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On Extended Generalized ϕ-Recurrent α-Kenmotsu Manifolds

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Abstract. In the present paper we study the extended generalized ϕ-recurrent α-Kenmotsu manifolds and discuss its different geometric properties. Among the results established here it is shown that an extended generalized ϕ-recurrent α-Kenmotsu manifold is an Einstein manifold and the curvature tensor have also been calculated. Finally an example of extended generalized ϕ-recurrent α-Kenmotsu manifolds have been constructed.

1 Introduction

A differentiable manifold M of dimension $(2n+1)$ is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(n) \times 1$, ([1], [19]) equivalently an almost contact structure is given by a triplet (ϕ, ξ, η) satisfying certain conditions (see Section 2). Many different type of almost contact structures are defined in the literature (Cosymplectic, almost Cosymplectic, Sasakian, Qusi-Sasakian, α-Kenmotsu, almost α-Kenmotsu,(see [8], [10], [11],[18]). These manifolds appear for the first time in [9], where they have been locally classified.

In 1977, Takahashi [17] introduced the notion of local ϕ-symmetry on Sasakian manifold. Generalizing the notion of local ϕ-symmetry of Takahashi [17], U. C. De, et al.([3], [4]) introduced the notion of ϕ-recurrent Sasakian manifolds and the notion of ϕ-recurrent Kenmotsu manifolds. In ([14], [15], [16]) Shaikh et al. also studied the locally ϕ-symmetry LP-Sasakian and locally ϕ-recurrent $(LCS)_n$-manifolds. Firstly, the notion of generalized recurrent manifold has been introduced by Dubey[7] and after studied by De and Guha[5]. Then again the notion of generalized Ricci-recurrent manifolds has been studied by De et al.[6].

A Riemannina manifold (M^n, g), $n \geq 2$-is called generalized recurrent if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are two non-vanishing 1-forms defined by $A(*) = g(*, \lambda_1)$ and $B(*) = g(*, \lambda_2)$ and the tensor G is defined by

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ being the Lie algebra of smooth vector fields. Here λ_1 and λ_2 are vector fields associated with 1-form A and B respectively. If the 1-form B vanishes, then (1, 1) tensor field turns into the notion of recurrent manifold.

A Riemannina manifold (M^n, g), $n \geq 2$, is called generalized Ricci-recurrent [6] if its Ricci tensor of type $(0,2)$ satisfies the condition

$$\nabla S = A \otimes S + B \otimes g. \tag{1.3}$$

In particular if $B = 0$, then (1.3) reduces to the notion of Ricci-recurrent manifolds [6].

Moreover, Özgür[9] studied generalized recurrent Kenmotsu manifolds and after that Basari and Murathan [2], introduced the notion of generalized ϕ-recurent Kenmotsu manifold. Extending the notion of Basari and Murathan [2] Shaikh et al. ([16], [13]) introduced the notion of extended generalized ϕ-recurrent β-Kenmotsu and LP-Sasakian manifolds. Recently Parakasha

[12] also studied the extended generalized ϕ-recurrent Sasakian manifolds. In [11], Öztürk, Aktan and Murathan studied α-Kenmotsu and generalized recurrent α-Kenmotsu manifolds. Motivated by the above studies, in this article we plan to study the extended generalized ϕ-recurrent α-Kenmostsu manifolds.

The present paper is organized as follows: Section 2 explores the some preliminaries about α-Kenmotsu manifolds. In Section 3 we discuss the extended generalized ϕ-recurrent α-Kenmotsu manifold and we obtain necessary and sufficient condition for such a manifold to be genralized Ricci-recurrent. Further we shown that an extended generalized ϕ-recurrent α-Kenmotsu manifold is Einstien manifold and also obtain curvature tensor R. In the last section the existence of an extended generalized ϕ-recurrent α-Kenmotsu manifold is ensured by an example.

2 Preliminaries

Let M be a real $(2n+1)$-dimensional C^∞-manifold and $\chi(M)$ the Lie algebra of C^∞-vector fields on M. An almost contact structure on M is defined by $(1,1)$-tensor field ϕ, a vector ξ and 1-form η on M such that for any point $p \in M$, we have

$$\phi_p{}^2 = -I + \eta_p \otimes \xi_p, \quad \phi_p(\xi_p) = 0,\ \eta_p \phi_p = 0,\ \eta_p(\xi_p) = 1, \tag{2.1}$$

where I denotes the identity transformation of the tangent space at a point p. Manifolds equipped with an almost contact structure are called almost contact manifolds.

A Riemannian manifold M with metric tensor g and with a triplet (ϕ, ξ, η) such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

and

$$g(\xi, X) = \eta(X) \tag{2.3}$$

is an almost contact metric manifold. Then M is said to have (ϕ, ξ, η, g)-structure.

An almost contact metric manifold M is said to be contact metric manifold. M is said to be α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real number constant, where the 2-from Φ is define as

$$\Phi(X, Y) = g(\phi X, Y). \tag{2.4}$$

We know that an almost contact metric manifold M is said to be normal if the Nijenhuis torsion tensor

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] + 2d\eta(X, Y)\xi,$$

vanishes for any $X, Y \in \chi(M)$. Remarking that a normal almost α-Kenmotsu manifold is said to be α-Kenmotsu manifold $(\alpha) \neq 0)$ [8].

Moreover, if the manifold M satisfies the following relations:

$$(\nabla_X \phi)(Y) = -\alpha[g(X, \phi Y)\xi + \eta(Y)\phi X], \tag{2.5}$$

and

$$\nabla_X \xi = -\alpha\phi^2 X, \tag{2.6}$$

then, $(M^{2n+1}, \phi, \xi, \eta, g)$ is called α-Kenmotsu manifold ([1], [8]), where ∇ denotes the Riemannian connection of g.
On an α-Kenmotsu manifold M, the following relations hold ([8], [19]).

$$(\nabla_X \eta)Y = \alpha g(\phi X, \phi Y), \tag{2.7}$$

$$R(X, Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X], \tag{2.8}$$

(2.9) $$R(\xi, X)Y = \alpha^2[-g(X,Y)\xi + \eta(Y)X],$$

(2.10) $$S(X,\xi) = -2n\alpha^2\eta(X),$$

(2.11) $$R(\xi, X)\xi = \alpha^2[X - \eta(X)\xi] = -\alpha^2\phi^2 X,$$

(2.12) $$g(R(\xi, X)Y, \xi) = \alpha^2[-g(X,Y) + \eta(X)\eta(Y)],$$

(2, 13) $$\eta(R(X,Y)Z) = \alpha^2[-g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]$$

for all $X, Y, Z \in \chi(M)$.

Since $g(QX,Y) = S(X,Y)$, we have

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y),$$

where Q is the Ricci operator. Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, (2.1) and (2,10), we obtain

(2.14) $$S(\phi X, \phi Y) = S(X,Y) + 2n\alpha^2\eta(X)\eta(Y).$$

Also, we have

(2.15) $$(\nabla_X \eta)(Y) = \alpha[g(X,Y) - \eta(X)\eta(Y)]$$

Now, we can state and prove some basic result in an α-Kenmotsu manifold.

Lemma 2.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an α-Kenmotsu manifold. Then for any vector fields X,Y and Z, the following relation holds*

(2.16) $$(\nabla_W R)(X,Y)\xi = \alpha^2[g(\phi W, \phi X)Y - g(\phi W, \phi Y)X] + \alpha R(X,Y)\phi^2 W$$

for any vector fields $X, Y, Z, W \in \chi(M)$.

Proof. Using (2.6), (2.7) and (2.8), we can easily obtain (2.16). □

Lemma 2.2. *In a Riemannian manifold (M^n, g) the following relation holds*

(2.17) $$g((\nabla_W R(X,Y)Z, U) = -g((\nabla_W R(X,Y)U, Z)$$

for any vector fields $X, Y, Z, W \in \chi(M)$.

Proof. It is easy and obvious and hence we omit the proof. □

3 Extended generalized ϕ-recurrent α-Kenmotsu manifold

Definition 3.1. An α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is said to be an extended generalized ϕ-recurrent α-Kenmotsu manifold if its curvature tensor R satisfies the following relation

(3.1) $$\phi^2((\nabla_W R(X,Y)Z) = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2(G(X,Y)Z)$$

for any vector fields $X, Y, Z, W \in \chi(M)$, where A and B are two non-vanishing 1-form such that $A(X) = g(X, \lambda_1)$ and $B(X) = g(X, \lambda_2)$. Here λ_1, λ_2 are vector fields associated with 1-forms A and B recpectively.

Theorem 3.1. *An extended generalized ϕ-recurrent α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is generalized Ricci-recurrent if and only if the sum of associated* 1*-froms A and B is zero.*

Proof. Let us consider an extended generalized ϕ-recurrent α-Kenmotsu manifold. Then using (2.1), in (3.1), we have

$$-(\nabla_W R)(X,Y)Z + \eta(\nabla_W R(X,Y)Z)\xi = A(W)[-R(X,Y)Z + \eta(R(X,Y)Z)\xi] + B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi], \tag{3.2}$$

from which it follows that

$$-g((\nabla_W R(X,Y)Z,U) + \eta((\nabla_W R(X,Y)Z)\eta(U)) = A(W)[-g(R(X,Y)Z,U) + \eta((R(X,Y)Z)\eta(U)] + B(W)[-g(G(X,Y)Z,U) + \eta((G(X,Y)Z)\eta(U)]. \tag{3.3}$$

Let $e_i : i = 1, 2, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of manifold M.
Setting $X = U = e_i$ in (3.3) and taking summation over i, $1 \le i \le 2n+1$, and then using (1.2), we get

$$-(\nabla_W S)(Y,Z) + g((\nabla_W R)(\xi,Y)Z,\xi) = A(W)[-S(Y,Z) + \eta(R(\xi,Y)Z)] + B(W)[-(2n-1)g(Y,Z) - \eta(Y)\eta(Z)]. \tag{3.4}$$

Using (2.8) , (2.16) and (2.17), we have

$$g((\nabla_W R)(\xi,Y)Z,\xi) = 0. \tag{3.5}$$

By the virtue of (2.9) and (3.5), it follows from (3.4) that

$$(\nabla_W S)(Y,Z) = A(W)S(Y,Z) + [(2n-1)B(W) - A(W)]g(Y,Z) + [A(W) + B(W)\eta(Y)\eta(Z)]. \tag{3.6}$$

If $A(W) + B(W) = (A+B)(W) = 0$, that is, the sum of associated 1-forms A and B is zero, then (3.6) reduces to

$$\nabla S = A \otimes S + \psi \otimes g, \tag{3.7}$$

where $\psi(W) = 2nB(W)$ for all $W \in \chi(M)$.
□

Theorem 3.2. An extended generalized ϕ-recurrent α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is an Einstein manifold. Moreover the associated 1-forms A and B are related by $A + B = 0$.

Proof. Setting $Z = \xi$ in (3.6) , using (2.3) and (2.10), we obtain

$$(\nabla_W S)(Y,\xi) = 2nA(W) + B(W)\eta(Y). \tag{3.8}$$

Also, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{3.9}$$

Using (2.7) and (2.10) in (3.9), we get

$$(\nabla_W S)(Y,\xi) = -2n\alpha^3 g(\phi W, \phi Y) - S(Y, -\alpha\phi^2 W). \tag{3.10}$$

By (3.8) and (3.10) we have

$$-2n\alpha^3 g(\phi W, \phi Y) - S(-\alpha\phi^2 W, Y) = 2nA(W) + B(W)\eta(Y). \tag{3.11}$$

Again setting $Y = \xi$ in (3.11), we get

$$A(W) + B(W) = 0, \quad \text{for all W} \in \chi(M). \tag{3.12}$$

By taking account of (3.12) in (3.11), we have

$$\alpha S(\phi^2 W, Y) = -2n\alpha^3 g(\phi W, Y) \tag{3.13}$$

Further, using (2.1) and (2.2), we get

$$\alpha S(W, Y) = -2n\alpha^3 g(W, Y). \tag{3.13}$$

or

$$S(W, Y) = -2n\alpha^2 g(W, Y). \tag{3.14}$$

From (3.12) and (3.14), the theorem follows.

It is known that an α-Kenmotsu manifold is Ricci-semi symmetric if and only if it is an Einstein manifold. From Theorem (3.2), we have the following.
□

Corollary 3.1. An extended generalized ϕ-recurrent α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is Ricci-semi symmetric.

Theorem 3.2. In an extended generalized ϕ-recurrent α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the eigen value of Ricci tensor S corresponding to the eigen vector λ_1 is

$$\frac{r - 2n\alpha^2(2n-1)}{2}.$$

Proof. Changing W,X,Y cyclically in (3.3) and adding them, we get by virtue of Bianchi identity and (3.12) that

$$\begin{aligned} &A(W)[\{g(R(X,Y)Z,U) - g(G(X,Y)Z,U)\} \\ &+ \{\eta(R(X,Y)Z) - \eta(G(X,Y)Z)\}\,\eta(U)] \\ &+ A(X)[\{g(R(Y,W)Z,U) - g(G(Y,W)Z,U)\} \\ &+ \{\eta(R(Y,W)Z) - \eta(G(Y,W)Z)\}\,\eta(U)] \\ &+ A(Y)[\{g(R(W,X)Z,U) - g(G(W,X)Z,U)\} \\ &+ \{\eta(R(W,X)Z) - \eta(G(W,X)Z)\}\,\eta(U)] = 0. \end{aligned} \tag{3.15}$$

Setting $Y = Z = e_i$ in (3.15) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} A(W)[S(X,U) - 2n\alpha^2 g((X,U)] - A(X)[S(U,W) - 2n\alpha^2 g((U,W)] \\ - A(R(W,X)U) - -A(R(W,X)\xi)\eta(U)) - A(X)g(W,U) \\ +A(W)g(X,U) - \{A(X)\eta(W) - A(W)\eta(X)\} = 0. \end{aligned}$$

Again setting $X = U = e_i$ in the above relation and taking summation over i, $1 \leq i \leq 2n+1$, we have

$$S(W, \lambda_1) = \frac{r - 2n\alpha^2(2n-1)}{2} g(W, \lambda_1).$$

This proves the theorem.
□

Theorem 3.4. An α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is an extended generalized ϕ-recurrent, if and only if the following relation holds:

$$(\nabla_W R)(X,Y)Z = \alpha^3[\{g(\phi W, \phi X)g(Y,Z) - g(\phi W, \phi Y)g(X,Z)\} - \alpha g(R(X,Y)W,Z) + \alpha\eta(W)g(R(X,Y)Z,\xi) + A(W)[R(X,Y)Z - \eta(R(X,Y)Z)\xi)] + B(W)[G(X,Y)Z - \eta(G(X,Y)Z,\xi)] \tag{3.16}$$

Proof. Using (2.16) and (2.17) in (3.2), we have (3.16). Conversely, applying ϕ^2 on both sides of (3.16), we get the relation (3.1). □

Theorem 3.5. In an extended generalized ϕ-recurrent α-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, the curvature tensor is of the form

$$\alpha R(X,Y)W = \alpha^3[g(\phi W, \phi X)Y - g(\phi W, \phi Y)X] + \alpha^2[(\alpha Y)\eta(W)\eta(X) - X]\eta(W)\eta(Y)] + [\alpha^2 A(W) + B(W)][-\eta(X)Y - \eta(Y)X]. \tag{3.17}$$

Proof. Setting $Z = \xi$ in (3.2), we get

$$(\nabla_W R)(X,Y)\xi = A(W)R(X,Y)\xi + B(W)G(X,Y)\xi. \tag{3.18}$$

By the virtue of (2.8) and (1.2), the above equation gives

$$(\nabla_W R)(X,Y)\xi = \alpha^3[g(\phi W, \phi X)Y - g(\phi W, \phi Y)X] - \alpha R(X,Y)\xi + \alpha\eta(W)R(X,Y)\xi. \tag{3.19}$$

From (2.16) and (3.19), we obtain (3.17).

4 Example of Extended Generalized ϕ-recurrent α-Kenmotsu manifold

Let us consider the manifold $M = \{(x,y,z) \in \Re^3\}$, where (x,y,z) are the standard coordinates in $\Re^3$. The basis are

$$e_1 = (k_1 e^{-\alpha}\frac{\partial}{\partial x} + k_2 e^{-\alpha z}\frac{\partial}{\partial y}), e_2 = (k_1 e^{-\alpha} z\frac{\partial}{\partial y} - k_2 e^{-\alpha} z\frac{\partial}{\partial y}), e_3 = \frac{\partial}{\partial z}$$

,

where $k_1^2 + k_2^2 \neq 0$, $\alpha \neq 0$ for constant k_1, k_2 and α. Here $\{e_1, e_2, e_3\}$ are linearly independent at each point of M. The Riemannian metric is defined as

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \quad g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M and ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \quad \eta(e_3) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$$[e_1, e_3] = \alpha e_1, \quad [e_2, e_3] = \alpha e_2, \quad [e_1, e_2] = 0.$$

Using Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z])$$

$$- g(Y,[X,Z]) + g(Z,[X,Y]).$$

Koszul's formula yields

$$\nabla_{e_1} e_1 = \alpha e_3, \qquad \nabla_{e_1} e_2 = -e_3, \qquad \nabla_{e_1} e_3 = \alpha e_1$$

$$\nabla_{e_2} e_1 = -e_3, \qquad \nabla_{e_2} e_2 = -\alpha e_3, \qquad \nabla_{e_2} e_3 = \alpha e_2$$

$$\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = 0$$

Thus it can be seen that M is an α-Kenmotsu manifold. Hence by simple calculation we can obtain the curvature tensor components

$$R(e_1,e_2)e_1 = \alpha(\alpha e_2 - e_1), \qquad R(e_1,e_2)e_2 = \alpha(e_2 - \alpha e_1),$$

$$R(e_1,e_2)e_3 = 0, \qquad R(e_1,e_3)e = \alpha^2 e_3,$$

$$R(e_1,e_3)e_2 = \alpha e_3, \qquad R(e_1,e_3)e_3 = -\alpha^2 e_1,$$

$$R(e_2,e_3)e_1 = \alpha e_3, \qquad R(e_2,e_3)e_2 = \alpha^2 e_3,$$

$$R(e_2,e_3)e_3 = -\alpha^2 e_2$$

and the components which can be obtained from these symmetry properties. Since $\{e_1,e_2,e_3\}$ form a basis of the α-Kenmotsu manifold, any vector field $X,Y,Z \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3,$$

$$Y = a_2e_1 + b_2e_2 + c_2e_3,$$

$$Z = a_3e_1 + b_3e_2 + c_3e_3,$$

noindent where $a_i, b_i, c_i \in \Re^+$ (the set of all positive real numbers), $i = 1,2,3$. Then

$$R(X,Y)Z = \alpha(e_2 - \alpha e_1)[(a_1b_2 - a_2b_1)b_3 + 3(a_1c_2 - a_2c_1)c_3] + \alpha(\alpha e_2 - e_1)[(a_1b_2 - a_2b_1)a_3 - 3(b_1c_2 - a_bc_1)b_3] + \alpha^2 e_3[(a_1c_2 - a_2c_1)a_3 + (b_1c_2 - b_2c_1)b_3], \tag{4.1}$$

$$G(X,Y)Z = (a_2a_3 + b_2b_3 + c_2c_3)(a_1e_1 + b_1e_2 + c_1e_3) - (a_2a_3 + b_1b_3 + c_1c_3)(a_2e_1 + b_2e_2 + c_2e_3). \tag{4.2}$$

By the virtue of (4.1) we have the following

$$(\nabla_{E_1} R)(X,Y)Z = -2\alpha^2(5b_1c_2 - b_2c_1)b_3e_3 - 10\alpha^2(a_1b_2 - a_2b_1)b_3e_3 - 2\alpha^2 e_3[5(a_1b_1 - a_2b_1)c_3 + (5b_1 - c_2 - b_2c_1)a_3]e_2, \tag{4.3}$$

$$(\nabla_{E_2} R)(X,Y)Z = -10\alpha^3 e_3[(a_1b_2 - a_2b_1)c_3 - (a_1c_2 - a_2c_1)b_3]e_1 - 10\alpha^3 e_3((a_1c_2 - a_2c_1)a_3 + 10\alpha^3(a_1b_2 - a_2b_1)a_3]e_3, \tag{4.4}$$

$$(\nabla_{E_3} R)(X,Y)Z = 0. \tag{4.5}$$

From (4.1) and (4.2), we get

$$\phi^2(R(X,Y)Z) = r_1e_1 + r_2e_2$$

and

$$\phi^2(G(X,Y)Z) = s_1e_1 + s_2e_2,$$

where

$$r_1 = \alpha(\alpha e_2 - e_1)[2(a_1b_2 - a_2b_1)b_3 + (a_1c_2 - a_2c_1)c_3],$$

$$r_2 = \alpha(e_2 - \alpha e1)[2(a_1b_2 - a_2b_1)b_3 - (b_1c_2 - b_2c_1)c_3],$$

$$s_1 = a_2(b_1b_3 + c_1c_3) - a_1(b_2b_3 + c_2c_3),$$
$$s_2 = b_2(a_1a_3 + c_1c_3) - b_1(a_2a_3 + c_2c_3).$$

Also from (4.3)-(4.5), we obtain

$$\phi^2((\nabla_{E_i}R)(X,Y)Z) = u_ie_1 + v_ie_2 \tag{4.6}$$

for $i = 1, 2, 3$, where

$$u_1 = -2\alpha^2 e_3(5b_1c_2 - b_2c_1)b_3, \tag{4.7}$$
$$v_1 = -2\alpha^2 e_3[(5a_1b_1 - a_2b_1)c_3 + (5b_1c_2 - b_2c_1)a_3],$$
$$u_2 = -10\alpha^3 e_3[(a_1b_2 - a_2b_1)c_3 - (a_1c_2 - a_2c_1)b_3],$$
$$v_2 = -10\alpha^3 e_3(a_1c_2 - a_2c_1)a_3,$$
$$u_3 = 0, \quad v_3 = 0.$$

Let us consider the 1-forms

$$A(e_i) = \frac{s_1u_i - s_2v_i}{r_1s_1 - r_2s_2} \text{ and } B(e_i) = \frac{r_1v_i - r_2u_i}{r_1s_1 - r_2s_2} \tag{4.8}$$

such that $r_1s_1 - r_2s_2 \neq 0$, $s_1u_i - s_2v_i \neq 0$ and $r_1v_i - r_2u_i \neq 0$ for $i = 1, 2, 3$ from (3.1), we have

$$\phi^2((\nabla_{e_i}R)(X,Y)Z) = A(e_i)\phi^2(R(X,Y)Z) + B(e_i)\phi^2(G(X,Y)Z)$$

for $i = 1, 2, 3$.
By the virtue of (4.6)-(4.8), it can be easily shown that the manifold satisfies the relation (4.8). Hence the manifold under consideration is a extended generalized ϕ-recurrent α-Kenmotsu manifold, which is neither ϕ-recurrent nor generalized ϕ-recurrent.

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