(u_1) = \mu_1(u_2)$. Then $\mu(w_{ij}) = \mu_1(u_1)\ \forall i, j$. Therefore, the length of the extra strong path joining u and v in G is the minimum length of the path joining u and v in G, which is less than or equal to $[\frac{n+1}{2}]$.

Subcase 2. $\mu_1(u_1) < \mu_1(u_2)$. Then $\mu(w_{1j}) = \mu_1(u_1)$ and $\mu(w_{2j}) = \mu_1(u_2)\ \forall i, j$.
If u or $v \in \{w_{1j}, 1 \leq j \leq n\}$, then all the paths joining u and v have weight $\mu_1(u_1)$. Therefore, the length of the extra strong path joining u and v is the minimum length of the path joining u and v in G which is $[\frac{n}{2}]$.

If u and $v \in \{w_{2j}, 1 \leq j \leq n\}$, then the extra strong path joining u and v lie in the subgraph $G_3 \square G_2$. So by Theorem 2.17 the strength of G is $[\frac{n}{2}]$.

□

Note 2.19. Let $G(V, \mu, \sigma)$ be a fuzzy graph. If W is a subset of V then $< W >$ denotes the maximal partial fuzzy subgraph of G on W.

Definition 2.20. The fuzzy book is defined as the Cartesian product of graphs G_1 with underlying crisp graph P_2 and fuzzy star graph S_n, where $n > 2$. Let $V(P_2) = \{u_1, u_2\}$ and $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. For $i = 2, 3, \ldots, n$, the maximal partial fuzzy subgraph $< \{ w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ with vertex set $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ is called a fuzzy page of the fuzzy book.

The underlying crisp graph of any fuzzy page is $P_2 \square P_2$.

Note 2.21. The crisp graph of the union of two fuzzy pages $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ and $< \{w_{11}, w_{21}, w_{1j}, w_{2j}\} >$ is $P_2 \square P_3, 2 \leq i \neq j \leq n$. It is called a fuzzy Domino graph.

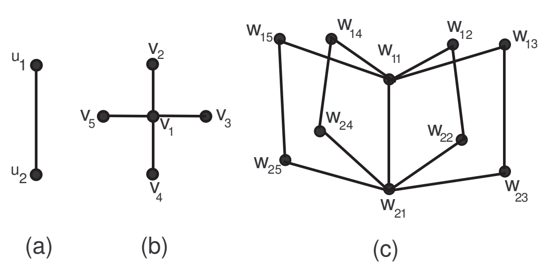

Figure 7. (a)The fuzzy path $G_1 = P_2$, (b) the fuzzy star graph $G_2 = S_5$, (c) the Cartesian product of G_1 and G_2

Theorem 2.22. *Let G_1 and G_2 be two strong fuzzy graphs with underlying crisp graphs the path P_2 and the star graph S_n respectively. Let $V(P_2) = \{u_1, u_2\}$ and $V(S_n) = \{v_1, v_2, \ldots, v_n\}$ with v_1 as the central vertex. Then the strength of the Cartesian product $G = G_1 \square G_2$ is* 3.

Proof. Let $\{w_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}\}$, where $n \geq 3$, be the vertex set of G. Clearly $w_{11}w_{21}$ is the common edge of the pages of $G_1 \square G_2$. Let u and v be two non-adjacent vertices of G (See Figure 7). Then u and v lie on the same page or different pages of G. For $i \neq j$, denote the partial fuzzy subgraph $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} > \cup < \{w_{11}, w_{21}, w_{1j}, w_{2j}\} >$ of $P_2\Box S_n$ by H_{ij}. Therefore any extra strong path joining u and v can be considered as a path in H_{ij} for some i and j. Since the underlying crisp graph of H_{ij} is $P_2\Box P_3$, the length of any extra strong path joining u and v in G is less than or equal to 3 , by Theorem 2.13.

In particular if $u = w_{12}$ and $v = w_{23}$, then any extra strong path joining u and v lie completely in H_{23} and hence has length exactly 3. Hence the theorem.

□

Now we are going to find the strength of the Cartesian product of fuzzy path and a fuzzy butterfly graph.

Theorem 2.23. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with crisp graphs the path P_2 with vertex set $\{u_1, u_2\}$ and the butterfly graph with vertex set $\{v_1, v_2, \ldots, v_5\}$ respectively. Then the strength of the Cartesian product $G(V, \mu, \sigma)$ of G_1 and G_2 is* 3.

Proof. First of all assume that the degree of the vertex v_1 of G_2 is 4 and $\mu_1(u_1) \leq \mu_1(u_2)$.

Let u and v be any two non-adjacent vertices of $G = G_1\Box G_2$ with vertex set $\{w_{11}, w_{12}, \ldots, w_{15}, w_{21}, w_{22}, \ldots, w_{25}\}$.

Case 1. $\mu_1(u_1)$ or $\mu_1(u_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_5)$.

Then all the $u - v$ paths passing through any of $w_{1j}, j = 1, 2, \ldots, 5$ have strength $\mu_1(u_1)$, because every edge incident with w_{1j} has weight $\mu_1(u_1)$. Therefore if at least one of u and v belongs to $\{w_{11}, w_{12}, \ldots, w_{15}\}$ then the extra strong $u - v$ paths are the shortest $u - v$ paths in the underlying crisp graph of G and therefore has length less than or equal to 3.

If $u, v \in \{w_{21}, w_{22}, \ldots, w_{25}\}$ then any extra strong $u-v$ path lie in the maximal partial fuzzy subgraph with vertex set $\{w_{21}, w_{22}, \ldots, w_{25}\}$ which is a strong fuzzy butterfly graph. Therefore, the length of any extra strong $u - v$ path in G is 2.

Case 2. $\mu_2(v_j)$ less than $\mu_1(u_1)$ for at least one j. Let us suppose that $\mu_2(v_j) \leq \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_5)$.

Subcase 1. $v_j = v_1$

Then all the paths passing through $w_{i1}, i = 1, 2$ have strength $\mu_2(v_1)$. The fuzzy graph of G can be viewed as the union of two fuzzy subgraphs H_1 and H_2, as shown in Figure 8. Note that $P_2\Box C_2$ is the underlying crisp graph of both H_1 and H_2.

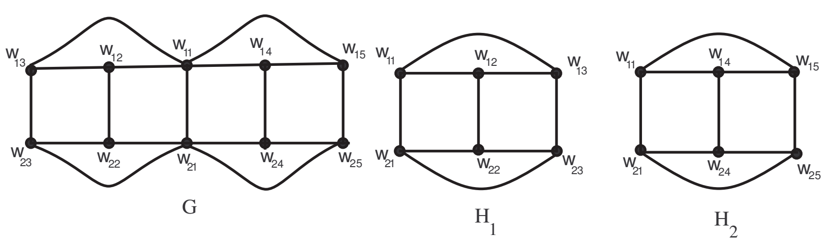

Figure 8. Cartesian product $G = G_1\Box G_2$ of a fuzzy path G_1 on 2 vertices and G_2, a fuzzy butterfly graph and the fuzzy subgraphs H_1 and H_2 of G

Suppose u and v belong to $V(H_1)$. Then any extra strong $u-v$ path lie in H_1, since $\mu(w_{11}) = \mu(w_{21}) = \mu_2(v_1)$, all the $u - v$ paths through w_{11} and w_{21} have the same strength. Therefore the length of the extra strong $u-v$ path is ≤ 2. Similarly if u and $v \in V(H_2)$ the length of any extra strong $u - v$ path is ≤ 2.

Let $u \in V(H_1)$ and $v \in V(H_2) \setminus V(H_1)$. In this case all the $u - v$ paths pass through w_{11} or w_{21} or both. Therefore all the $u - v$ paths have same strength. Hence the length of the extra strong path joining u and v is less than or equal to the minimum distance between u and v in G which is 3.

Subcase 2. $v_j \neq v_1$

Without loss of generality assume that $v_j = v_2$. Then by our assumption, $\mu_2(v_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_5)$,

Let u or $v \in V(H_1)$. If at least one of the vertices u and $v \in \{w_{12}, w_{22}\}$, then all the $u - v$ paths have strength $\mu_2(v_2)$. So the length of any extra strong $u - v$ path in G is ≤ 3. If u and $v \notin \{w_{12}, w_{22}\}$ then all the extra strong $u - v$ paths lie in the graph H in Figure 9, which is obtained by deleting the vertices w_{12}, w_{22} from G.

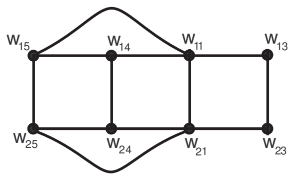

Figure 9. A fuzzy subgraph H of G

In this case if u and $v \in V(H_1)$ then either $u = w_{13}$, and $v = w_{21}$ or $u = w_{11}$ and $v = w_{23}$. In both these cases if a path joining u and v pass through a vertex of H_2 then it must pass through w_{11} and w_{21} and any such path have strength $\leq \mu(w_{11}) \wedge \mu(w_{21})$. Thus each extra strong path lies in the maximal partial fuzzy subgraph with vertex set $\{w_{11}, w_{21}, w_{13}, w_{23}\}$. Hence the length of the extra strong $u - v$ path is 2 by Theorem 2.12. Now suppose u and $v \in V(H_2)$, if any of the $u - v$ path through w_{13} (or w_{23}), definitely will pass through w_{23} (or w_{13}), w_{11} and w_{21}. Any such path has strength $\leq \mu(w_{11}) \wedge \mu(w_{21})$. So every extra strong path lies in H_2. Therefore, the length of any extra strong $u - v$ path is 2.

If $u = w_{13}$ and $v = w_{25}$ then any $u - v$ path in H has length ≥ 3, Also any $u - v$ path through the vertices w_{14} or w_{24} has length > 3 and strength $\leq$ any other $u - v$ path in H. Therefore the strength of the $u - v$ path is the minimum distance between u and v, which is 3. Hence we can conclude that $\mathscr{S}(G) = 3$. □

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Received: March 30, 2017.
Accepted: December 21, 2017.