Palestine Journal of Mathematics

Vol. 8(1)(2019) , 169–183

WRIGHT GENERALIZED HYPERGEOMETRIC INEQUALITIES OF UNIVALENT HARMONIC MAPPINGS DEFINED BY SHEARING OF ANALYTIC FUNCTIONS

V. K. Gupta and P. Sharma

Communicated by Ayman Badawi

MSC 2010 Classifications: 30C45, 30C55.

Keywords and phrases: Univalent harmonic functions, Convex functions, Wright's generalized hypergeometric function, Subordination.

Abstract. In this paper, using the shear construction method, Inequalities that are both necessary and sufficient for the harmonic shears of analytic functions involving Wright's generalized hypergeometric functions are derived. As in special case, some inequalities for harmonic shears of analytic functions involving generalized hypergeometric functions are also obtained.

1 Introduction and preliminaries

Let $\mathcal{S}_H$ denotes a class of functions f which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ and are normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Since Δ is simply connected, a function $f \in \mathcal{S}_H$ has the canonical representation given by $h + \overline{g}$, where h and g are the members of linear space $A(\Delta)$ of all analytic functions in Δ and where h and g can be written as a power series representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{1.1}$$

We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for a harmonic function of the form $f = h + \overline{g}$ to be locally univalent and sense preserving in Δ is that $|g'(z)| < |h'(z)|$ for all z in Δ. The analytic dilatation of a harmonic mapping $f = h + \overline{g}$ is defined by $\omega(z) = (g'(z)/h'(z))$. Thus if f is locally univalent and sense preserving, then $|\omega(z)| < 1$ in Δ.

A subclass $T\mathcal{S}_H$ of $\mathcal{S}_H$ is well known in the literature. A function $f = h + \overline{g}$ is said to be in the class $T\mathcal{S}_H$ if h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = \sum_{n=1}^{\infty} |b_n| z^n, |b_1| < 1. \tag{1.2}$$

In case $g(z) = 0$, $\forall z \in \Delta$, the class $\mathcal{S}_H$ reduces to a well known class $\mathcal{S}$ of univalent functions and the class $\mathcal{TS}_H$ reduces to $\mathcal{T}$ introduced and studied by Silverman [18, 19]. We further denote a subclass $\mathcal{TS}_H^0$ of $\mathcal{TS}_H$ for which $f_{\overline{z}}(0) = 0$.

A domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex in the direction α $(0 \le \alpha < 2\pi)$, if for all $a \in \mathbb{C}$, the set $\mathbb{D} \cap \{a + te^{i\alpha} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex in the horizontal direction (or a CHD domain) if its intersection with each horizontal line is connected (or empty). The domains which are convex in every direction are called convex domains.

We say a univalent harmonic function f is convex in the direction α $(0 \le \alpha < 2\pi)$ if the domain $f(\mathbb{D})$ is convex in the direction α. In particular, a univalent harmonic function f is called a CHD map if its range is a CHD domain.

Construction of a univalent harmonic mapping f with prescribed dilatation ω can be done effectively by a method known as the "*shear construction*" method which was devised by Clunie and Sheil-Small [7] (see also [8, 9, 10, 14]). The basic shear construction theorem of a harmonic univalent function discovered by Clunie and Sheil-Small [7] is as follows.

Theorem A: For analytic functions h and g, assume the harmonic function $f = h + \overline{g}$ is locally univalent in a simply connected domain $\mathbb{D}$. Then a univalent function f maps $\mathbb{D}$ onto a CHD domain if and only if the analytic function $h - g$ is univalent and maps $\mathbb{D}$ onto a CHD domain.

For more details on "shear construction" method one may refers [5, 7, 8, 9, 10, 14, 17].

We also have following result of Clunie and Sheil-Small [7].

Theorem B: A functions $f = h + \overline{g}$ is harmonic convex if and only if the analytic functions $h - e^{i\alpha}g, 0 \le \alpha < 2\pi$, are convex in the direction $\frac{\alpha}{2}$ and f is suitably normalized.

The following two subclasses $\mathcal{T}[A,B]$ and $\mathcal{C}[A,B]$ of the class T introduced and studied by Silverman [18, 19].

Definition 1.1. [15] A function $h \in T$ of the form given in (1.2) is said to be in $\mathcal{T}[A,B]$ if, for some constant A and B such that $-1 \le B < A \le 1$, it satisfies

$$\sum_{n=2}^{\infty}\left\{(n-1)\frac{1-B}{A-B}+1\right\}|a_n| \le 1;$$

and is said to be in the class $\mathcal{C}[A,B]$, if $zh' \in \mathcal{T}[A,B]$.

It was observed in [15] that the functions of the classes $\mathcal{T}[A,B]$ and $\mathcal{C}[A,B]$ are univalent. Note that the class $\mathcal{T}[1,-1] = T^*$ was studied in [18, 19].

Adopting the "shear construction" method, introduced by Clunie and Sheil-Small [7] (see also [8, 9, 10, 14]), Sharma, Ahuja and Gupta in 2014 defined two classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$ as follows:

Definition 1.2. [17] Let a function ϕ_α defined by

$$\phi_\alpha(z) = H_\alpha(z) - e^{2i\alpha}G_\alpha(z) \tag{1.3}$$

be convex in the direction $\alpha \in \{0, \pi/2\}$, where

$$H_\alpha(z) = z - \sum_{n=2}^{\infty}\frac{|a_n|}{1-e^{2i\alpha}|b_1|}z^n,\ G_\alpha(z) = \sum_{n=2}^{\infty}\frac{|b_n|}{1-e^{2i\alpha}|b_1|}z^n \tag{1.4}$$

are analytic in Δ, $|b_1| < 1$ and $\alpha \in \{0, \pi/2\}$). Then the harmonic shear $F_\alpha = H_\alpha + \overline{G_\alpha}$ of ϕ_α, is said to be in the class $\mathcal{T}_H[A,B]$ if $\phi_\alpha \in \mathcal{T}[A,B]$. Further, we say that $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in the class $\mathcal{C}_H[A,B]$ if $z\phi'_\alpha(z) \in \mathcal{T}[A,B]$.

They [17] also observe that the analytic function ϕ_α considered in (1.3) may also be expressed as

$$\phi_\alpha(z) = \frac{h(z) - e^{2i\alpha}g(z)}{1-e^{2i\alpha}|b_1|}$$

where h and g are of the form (1.2).

Here it is worth mentioning that for a CHD map ϕ_0 defined by

$$\phi_0(z) = H_0(z) - G_0(z),$$

where

$$H_0(z) = z - \sum_{n=2}^{\infty}\frac{|a_n|}{1-|b_1|}z^n, G_0(z) = \sum_{n=2}^{\infty}\frac{|b_n|}{1-|b_1|}z^n \tag{1.5}$$

are analytic in Δ, $|b_1| < 1$, there exists a dilatation ω_0, such that the harmonic shear $F_0 = H_0 + \overline{G_0}$ of ϕ_0 may be obtained by solving the differential equations:

$$H_0' - G_0' = \phi_0',\ \omega_0 H_0' - G_0' = 0.$$

Also, for a map $\phi_{\pi/2}$ convex in vertical direction, defined by

$$\phi_{\pi/2}(z) = H_{\pi/2}(z) + G_{\pi/2}(z),$$

where

$$H_{\pi/2}(z) = z - \sum_{n=2}^{\infty} \frac{|a_n|}{1+|b_1|} z^n, G_{\pi/2}(z) = \sum_{n=2}^{\infty} \frac{|b_n|}{1+|b_1|} z^n \tag{1.6}$$

are analytic in Δ, there exists a dilatation $\omega_{\pi/2}$, such that the harmonic shear $F_{\pi/2} = H_{\pi/2} + \overline{G_{\pi/2}}$ of $\phi_{\pi/2}$ may be obtained by solving the differential equations:

$$H'_{\pi/2} + G'_{\pi/2} = \phi'_{\pi/2},\ \omega_{\pi/2} H'_{\pi/2} - G'_{\pi/2} = 0.$$

The Wgh functions have an increasingly significant role in various types of applications (see [20, 21]). Generalized hypergeometric functions, generalized Mittag-Leffler functions and Bessel-Maitland (Wright generalized Bessel) functions are some special cases of Wgh functions; one may refer to [22, 23]. Several results on harmonic functions by involving hypergeometric functions have recently been studied in [1] to [4]. Involvement of the Wright generalized hypergeometric function (Wgh) in the harmonic functions has recently been investigated amongst others in [6, 12, 13, 16].

Let $A_i > 0$ $(i = 1, ..., p)$ and $B_i > 0$ $(i = 1, ..., q)$ such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$. Following the definition and terminology in [20, 22, 24], a Wright's generalized hypergeometric (Wgh) function for non-negative integers p and q, $\alpha_i \in \mathbb{C}$ $\left(\frac{\alpha_i}{A_i} \neq 0, -1, -2, ...; i = 1, ..., p\right)$ and $\beta_i \in \mathbb{C}$ $\left(\frac{\beta_i}{B_i} \neq 0, -1, -2, ...; i = 1, ..., q\right)$ is defined by

$${}_p\psi_q\left([(\alpha_i, A_i)]\,; z\right) \equiv {}_p\psi_q\left[\begin{pmatrix}(\alpha_i, A_i)_{1,p}\\(\beta_i, B_i)_{1,q}\end{pmatrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{q}\Gamma(\beta_i + nB_i)} \frac{z^n}{n!}, z \in \Delta. \tag{1.7}$$

By involving Wgh functions as defined by (1.7), consider an analytic function $\Phi_1(z)$ defined by

$$\Phi_1(z) = \frac{W_1(z) - e^{2i\alpha} W_2(z)}{1 - e^{2i\alpha} d_1}, z \in \Delta \tag{1.8}$$

where

$$W_1(z) = z \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\, {}_p\psi_q\left[\begin{pmatrix}(\alpha_i, A_i)_{1,p}\\(\beta_i, B_i)_{1,q}\end{pmatrix}; z\right], \tag{1.9}$$

$$W_2(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\, {}_r\psi_s\left[\begin{pmatrix}(\gamma_i, C_i)_{1,r}\\(\delta_i, D_i)_{1,s}\end{pmatrix}; z\right] - 1 \tag{1.10}$$

and

$$d_1 = \frac{\prod_{i=1}^{r}(\gamma_i)_{C_i}}{\prod_{i=1}^{s}(\delta_i)_{D_i}} \tag{1.11}$$

for positive integers $A_i, B_i, C_i,$ and D_i and for $\alpha_i > -A_i$ $(i = 1, ..., p)$, satisfying $\prod_{i=1}^{p}(\alpha_i)_{A_i} < 0$, and $\beta_i. > 0$ $(i = 1, ..., q)$, $\gamma_i > 0$ $(i = 1, ..., r)$, $\delta_i > 0$ $(i = 1, ..., s)$ with

$$\frac{\prod_{i=1}^{r}(\gamma_i)_{nC_i}}{\prod_{i=1}^{s}(\delta_i)_{nD_i}} < \frac{n\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right|\prod_{i=1}^{p}(\alpha_i + A_i)_{(n-2)A_i}}{\prod_{i=1}^{q}(\beta_i)_{(n-1)B_i}}, n \geq 2; \frac{\prod_{i=1}^{r}(\gamma_i)_{C_i}}{\prod_{i=1}^{s}(\delta_i)_{D_i}} < 1.$$

In view of the parametric constraints cosidered above and $\prod_{i=1}^{p}(\alpha_i)_{A_i} < 0$, we have

$$\prod_{i=1}^{p}\Gamma(\alpha_i) = \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}{\prod_{i=1}^{p}(\alpha_i)_{A_i}} = -\frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right|}$$

and hence, the function $\Phi_\alpha(z)$ defined by (1.8) may also be written in the form

$$\Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \tag{1.12}$$

where

$$\mathcal{H}_\alpha(z) = z - \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right| \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)} \sum_{n=2}^{\infty} \frac{\theta_n}{1 - e^{2i\alpha}d_1} z^n, \tag{1.13}$$

$$\mathcal{G}_\alpha(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} \sum_{n=2}^{\infty} \frac{\phi_n}{1 - e^{2i\alpha}d_1} z^n \tag{1.14}$$

and

$$\theta_n = \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + (n-1)A_i)}{\prod_{i=1}^{q}\Gamma(\beta_i + (n-1)B_i)} \frac{1}{(n-1)!}, \quad \phi_n = \frac{\prod_{i=1}^{r}\Gamma(\gamma_i + nC_i)}{\prod_{i=1}^{s}\Gamma(\delta_i + nD_i)} \frac{1}{n!}, \tag{1.15}$$

and d_1 is given by (1.11). Using $\Phi_\alpha(z)$ defined by (1.12), we get a harmonic shear $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha}$ and obtain following results.

Based on the above defined classes, Sharma and gupta [17] proved following results observing various equivalent class conditions considered in [17], we mention these results in form of following Lemmas.

Lemma 1.3. *Under the parametric conditions stated as above, let $\mathcal{H}_\alpha$ and $\mathcal{G}_\alpha$, respectively, be functions of the form (1.13) and (1.14) with θ_n, ϕ_n given by (1.15). Let*

$$\Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A, B]$$

be convex in the direction $\alpha \in \{0, \pi/2\}$ and let $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ be its harmonic shear, convex in the same direction α. Then $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{T}_H[A, B]$ if and only if the inequality

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right| \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \theta_n \\ &+ e^{2i\alpha} \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \phi_n \\ \leq\ & 1 \end{aligned} \tag{1.16}$$

is satisfied.

Lemma 1.4. *Under the hypothesis of Theorem 1.3, the function* $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{C}_H[A,B]$ *if and only if the inequality*

$$\sum_{n=2}^{\infty} \frac{\left|\prod_{i=1}^{p} (\alpha_i)_{A_i}\right| \prod_{i=1}^{q} \Gamma(\beta_i)}{\prod_{i=1}^{p} \Gamma(\alpha_i + A_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n \tag{1.17}$$

$$+e^{2i\alpha}\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n$$

$$\leq \quad 1$$

is satisfied.

In the following two Lemmas 1.5 and 1.6, we consider an analytic function $\Psi_\alpha(z)$ defined by

$$\Psi_\alpha(z) = \frac{z\left(2 - \frac{W_1(z)}{z}\right) - e^{2i\alpha}W_2(z)}{1 - e^{2i\alpha}d_1} \quad (z \in \mathbb{U}), \tag{1.18}$$

where $W_1(z)$ and $W_2(z)$ are of the form (1.9) and (1.10), d_1 is given by (1.11) for positive integers A_i, B_i, C_i, D_i and for $\alpha_i > 0 \ (i = 1, ..., p)$, $\beta_i. > 0 \ (i = 1, ..., q)$, $\gamma_i > 0 \ (i = 1, ..., r)$, $\delta_i > 0 \ (i = 1, ..., s)$ with

$$\frac{\prod_{i=1}^{r} (\gamma_i)_{nC_i}}{\prod_{i=1}^{s} (\delta_i)_{nD_i}} < \frac{n\prod_{i=1}^{p} (\alpha_i)_{(n-1)A_i}}{\prod_{i=1}^{q} (\beta_i)_{(n-1)B_i}} \ (n \geq 1), \quad \frac{\prod_{i=1}^{r} (\gamma_i)_{nC_i}}{\prod_{i=1}^{s} (\delta_i)_{nD_i}} < 1.$$

The function $\Psi_\alpha(z)$ may also be written in the form

$$\Psi_\alpha(z) = \mathcal{L}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z), \tag{1.19}$$

where

$$\mathcal{L}_\alpha(z) = z - \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} \sum_{n=2}^{\infty} \frac{\theta_n}{1 - e^{2i\alpha}d_1} z^n, \tag{1.20}$$

$\mathcal{G}_\alpha(z)$, d_1 and θ_n are given by (1.14), (1.11) and (1.15).

Lemma 1.5. *Let* $\mathcal{L}_\alpha$ *and* $\mathcal{G}_\alpha$ *be given by (1.20) and (1.14), respectively, with* θ_n, ϕ_n *given by (1.15) with positive values of* $\alpha_i, \beta_i, \gamma_i$ *and* δ_i. *Let*
$\Psi_\alpha(z) = \mathcal{L}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A,B]$ *be convex in the direction* $\alpha \in \{0, \pi/2\}$ *and*
$\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ *be its harmonic shear, convex in the same direction* α. *Then, the function* $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{T}_H[A,B]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} \left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} \left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \leq 1$$

is satisfied.

Lemma 1.6. *Under the hypothesis of Lemma 1.5,* $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{C}_H[A,B]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \leq 1,$$

is satisfied.

Since, wright's generalized hypergeometric (Wgh) function defined by (1.7) is entire function if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i > 0$, also it is analytic for $|z| < \frac{\prod_{i=1}^{q}(B_i)^{B_i}}{\prod_{i=1}^{p}(A_i)^{A_i}}$ if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$. However if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ and $|z| = \frac{\prod_{i=1}^{q}(B_i)^{B_i}}{\prod_{i=1}^{p}(A_i)^{A_i}}$, then wgh function is analytic for $\Re\left\{\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i\right\} + \frac{p-q}{2} > \frac{1}{2}$ (for more details one may refer to [11]).

Throughout this paper, we consider (1.7) Wgh functions ${}_p\psi_q\left[\left(\begin{smallmatrix}(\alpha_i, A_i)1,p\\(\beta_i, B_i)1,q\end{smallmatrix}\right); z\right]$ and ${}_r\psi_s\left[\left(\begin{smallmatrix}(\gamma_i, C_i)1,r\\(\delta_i, D_i)1,s\end{smallmatrix}\right); z\right]$ with additional condition that A_i, B_i, C_i, and D_i are positive integers satisfying the condition $\prod_{i=1}^{q}(B_i)^{B_i} \geq \prod_{i=1}^{p}(A_i)^{A_i}$, in the case

$$1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0,$$

$\prod_{i=1}^{s}(D_i)^{D_i} \geq \prod_{i=1}^{r}(C_i)^{C_i}$, in the case

$$1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0,$$

which ensure that the wgh functions are defined at $z = 1$.

In this paper, using the shear construction method, Wright's generalized hypergeometric inequalities that are both necessary and sufficient for the harmonic univalent functions $F_\alpha = H_\alpha + \overline{G_\alpha} \in TS_H^0$ which are the harmonic shear of analytic functions for the classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$ are derived which are convex in the direction $\alpha \in \{0, \pi/2\}$ (that is convex in the horizontal direction or vertical direction). Further these necessary and sufficient Wright's generalized hypergeometric inequalities for another harmonic univalent functions $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ for the classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$ are obtained. As in special case, some inequalities for harmonic shears of analytic functions involving generalized hypergeometric functions are also obtained.

2 Main Results

Theorem 2.1. *Let under the hypothesis of Lemma 1.3, $\mathcal{H}_\alpha$ and $\mathcal{G}_\alpha$, respectively be of the form (1.13) and (1.14) and let $\Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A,B]$ be convex in the direction $\alpha \in \{0, \pi/2\}$. Let $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ be the harmonic shear of Φ_α in the same direction α. Then in case*
$\prod_{i=1}^{q}(B_i)^{B_i} = \prod_{i=1}^{p}(A_i)^{A_i}$, $\prod_{i=1}^{s}(D_i)^{D_i} = \prod_{i=1}^{r}(C_i)^{C_i}$, $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$, $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$, *under the validity condition*

$$\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i + \frac{p-q}{2} > \frac{3}{2}, \sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i + \frac{r-s}{2} > \frac{3}{2}, \tag{2.1}$$

$\mathcal{F}_\alpha \in \mathcal{T}_H[A,B]$ if and only if

$$\begin{aligned} &\lambda_1\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i, A_i)];1\right) + {}_p\psi_q\left([(\alpha_i, A_i)];1\right)\right] \\ &+e^{2i\alpha}\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+C_i, C_i)];1\right) + \frac{A-1}{A-B}\,{}_r\psi_s\left([(\gamma_i, C_i)];1\right)\right] \\ \leq\ & e^{2i\alpha}\frac{A-1}{A-B} \end{aligned} \tag{2.2}$$

holds, where

$$\lambda_1 = \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right| \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}, \quad \mu = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}. \tag{2.3}$$

Proof. To show that $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0 \in \mathcal{T}_H[A,B]$, by Lemma 1.3 we need to show

$$S_1 := \sum_{n=2}^{\infty}\lambda_1\left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty}\mu\left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \le 1,$$

where θ_n and ϕ_n are given by (1.15). Under the validity condition (2.1) which ensures the convergence of

$${}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right), {}_p\psi_q\left([(\alpha_i,A_i)];1\right), {}_r\psi_s\left([(\gamma_i+C_i,C_i)];1\right), {}_r\psi_s\left([(\gamma_i,C_i)];1\right),$$

we obtain from (1.15),

$$\sum_{n=2}^{\infty}(n-1)\theta_n = {}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right), \quad \sum_{n=2}^{\infty}\theta_n = {}_p\psi_q\left([(\alpha_i,A_i)];1\right) - \frac{\prod_{i=1}^{p}\Gamma(\alpha_i)}{\prod_{i=1}^{q}\Gamma(\beta_i)} \tag{2.4}$$

and

$$\sum_{n=1}^{\infty}(n-1)\phi_n = {}_r\psi_s\left([(\gamma_i+C_i,C_i)];1\right) - {}_r\psi_s\left([(\gamma_i,C_i)];1\right) + \frac{\prod_{i=1}^{r}\Gamma(\gamma_i)}{\prod_{i=1}^{s}\Gamma(\delta_i)}, \tag{2.5}$$

$$\sum_{n=1}^{\infty}\phi_n = {}_r\psi_s\left([(\gamma_i,C_i)];1\right) - \frac{\prod_{i=1}^{r}\Gamma(\gamma_i)}{\prod_{i=1}^{s}\Gamma(\delta_i)}.$$

Hence, on using (2.4) and (2.5), we get

$$S_1 \le 1$$

if and only if (2.2) holds. □

Theorem 2.2. *Under the same hypothesis of Theorem 2.1, let* $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ *be the harmonic shear of* Φ_α *in the direction* α. *Then in case* ${}_{i=1}^{q}(B_i)^{B_i} = \prod_{i=1}^{p}(A_i)^{A_i}, {}_{i=1}^{s}(D_i)^{D_i} = \prod_{i=1}^{r}(C_i)^{C_i}$, $1+\sum_{i=1}^{q}B_i - \sum_{i=1}^{p}A_i = 0, 1+\sum_{i=1}^{s}D_i - \sum_{i=1}^{r}C_i = 0$, *under the validity condition*

$$\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i + \frac{p-q}{2} > \frac{5}{2}, \quad \sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i + \frac{r-s}{2} > \frac{5}{2} \tag{2.6}$$

$\mathcal{F}_\alpha \in \mathcal{C}_H[A,B]$ *if and only if*

$$\lambda_1\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+2A_i,A_i)];1\right) + \frac{A-3B+2}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right)\right.$$
$$\left. + {}_p\psi_q\left([(\alpha_i,A_i)];1\right)\right] \tag{2.7}$$
$$+e^{2i\alpha}\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+2C_i,C_i)];1\right) + {}_r\psi_s\left([(\gamma_i+C_i,C_i)];1\right)\right]$$
$$\le 0 \tag{2.8}$$

holds, where λ_1 *and* μ *are given by (2.3).*

Proof. To show that $\mathcal{F}_\alpha \in \mathcal{C}_H[A,B]$, by Lemma 1.4 we need to show that

$$S_2 := \sum_{n=2}^{\infty} \lambda_1 n \left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty}\mu n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \le 1,$$

where λ_1 and μ are given by (2.3). After some simple calculations S_2 can be written as

$$\begin{aligned} S_2 &= \sum_{n=2}^{\infty}\lambda_1\left\{(n-2)(n-1)\frac{1-B}{A-B}+(n-1)\frac{A-3B+2}{A-B}+1\right\}\theta_n + \\ &\quad \mu e^{2i\alpha}\sum_{n=1}^{\infty}\left\{n(n-1)\frac{1-B}{A-B}+n\right\}\phi_n. \end{aligned}$$

Under the validity condition (2.6), using (2.4), (2.5) and

$$\begin{aligned} \sum_{n=2}^{\infty}(n-2)(n-1)\theta_n &= {}_p\psi_q\left([(\alpha_i+2A_i, A_i)];1\right), \\ \sum_{n=1}^{\infty}n(n-1)\phi_n &= {}_r\psi_s\left([(\gamma_i+2C_i, C_i)];1\right) \\ \sum_{n=1}^{\infty}n\phi_n &= {}_r\psi_s\left([(\gamma_i+C_i, C_i)];1\right), \end{aligned}$$

we get

$$S_2 \le 1$$

if and only if (2.8) holds. □

Now, we give following Theorems 2.3 and 2.4 giving Wgh inequalities for function $\mathcal{E}_\alpha \in TS_H^0$ considered in Lemma 1.5 to be in $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$, respectively. Proof of these Theorems is similar to the proof of Theorems 2.1 and 2.2 hence, we may omit the proof.

Theorem 2.3. *Let under the hypothesis of Lemma 1.5, $\mathcal{L}_\alpha$ and $\mathcal{G}_\alpha$ be given, respectively, by (1.20) and (1.14). Let $\Psi_\alpha(z) = \mathcal{L}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A,B]$ be convex in the direction $\alpha \in \{0, \pi/2\}$ and $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ be the harmonic shear of $\Psi_\alpha(z)$ in the same direction α. Then in case*
${}_{i=1}^{q}(B_i)^{B_i} = \prod_{i=1}^{p}(A_i)^{A_i}, {}_{i=1}^{s}(D_i)^{D_i} = \prod_{i=1}^{r}(C_i)^{C_i}$,
$1+\sum_{i=1}^{q}B_i - \sum_{i=1}^{p}A_i = 0, 1+\sum_{i=1}^{s}D_i - \sum_{i=1}^{r}C_i = 0$, *under the validity condition*

$$\Re\left\{\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i\right\}+\frac{p-q}{2} > \frac{3}{2}, \Re\left\{\sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i\right\}+\frac{r-s}{2} > \frac{3}{2},$$

$\mathcal{E}_\alpha \in \mathcal{T}_H[A,B]$ if and only if

$$\begin{aligned} &\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i, A_i)];1\right)+\,{}_p\psi_q\left([(\alpha_i, A_i)];1\right)\right] \\ &+e^{2i\alpha}\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+C_i, C_i)];1\right)+\frac{A-1}{A-B}\,{}_r\psi_s\left([(\gamma_i, C_i)];1\right)\right] \\ \le\ & 2+\frac{A-1}{A-B}e^{2i\alpha} \end{aligned}$$

holds, where

$$\lambda_2 = \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} \tag{2.9}$$

and μ is given by (2.3).

Theorem 2.4. *Let under the hypothesis of Theorem 2.3, $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in TS_H^0$ be the harmonic shear of Ψ_α in the same direction α. Then in case*
$_{i=1}^{q}(B_i)^{B_i} = \prod_{i=1}^{p}(A_i)^{A_i}, {}_{i=1}^{s}(D_i)^{D_i} = \prod_{i=1}^{r}(C_i)^{C_i}$,
$1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0, 1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$, *under the validity condition*

$$\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i + \frac{p-q}{2} > \frac{5}{2}, \sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i + \frac{r-s}{2} > \frac{5}{2},$$

$\mathcal{E}_\alpha \in \mathcal{C}_H[A,B]$ if and only if

$$\begin{aligned}
&\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left(\left[(\alpha_i+2A_i,A_i)\right];1\right)+\frac{A-3B+2}{A-B}\,{}_p\psi_q\left(\left[(\alpha_i+A_i,A_i)\right];1\right)\right.\\
&\left.+\,{}_p\psi_q\left(\left[(\alpha_i,A_i)\right];1\right)\right]\\
&+e^{2i\alpha}\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left(\left[(\gamma_i+2C_i,C_i)\right];1\right)+{}_r\psi_s\left(\left[(\gamma_i+C_i,C_i)\right];1\right)\right]\\
&\leq 2
\end{aligned}$$

holds, where λ_2 and μ are given respectively, by (2.9) and (2.3).

Now in particular, if $\alpha = 0$, the function $\Phi_\alpha(z)$ defined by (1.8) will be denoted by

$$\Phi_0(z) = \mathcal{H}_0(z) - \mathcal{G}_0(z), \tag{2.10}$$

where

$$\mathcal{H}_0(z) = z - \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right| \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i)}\sum_{n=2}^{\infty}\frac{\theta_n}{1-d_1}z^n,\ \mathcal{G}_0(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\sum_{n=2}^{\infty}\frac{\phi_n}{1-d_1}z^n \tag{2.11}$$

and d_1 is given by (1.11). Using $\Phi_0(z)$ defined by (2.10), we get a harmonic shear $\mathcal{F}_0 = \mathcal{H}_0 + \overline{\mathcal{G}_0}$.

Also, taking $\alpha = \pi/2$, the function $\Phi_\alpha(z)$ defined by (1.8) is denoted by

$$\Phi_{\pi/2}(z) = \mathcal{H}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z), \tag{2.12}$$

where

$$\mathcal{H}_{\pi/2}(z) = z - \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right| \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i)}\sum_{n=2}^{\infty}\frac{\theta_n}{1+d_1}z^n,\ \mathcal{G}_{\pi/2}(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\sum_{n=2}^{\infty}\frac{\phi_n}{1+d_1}z^n \tag{2.13}$$

and d_1 is given by (1.11). Using $\Phi_{\pi/2}(z)$ defined by (2.12), we get a harmonic shear $\mathcal{F}_{\pi/2} = \mathcal{H}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}}$.

For the harmonic shear $\mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$, we get following results from Theorems 2.1 and 2.2 on taking $\alpha = 0$ and $\alpha = \pi/2$, for CHD map and for the map convex in vertical direction.

Corollary 2.5. *Under the hypothesis of Lemma 1.3, with $\alpha = 0, \mathcal{H}_0$ and $\mathcal{G}_0$be of the form (2.11). Let $\Phi_0(z) = \mathcal{H}_0(z) - \mathcal{G}_0(z) \in \mathcal{T}[A,B]$ be convex in the horizontal direction. Let $\mathcal{F}_0 = \mathcal{H}_0 + \overline{\mathcal{G}_0} \in TS_H^0$ be the harmonic shear of $\Phi_0(z)$ in the same direction. Then $\mathcal{F}_0 \in \mathcal{T}_H[A,B]$ if and only if*

$$\begin{aligned}
&\lambda_1\left[\frac{1-B}{A-B}\,{}_p\psi_q\left(\left[(\alpha_i+A_i,A_i)\right];1\right)+\,{}_p\psi_q\left(\left[(\alpha_i,A_i)\right];1\right)\right]\\
&+\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left(\left[(\gamma_i+C_i,C_i)\right];1\right)+\frac{A-1}{A-B}\,{}_r\psi_s\left(\left[(\gamma_i,C_i)\right];1\right)\right]\\
&\leq \frac{A-1}{A-B}
\end{aligned}$$

holds, where λ_1 and μ are given by (2.3).

Corollary 2.6. *Under the hypothesis of Lemma 1.3, with* $\alpha = \pi/2$, $\mathcal{H}_{\pi/2}$ *and* $\mathcal{G}_{\pi/2}$ *be of the form (2.13). Let* $\Phi_{\pi/2}(z) = \mathcal{H}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z) \in \mathcal{T}[A,B]$ *be convex in the vertical direction. Let* $\mathcal{F}_{\pi/2} = \mathcal{H}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}} \in TS_H^0$ *be the harmonic shear of* $\Phi_{\pi/2}(z)$ *in the vertical direction.Then* $\mathcal{F}_{\pi/2} \in \mathcal{T}_H[A,B]$ *if and only if*

$$\begin{aligned}
&\lambda_1 \left[\frac{1-B}{A-B}\, {}_p\psi_q\left([(\alpha_i + A_i, A_i)]; 1\right) + {}_p\psi_q\left([(\alpha_i, A_i)]; 1\right) \right] \\
&-\mu \left[\frac{1-B}{A-B}\, {}_r\psi_s\left([(\gamma_i + C_i, C_i)]; 1\right) + \frac{A-1}{A-B}\, {}_r\psi_s\left([(\gamma_i, C_i)]; 1\right) \right] \\
\leq\ & \frac{1-A}{A-B}
\end{aligned}$$

holds, where λ_1 *and* μ *are given by (2.3).*

Corollary 2.7. *Under the hypothesis of Lemma 1.3, with* $\alpha = 0$, $\mathcal{H}_0$ *and* $\mathcal{G}_0$ *be of the form (2.11) Let* $\Phi_0(z) = \mathcal{H}_0(z) - \mathcal{G}_0(z) \in \mathcal{T}[A,B]$ *be convex in the horizontal direction. Let* $\mathcal{F}_0 = \mathcal{H}_0 + \overline{\mathcal{G}_0} \in TS_H^0$ *be the harmonic shear of* $\Phi_0(z)$ *in the horizontal direction. Then* $\mathcal{F}_0 \in \mathcal{C}_H[A,B]$ *if and only if*

$$\begin{aligned}
&\lambda_1 \left[\frac{1-B}{A-B}\, {}_p\psi_q\left([(\alpha_i + 2A_i, A_i)]; 1\right) + \frac{A-3B+2}{A-B}\, {}_p\psi_q\left([(\alpha_i + A_i, A_i)]; 1\right) \right. \\
&\left. + {}_p\psi_q\left([(\alpha_i, A_i)]; 1\right) \right] \\
&+\mu \left[\frac{1-B}{A-B}\, {}_r\psi_s\left([(\gamma_i + 2C_i, C_i)]; 1\right) + {}_r\psi_s\left([(\gamma_i + C_i, C_i)]; 1\right) \right] \\
&\leq 0
\end{aligned}$$

holds, where λ_1 *and* μ *are given by (2.3).*

Corollary 2.8. *Under the hypothesis of Lemma 1.3, with* $\alpha = \pi/2$, $\mathcal{H}_{\pi/2}$ *and* $\mathcal{G}_{\pi/2}$ *be of the form (2.13). Let* $\Phi_{\pi/2}(z) = \mathcal{H}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z) \in \mathcal{T}[A,B]$ *be convex in the vertical direction. Let* $\mathcal{F}_{\pi/2} = \mathcal{H}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}} \in TS_H^0$ *be the harmonic shear of* $\Phi_{\pi/2}(z)$ *in the same direction.Then* $\mathcal{F}_{\pi/2} \in \mathcal{C}_H[A,B]$ *if and only if*

$$\begin{aligned}
&\lambda_1 \left[\frac{1-B}{A-B}\, {}_p\psi_q\left([(\alpha_i + 2A_i, A_i)]; 1\right) + \frac{A-3B+2}{A-B}\, {}_p\psi_q\left([(\alpha_i + A_i, A_i)]; 1\right) \right. \\
&\left. + {}_p\psi_q\left([(\alpha_i, A_i)]; 1\right) \right] \\
&-\mu \left[\frac{1-B}{A-B}\, {}_r\psi_s\left([(\gamma_i + 2C_i, C_i)]; 1\right) + {}_r\psi_s\left([(\gamma_i + C_i, C_i)]; 1\right) \right] \\
&\leq 0
\end{aligned}$$

holds, where λ_1 *and* μ *are given by (2.3).*

Further, taking $\alpha = 0$, $\Psi_\alpha(z)$ (1.19) is denoted by

$$\Psi_0(z) = \mathcal{L}_0(z) - \mathcal{G}_0(z), \tag{2.14}$$

where

$$\mathcal{L}_0(z) = z - \frac{\prod_{i=1}^{q} \Gamma(\beta_i)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \sum_{n=2}^{\infty} \frac{\theta_n}{1-d_1} z^n, \tag{2.15}$$

$\mathcal{G}_0(z)$, d_1 and θ_n are given by (1.14), (1.11) and (1.15). Using $\Psi_0(z)$ defined by (2.14), we get a harmonic shear $\mathcal{E}_0 = \mathcal{L}_0 + \overline{\mathcal{G}_0}$.

Also, taking $\alpha = \pi/2$, the function $\Psi_{\pi/2}(z)$ may also be written as

$$\Psi_{\pi/2}(z) = \mathcal{L}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z), \tag{2.16}$$

where

$$\mathcal{L}_{\pi/2}(z) = z - \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\sum_{n=2}^{\infty}\frac{\theta_n}{1+d_1}\, z^n, \tag{2.17}$$

$\mathcal{G}_{\pi/2}(z)$, d_1 and θ_n are given by (1.14), (1.11) and (1.15). Using $\Psi_{\pi/2}(z)$ defined by (2.16), we get a harmonic shear $\mathcal{E}_{\pi/2} = \mathcal{L}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}}$.

For the harmonic shear $\mathcal{E}_0$ and $\mathcal{E}_{\pi/2}$, we get following inequalities from Theorems 2.3 and 2.4, by taking $\alpha = 0$ and $\alpha = \pi/2$, for CHD map and for the map convex in vertical direction.

Corollary 2.9. *Under the hypothesis of Lemma 1.5, with $\alpha = 0$, $\mathcal{L}_0$ be of the form (2.15) and $\Psi_0(z) = \mathcal{L}_0(z) - \mathcal{G}_0(z) \in \mathcal{T}[A,B]$ be convex in the horizontal direction. Let $\mathcal{E}_0 = \mathcal{L}_0 + \overline{\mathcal{G}_0} \in TS_H^0$ be the harmonic shear of $\Psi_0(z)$ in the same direction. Then $\mathcal{E}_0 \in \mathcal{T}_H[A,B]$ if and only if*

$$\begin{aligned}
&\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right) + {}_p\psi_q\left([(\alpha_i,A_i)];1\right)\right]\\
&+\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+C_i,C_i)];1\right) + \frac{A-1}{A-B}\,{}_r\psi_s\left([(\gamma_i,C_i)];1\right)\right]\\
\le\;& \frac{3A-2B-1}{A-B}
\end{aligned}$$

holds, where λ_2 and μ are given respectively, by (2.9) and (2.3).

Corollary 2.10. *Under the hypothesis of Lemma 1.5, with $\alpha = \pi/2$, $\mathcal{L}_{\pi/2}$ be of the form (2.17), and $\Psi_{\pi/2}(z) = \mathcal{L}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z) \in \mathcal{T}[A,B]$ be convex in the vertical direction. Let $\mathcal{E}_{\pi/2} = \mathcal{L}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}(z)} \in TS_H^0$ be the harmonic shear of $\Psi_{\pi/2}(z)$ in the same direction. Then $\mathcal{E}_{\pi/2} \in \mathcal{T}_H[A,B]$ if and only if*

$$\begin{aligned}
&\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right) + {}_p\psi_q\left([(\alpha_i,A_i)];1\right)\right]\\
&-\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+C_i,C_i)];1\right) + \frac{A-1}{A-B}\,{}_r\psi_s\left([(\gamma_i,C_i)];1\right)\right]\\
\le\;& \frac{A-2B+1}{A-B}
\end{aligned}$$

holds, where λ_2 and μ are given respectively, by (2.9) and (2.3).

Corollary 2.11. *Under the hypothesis of Lemma 1.5, with $\alpha = 0$, $\mathcal{L}_0$ be of the form (2.15), and $\Psi_0(z) = \mathcal{L}_0(z) - \mathcal{G}_0(z) \in \mathcal{T}[A,B]$ be convex in the horizontal direction. Let $\mathcal{E}_0 = \mathcal{L}_0 + \overline{\mathcal{G}_0} \in TS_H^0$ be the harmonic shear of $\Psi_0(z)$ in the same direction. Then $\mathcal{E}_0 \in \mathcal{C}_H[A,B]$ if and only if*

$$\begin{aligned}
&\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left([(\alpha_i+2A_i,A_i)];1\right) + \frac{A-3B+2}{A-B}\,{}_p\psi_q\left([(\alpha_i+A_i,A_i)];1\right)\right.\\
&\left.+\,{}_p\psi_q\left([(\alpha_i,A_i)];1\right)\right]\\
&+\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left([(\gamma_i+2C_i,C_i)];1\right) +_r \psi_s\left([(\gamma_i+C_i,C_i)];1\right)\right]\\
&\le 2
\end{aligned}$$

holds, where λ_2 and μ are given respectively, by (2.9) and (2.3).

Corollary 2.12. *Under the hypothesis of Lemma 1.5, with $\alpha = \pi/2$, $\mathcal{L}_{\pi/2}$ be of the form (2.17), and $\Psi_{\pi/2}(z) = \mathcal{L}_{\pi/2}(z) + \mathcal{G}_{\pi/2}(z) \in \mathcal{T}[A,B]$ be convex in the vertical direction. Let $\mathcal{E}_{\pi/2} = \mathcal{L}_{\pi/2} + \overline{\mathcal{G}_{\pi/2}(z)} \in TS_H^0$ be the harmonic shear of $\Psi_{\pi/2}(z)$ in the same direction. Then $\mathcal{E}_{\pi/2} \in \mathcal{C}_H[A,B]$*

if and only if

$$\lambda_2\left[\frac{1-B}{A-B}\,{}_p\psi_q\left(\left[(\alpha_i+2A_i,A_i)\right];1\right)+\frac{A-3B+2}{A-B}\,{}_p\psi_q\left(\left[(\alpha_i+A_i,A_i)\right];1\right)\right.$$
$$\left.+\,{}_p\psi_q\left(\left[(\alpha_i,A_i)\right];1\right)\right]$$
$$-\mu\left[\frac{1-B}{A-B}\,{}_r\psi_s\left(\left[(\gamma_i+2C_i,C_i)\right];1\right)+\,{}_r\psi_s\left(\left[(\gamma_i+C_i,C_i)\right];1\right)\right]$$
$$\leq 2$$

holds, where λ_2 and μ are given respectively, by (2.9) and (2.3).

3 Special Cases

Taking $A_i=1\ (i=1,...,p)\,,B_i=1\ (i=1,...,q)\,,C_i=1\ (i=1,...,r)\,,$ $D_i=1\ (i=1,...,s)\,,$ we define generalized hypergeometric (gh) functions as special case of Wgh functions given in (1.7), as follows:

$${}_pF_q(\alpha_1,...,\alpha_p,\beta_1,...,\beta_q;z)=$$

$${}_pF_q\left([\alpha_i];z\right)=\frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\,{}_p\psi_q\left[\begin{pmatrix}(\alpha_i,1)_{1,p}\\(\beta_i,1)_{1,q}\end{pmatrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n\ z^n}{\prod_{i=1}^{q}(\beta_i)_n n!}\quad(p\leq q+1)$$

$${}_rF_s\left([\gamma_i];z\right)=\frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\,{}_r\psi_s\left[\begin{pmatrix}(\gamma_i,1)_{1,r}\\(\delta_i,1)_{1,s}\end{pmatrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_n\ z^n}{\prod_{i=1}^{s}(\delta_i)_n\ n!}\quad(r\leq s+1)\,.$$

Denote

$$F_1(z):=z\,{}_pF_q\left([\alpha_i];z\right)\ \text{and}\ F_2(z):=\,{}_rF_s\left([\gamma_i];z\right)-1\tag{3.1}$$

which are analytic at $z=1$ if (in case $p=q+1,r=s+1$) $\Re\left(\sum_{i=1}^{q}\beta_i-\sum_{i=1}^{p}\alpha_i\right)>0,$ and $\Re\left(\sum_{i=1}^{s}\delta_i-\sum_{i=1}^{r}\gamma_i\right)>0$, the symbol $(\lambda)_n$ is the Pochhammer symbol defined in terms of gamma function by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\left\{\begin{array}{c}1,\quad n=0,\lambda\neq 0\\ \lambda(\lambda+1)\ldots(\lambda+n-1)\,,n\in\mathbb{N}\end{array}\right\}.$$

Define an analytic function Ω_α as follows,

$$\Omega_\alpha(z)=\frac{F_1(z)-e^{2i\alpha}\,F_2(z)}{1-e^{2i\alpha}c_1}\quad(z\in\mathbb{U})\,,\tag{3.2}$$

where

$$c_1=\frac{\prod_{i=1}^{r}\Gamma(\gamma_i)}{\prod_{i=1}^{s}\Gamma(\delta_i)}.\tag{3.3}$$

The function $\Omega_\alpha(z)$ defined by (3.2) may also be written in the form

$$\Omega_\alpha(z)=\mathbb{H}_\alpha(z)-e^{2i\alpha}\mathbb{G}_\alpha(z),$$

where

$$\mathbb{H}_\alpha(z)=z-\sum_{n=2}^{\infty}\frac{\xi_n}{1-e^{2i\alpha}c_1}\,z^n,\quad\mathbb{G}_\alpha(z)=\sum_{n=2}^{\infty}\frac{\zeta_n}{1-e^{2i\alpha}c_1}z^n\tag{3.4}$$

and

$$\xi_n = \frac{\prod_{i=1}^{p} (\alpha_i + 1)_{n-1}}{\prod_{i=1}^{q} (\beta_i)_{n-1}} \frac{1}{(n-1)!}, \quad \zeta_n = \frac{\prod_{i=1}^{r} \Gamma(\gamma_i)_n}{\prod_{i=1}^{s} \Gamma(\delta_i)_n} \frac{1}{n!}. \tag{3.5}$$

Taking $A_i = 1 \ (i = 1, ..., p), B_i = 1 \ (i = 1, ..., q), C_i = 1 \ (i = 1, ..., r), D_i = 1 \ (i = 1, ..., s)$ in Theorems 2.1, 2.2, 2.3 also in 2.4, we get inequalities involving generalized hypergeometric functions and various special form of hypergeometric functions in particular.

Theorem 3.1. *Let under the parametric conditions considered above, $\mathbb{H}_\alpha$ and $\mathbb{G}_\alpha$ be of the form (3.4) and $\Omega_\alpha(z) = \mathbb{H}_\alpha(z) - e^{2i\alpha}\mathbb{G}_\alpha(z) \in \mathcal{T}[A, B]$ be convex in the direction $\alpha \in \{0, \pi/2\}$. Let $\mathbb{F}_\alpha = \mathbb{H}_\alpha(z) + \overline{\mathbb{G}_\alpha(z)}$ be the harmonic shear of $\Omega_\alpha(z)$ in the same direction α. Suppose $\alpha_i > -1 \ (i = 1, ..., p)$, such that $\prod_{i=1}^{p} \alpha_i < 0$, $\beta_i > 0 \ (i = 1, ..., q), \gamma_i > 0 \ (i = 1, ..., r), \delta_i > 0$ $(i = 1, ..., s)$. Then under the validity condition (in the case $p = q + 1$ and $r = s + 1$)*

$$\sum_{i=i}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i > 1 \ \text{ and } \ \sum_{i=i}^{s} \delta_i - \sum_{i=i}^{r} \gamma_i > 1,$$

$\mathbb{F}_\alpha \in \mathcal{T}_H[A, B]$ if and only if

$$\begin{aligned} & \left[\frac{1-B}{A-B} \, {}_pF_q([(\alpha_i+1)];1) + \, {}_pF_q([(\alpha_i,)];1)\right] \\ & + e^{2i\alpha}\left[\frac{1-B}{A-B} \, {}_rF_s([(\gamma_i+1)];1) + \frac{A-1}{A-B} \, {}_rF_s([(\gamma_i)];1)\right] \\ \leq \quad & e^{2i\alpha}\frac{A-1}{A-B} \end{aligned}$$

holds.

Theorem 3.2. *Let under the hypothesis of Theorem 3.1, $\mathbb{F}_\alpha = \mathbb{H}_\alpha(z) + \overline{\mathbb{G}_\alpha(z)}$ be the harmonic shear of $\Omega_\alpha(z)$ in the direction $\alpha \in \{0, \pi/2\}$. Suppose*
$\alpha_i > -1 \ (i = 1, ..., p)$ such that $\prod_{i=1}^{p} \alpha_i < 0$, $\beta_i > 0 \ (i = 1, ..., q), \gamma_i > 0 \ (i = 1, ..., r), \delta_i > 0$ $(i = 1, ..., s)$. Then under the validity condition (in the case $p = q + 1$ and $r = s + 1$)

$$\sum_{i=i}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i > 2 \ \text{ and } \ \sum_{i=i}^{s} \delta_i - \sum_{i=i}^{r} \gamma_i > 2,$$

$\mathbb{F}_\alpha \in \mathcal{C}_H[A, B]$ if and only if

$$\begin{aligned} & \left[\frac{1-B}{A-B} \, {}_pF_q([(\alpha_i+2)];1) + \frac{A-3B+2}{A-B} \, {}_pF_q([(\alpha_i+1)];1) + \, {}_pF_q([(\alpha_i)];1)\right] \\ & + e^{2i\alpha}\left[\frac{1-B}{A-B} \, {}_rF_s([(\gamma_i+2)];1) + \, {}_rF_s([\gamma_i+1)];1)\right] \\ \leq \quad & 0 \end{aligned}$$

holds.

We next consider an analytic function $\Upsilon_\alpha(z)$ defined by

$$\Upsilon_\alpha(z) = \frac{z\left(2 - \frac{F_1(z)}{z}\right) - e^{2i\alpha}F_2(z)}{1 - e^{2i\alpha}c_1} \quad (z \in \mathbb{U}),$$

where $F_1(z)$ and $F_2(z)$ are given by (3.1) with $\alpha_i > 0 \ (i = 1, ..., p), \beta_i. > 0 \ (i = 1, ..., q)$, $\gamma_i > 0 \ (i = 1, ..., r), \delta_i > 0 \ (i = 1, ..., s)$ satisfy the condition

$$\frac{\prod\limits_{i=1}^{r} (\gamma_i)_n}{\prod\limits_{i=1}^{s} (\delta_i)_n} < \frac{n \prod\limits_{i=1}^{p} (\alpha_i)_{(n-1)}}{\prod\limits_{i=1}^{q} (\beta_i)_{(n-1)}} \ (n \geq 1)$$

and c_1 is given by (3.3). The function $\Upsilon_\alpha(z)$ may also be written in the form

$$\Upsilon_\alpha(z) = \mathbb{J}_\alpha(z) - e^{2i\alpha}\mathbb{G}_\alpha(z),$$

where

$$\mathbb{J}_\alpha(z) = z - \sum_{n=2}^{\infty} \frac{\xi_n}{1 - e^{2i\alpha}c_1} z^n, \tag{3.6}$$

$\mathbb{G}_\alpha(z)$, c_1 and ξ_n are given, respectively, by (3.4), (3.3) and (3.5).

Theorem 3.3. *Let under the parametric conditions considered above, $\mathbb{J}_\alpha$ and $\mathbb{G}_\alpha$ be given, respectively, by (3.4) and (3.6).*
Let $\Upsilon_\alpha(z) = \mathbb{J}_\alpha(z) - e^{2i\alpha}\mathbb{G}_\alpha(z) \in \mathcal{T}[A, B]$ be convex in the direction $\alpha \in \{0, \pi/2\}$. Let $\mathbb{E}_\alpha = \mathbb{J}_\alpha(z) + \overline{\mathbb{G}_\alpha(z)}$ be the harmonic shear of $\Upsilon_\alpha(z)$ in the same direction α. Suppose $\alpha_i > 0 \ (i = 1, ..., p), \beta_i > 0 \ (i = 1, ..., q), \gamma_i > 0 \ (i = 1, ..., r), \delta_i > 0 \ (i = 1, ..., s)$. Then under the validity condition (in the case $p = q + 1$ and $r = s + 1$)

$$\sum_{i=i}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i > 1 \ \text{ and } \ \sum_{i=i}^{s} \delta_i - \sum_{i=i}^{r} \gamma_i > 1,$$

$\mathbb{E}_\alpha \in \mathcal{T}_H[A, B]$ if and only if

$$\begin{aligned} & \left[\frac{1-B}{A-B}\, {}_pF_q\left([(\alpha_i+1)]; 1\right) + {}_pF_q\left([(\alpha_i)]; 1\right)\right] \\ & +e^{2i\alpha}\left[\frac{1-B}{A-B}\, {}_rF_s\left([(\gamma_i+1)]; 1\right) + \frac{A-1}{A-B}\, {}_rF_s\left([(\gamma_i)]; 1\right)\right] \\ \leq \quad & \frac{3A-2B-1}{A-B} \end{aligned}$$

holds.

Theorem 3.4. *Under the hypothesis of Theorem 3.3, $\mathbb{E}_\alpha = \mathbb{J}_\alpha(z) + \overline{\mathbb{G}_\alpha(z)}$ be the harmonic shear of $\Upsilon_\alpha(z)$ in the direction $\alpha \in \{0, \pi/2\}$. Suppose $\alpha_i > 0 \ (i = 1, ..., p), \beta_i > 0 \ (i = 1, ..., q), \gamma_i > 0 \ (i = 1, ..., r), \delta_i > 0 \ (i = 1, ..., s)$. Then under the validity condition (in the case $p = q + 1$ and $r = s + 1$)*

$$\sum_{i=i}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i > 2 \ \text{ and } \ \sum_{i=i}^{s} \delta_i - \sum_{i=i}^{r} \gamma_i > 2,$$

$\mathbb{E}_\alpha \in \mathcal{C}_H[A, B]$ if and only if

$$\begin{aligned} & \left[\frac{1-B}{A-B}\, {}_pF_q\left([(\alpha_i+2)]; 1\right) + \frac{A-3B+2}{A-B}\, {}_pF_q\left([(\alpha_i+1)]; 1\right) + {}_pF_q\left([(\alpha_i)]; 1\right)\right] + \\ & e^{2i\alpha}\left[\frac{1-B}{A-B}\, {}_rF_s\left([(\gamma_i+2)]; 1\right) + {}_rF_s\left([(\gamma_i+1)]; 1\right) - {}_rF_s\left([(\gamma_i)]; 1\right)\right] \\ \leq \quad & 2 \end{aligned}$$

holds.

References

[1] O.P. Ahuja, Planer harmonic convolution operators generated by hypergeometric functions, Integ. Trans. Spec. Funct., **18** (3), 165-177 (2007).

[2] O.P. Ahuja, Connections between various subclasses of planar harmonic mappings involving hypergeometric functions, Appl. Math. Comput., **198** , 305-316 (2008).

[3] O.P. Ahuja, Harmonic starlikeness and convexity of integral operators generated by hypergeometric series, Integ. Trans. Spec. Funct., **20** (8), 629-641 (2009).

[4] O.P. Ahuja, Inclusion theorems involving uniformly harmonic starlike mappings and hypergeometric functions, Analele Universitatii Oradea, Fasc. Matematica, **18**, 5-18 (2011).

[5] O.P. Ahuja, Use of theory of conformal mappings in harmonic univalent mappings with directional convexity, Bull. Malays. Math. Sci. Soc., (2) **35** (3), 775-784 (2012).

[6] O.P. Ahuja and P. Sharma**,** Inclusion theorems involving Wright's generalized hypergeometric functions and harmonic univalent functions, Acta Universitatis Apulensis, **32** , 111-128 (2012).

[7] J. Clunie and T. Sheil-small, Harmonic univalent functions, Ann. Acad. Aci. Fenn. Ser. A Math., **9** , 3-25 (1984).

[8] M. Dorff, J. Szynal, Harmonic shears of elliptic integrals, Rocky Mountain J. Math., **35** , 485-499 (2005).

[9] K. Driver, P. Duren, Harmonic shears of regular polygons by hypergeometric functions, J. Math. Anal. Appl., **239** , 72-84 (1999).

[10] W. Hengartner, G. Schober, On schlicht mappings to domains convex in one direction, Comment. Math. Helv., **45** , 303-314 (1970).

[11] A.A. Kilbas, M. Saigo and J.J. Trujillo, On the generalized Wright function, Fract. Calc. Appl. Anal., 5 (4), 437–460 (2002).

[12] G. Murugusundaramoorthy, R.K. Raina, On a subclass of harmonic functions associated with the Wright's generalized hypergeometric functions, Hacettepe Journal of Mathematics and Statistics, **38** (2) , 129-136 (2009).

[13] R.K. Raina and P. Sharma, Harmonic univalent functions associated with Wright's generalized hypergeometric functions, Integ. Trans. Spec. Funct., **22** (8), 561-572 (2011).

[14] L. Schaubroeck, Growth, distortion and coefficient bounds for plane harmonic mappings convex in one direction, Rocky Mountain J. Math., **31** , 625-639 (2001).

[15] P. Sharma, Univalent Wright's generalized hypergeometric functins, Journal of Inequalities and Special Functions, **3** (1), 28-39 (2012).

[16] P. Sharma, Some Wgh inequalities for univalent harmonic analytic functions, Applied Mathematics, **1** , 464-469 (2010).

[17] P. Sharma , Om P. Ahuja and V. K. Gupta , Univalent Harmonic Functions with Domains Convex in Horizontal and Vertical Directions, Acta Universitatis Apulensis, **39**, 1-19 (2014).

[18] H. Silverman, A survey with open problems on univalent functions whose coeffcients are negative, Rocky Mountain J. Math., **21** (3), 1099-1124 (1991).

[19] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., **51**, 109-116 (1975).

[20] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian HypergeometricSeries, Horwood Ser. Math. Appl., Horwood, Chichester, 1985.

[21] H.M. Srivastava, R.K. Saxena, C. Ram, A unified presentation of the Gamma-type functions occurring in diffraction theory and associated probability distributions, Appl. Math. Comput., **162** , 931-947 (2005).

[22] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood limited, Chichester), 1984.

[23] H.M. Srivastava, Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators, Applicable Analysis and Discrete Mathematics, **1** , 56-71 (2007).

[24] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function, Proc. London Math. Soc., **46** , 389-408 (1946).

Author information

V. K. Gupta and P. Sharma, Department of Mathematics & Astronomy University of Lucknow
Lucknow, 226007, UP INDIA, India.
E-mail: `vim987@gmail.com`

Received: April 23, 2017.
Accepted: December 26, 2017.