ON GENERALIZED CR-LIGHTLIKE SUBMANIFOLDS

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Abstract. The aim of the article is to find the conditions for the integrability of the distributions which are defined on a generalized Cauchy-Riemann (GCR) lightlike submanifold of a para-Sasakian manifold. Also, we find a condition for the induced connection to be a metric connection on the GCR-lightlike submanifold.

1 Introduction

Submanifold theory, especially theory of lightlike (or null) submanifolds, is one of the important research area in semi-Riemannian geometry. A submanifold $M$ of a semi-Riemannian manifold $\bar{M}$ is called a lightlike (null) submanifold if the induced metric tensor field on the submanifold is degenerate. If the induced metric tensor is degenerate then the classical theory of Riemannian submanifolds fails since the tangent bundle and the normal bundle of the submanifold have a non-zero intersection. To overcome this problem D. N. Küpeli [25] (intrinsic approach) and K. L. Duggal-A. Bejancu [15] (extrinsic approach) were introduced some new methods and studied lightlike submanifolds (see also [16]). On this topic, some applications of the theory to mathematical physics is inspired, especially general relativity [21], black hole horizons [16] and electromagnetism [15].

A CR-submanifold of Kaehler manifold was defined by A. Bejancu [10, 11], as a result of generalization of invariant and anti-invariant submanifolds. Contact CR-submanifolds of Sasakian manifolds were studied in [12].

In [17], K. L. Duggal-B. Şahin defined screen real, screen CR-lightlike, invariant and contact CR-lightlike submanifolds of indefinite Sasakian manifolds. Later on study of generalized CR-lightlike submanifolds of indefinite Sasakian manifolds were initiated in [18]. The authors in [22] gave some necessary and sufficient conditions on integrability of various distributions of GCR-lightlike submanifold of an indefinite Sasakian manifold. Moreover, CR-lightlike submanifolds of a Kaehlerian manifolds were studied in [19]. Also, a general notion of paraccontact CR-lightlike submanifolds was introduced in [5, 6]. Recently, a huge number of research papers has appeared on lightlike submanifolds and its applications (for further read we refer [28, 26, 4, 23, 3] and references therein).

On a semi-Riemannian manifold $\bar{M}$, S. Kaneyuki-M. Konzai [24] introduced a structure which is known the almost paracontact structure and then they characterized the almost para-complex structure on $\mathbb{M}^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [30] studied paracontact metric manifolds. The study of paracontact geometry has been continued by several papers ([8, 9, 2, 13, 14, 7, 20, 29]) which contain role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

The purpose of this article is to investigate the conditions for the integrability of the distributions which are defined on GCR-lightlike submanifolds of para-Sasakian manifolds. Also, we obtain a condition for the induced connection to be a metric connection on the GCR-lightlike submanifolds of a para-Sasakian manifold.
2 Preliminaries

2.1 Lightlike Submanifolds

Let \((\tilde{M}^{n+m}, \tilde{g})\) be a semi-Riemannian manifold with index \(q\), such that \(m, n \geq 1, 1 \leq q \leq m + n - 1\) and \((\tilde{M}^m, g)\) be a submanifold of \(\tilde{M}\), where \(g\) is the induced metric from \(\tilde{g}\) on \(\tilde{M}\). In this case, \(M\) is called a lightlike (null) submanifold of \(\tilde{M}\) if \(g\) is degenerate on \(M\). Now let us consider a degenerate metric \(g\) on \(M\). Thus \(TM^\perp\) is a degenerate \(n\)-dimensional subspace of \(T_x\tilde{M}\) and orthogonal subspaces \(T_xM\) and \(T_xM^\perp\) are degenerate but no longer complementary. So, there exists a subspace \(\text{Rad}T_x\tilde{M} = T_xM \cap T_xM^\perp\) which is called radical space. If the mapping \(\text{Rad}T_x\tilde{M} : x \in M \to \text{Rad}T_x\tilde{M}\) defines a distribution, namely radical distribution, on \(M\) of rank \(r > 0\) then the submanifold \(M\) is called an \(r\)-lightlike submanifold [15].

Let \(S(TM)\) be the screen distribution which is a semi-Riemannian complementary distribution of \(\text{Rad}T\tilde{M}\) in \(TM\). So one can write

\[
TM = S(TM) \perp \text{Rad}TM,
\]

and \(S(TM^\perp)\) is a complementary vector subbundle to \(\text{Rad}TM\) in \(TM^\perp\). Let \(\text{tr}(TM)\) and \(\text{ltr}(TM)\) be complementary (but not orthogonal) vector bundles to \(TM\) in \(\tilde{M}\) and \(\text{Rad}TM\) in \(S(TM^\perp)^\perp\), respectively. In this case we arrive at

\[
\text{tr}(TM) = S(TM^\perp) \perp \text{ltr}(TM),
\]

\[
\tilde{TM} = TM \oplus \text{tr}(TM) = \{\text{Rad}TM \oplus \text{ltr}(TM)\} \perp S(TM) \perp S(TM^\perp). \tag{2.3}
\]

**Theorem 2.1.** [15] Let \((M, g, S(TM))\) be a lightlike submanifold of a semi-Riemannian manifold \((\tilde{M}, \tilde{g})\). Then there exist a complementary vector subbundle \(\text{ltr}(TM)\) of \(\text{Rad}TM\) in \(S(TM^\perp)^\perp\) and a basis of \(\Gamma(\text{ltr}(TM))|_U\) consisting of smooth section \(\{N_i\}\) of \(S(TM^\perp)^\perp|_U\), where \(U\) is a coordinate neighborhood of \(M\), such that

\[
\tilde{g}(N_i, E_i) = 1, \quad \tilde{g}(N_i, N_j) = 0, \quad i, j \leq n,
\]

where \(\{E_1, E_2, ..., E_n\}\) is a lightlike basis of \(\Gamma(\text{Rad}TM)\).

For a lightlike submanifold \((M, g, S(TM))\) we have following four cases:

* If \(r < \min\{m, n\}\) then \(M\) is a \(r\)-lightlike submanifold,
* If \(r = n < m\), \(S(TM^\perp) = \{0\}\) then \(M\) is a coisotropic lightlike submanifold,
* If \(r = m < n\), \(S(TM) = \{0\}\) then \(M\) is a isotropic lightlike submanifold,
* If \(r = m = n\), \(S(TM) = \{0\}\) = \(S(TM^\perp)\) then \(M\) is a totally null submanifold.

By use of (2.3), the Gauss and Weingarten formulas are defined by

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}
\]

\[
\nabla_X U = -A_U X + \nabla_X^t U, \quad \forall X \in \Gamma(TM), U \in \Gamma(\text{tr}(TM)), \tag{2.6}
\]

respectively, where \(\{\nabla_X Y, A_U X\}\) belongs to \(\Gamma(TM)\) and \(\{h(X, Y), \nabla_X^t U\}\) belongs to \(\Gamma(\text{tr}(TM))\). \(\nabla\) and \(\nabla^t\) are linear connections on \(M\) and on the vector bundle \(\text{tr}(TM)\), respectively.

In view of (2.2), we consider the projection morphisms \(L\) and \(S\) of \(\text{tr}(TM)\) on \(\text{ltr}(TM)\) and \(S(TM^\perp)\). Therefore (2.5) and (2.6) become

\[
\tilde{\nabla}_X Y = \nabla_X Y + h_1(X, Y) + h_2(X, Y), \quad X, Y \in \Gamma(TM), \tag{2.7}
\]

\[
\nabla_X N = -A_N X + \nabla_X^t N + D^N(X, N), \quad X \in \Gamma(TM), N \in \Gamma(\text{ltr}(TM)), \tag{2.8}
\]

\[
\tilde{\nabla}_X W = -A_W X + \nabla_X W + D^W(X, W), \quad X \in \Gamma(TM), W \in \Gamma(S(TM^\perp)), \tag{2.9}
\]

where \(h_1(X, Y) = L(h(X, Y)), h_2(X, Y) = S(h(X, Y)), \nabla_X^t N, D^t(X, W) \in \Gamma(\text{ltr}(TM)), \nabla_X W, D^W(X, N) \in \Gamma(S(TM^\perp))\) and \(\nabla_X Y, A_N X, A_W X \in \Gamma(TM)\).
Assume that $P$ is a projection of $TM$ on $S(TM)$. Then by use of $(2.1)$, we have
\[
\nabla_XPY = \nabla_X^P PY + h^*(X, PY), \quad X, Y \in \Gamma(TM),
\]
(2.10)
\[
\nabla_XE = -A^*_E X + \nabla_X^S E, \quad X \in \Gamma(TM), \quad E \in \Gamma(RadTM),
\]
(2.11)
where $\{\nabla_X^P PY, A^*_E X\}$ belongs to $\Gamma(S(TM))$ and $\{h^*(X, PY), \nabla_X^S E\}$ belongs to $\Gamma(RadTM)$. Using $(2.10)$ and $(2.11)$, we get
\[
\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY),
\]
(2.12)
\[
\bar{g}(h^l(X, PY), E) = \bar{g}(A^*_E X, PY),
\]
(2.13)
\[
\bar{g}(h^l(X, E), E) = 0, \quad A^*_E E = 0.
\]
(2.14)

In general, the induced connection $\nabla$ on $M$ is not a metric connection. Since $\bar{\nabla}$ is a metric connection then from $(2.7)$, we have
\[
(\nabla \times g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).
\]
(2.15)
However, $\nabla^*$ is a metric connection on $S(TM)$.

### 2.2 Almost paracontact metric manifolds

A paracontact manifold $\tilde{M}$ is a differentiable manifold equipped with a $1$–form $\eta$, a characteristic vector field $\xi$ and a tensor field $\bar{\phi}$ of type $(1, 1)$ such that [24]:
\[
\eta(\xi) = 1,
\]
(2.16)
\[
\bar{\phi}^2 = I - \eta \otimes \xi,
\]
(2.17)
\[
\bar{\phi}\xi = 0,
\]
(2.18)
\[
\eta \circ \bar{\phi} = 0.
\]
(2.19)

Moreover, if the manifold $\tilde{M}$ is equipped with a semi-Riemannian metric $\bar{g}$ of signature $(n + 1, n)$, which is called compatible metric, satisfying [30]
\[
\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y),
\]
(2.20)
then we say that $\tilde{M}$ is an almost paracontact metric manifold with an almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$.

From the definition, one can see that [30]
\[
\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y),
\]
(2.21)
\[
\bar{g}(X, \xi) = \eta(X).
\]
(2.22)
If $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$, then the almost paracontact metric manifold is said to be a paracontact metric manifold.

For an almost paracontact metric manifold $(\tilde{M}, \bar{\phi}, \xi, \eta, \bar{g})$, one can always find a local orthonormal basis $(X_i, \bar{\phi}X_i, \xi)$, $i = 1, 2, ..., n$, which is called $\bar{\phi}-basis$ [30].

An almost paracontact metric manifold $(\tilde{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is a para-Sasakian manifold if and only if [30]
\[
(\nabla_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X,
\]
(2.23)
where $\bar{\nabla}$ is a Levi-Civita connection on $\tilde{M}$.

From $(2.23)$, one arrive at
\[
\bar{\nabla}_X \xi = -\bar{\phi}X.
\]
(2.24)
Example 2.2. [1] Let $\tilde{M} = \mathbb{R}^{2n+1}$ be the $(2n + 1)$-dimensional real number space with standard coordinate system $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, z)$. Defining

$$\phi \frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \quad \phi \frac{\partial}{\partial y_\alpha} = -\frac{\partial}{\partial x_\alpha}, \quad \phi \frac{\partial}{\partial z} = 0,$$

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz,$$

$$g = \eta \otimes \eta + \sum_{\alpha=1}^{n}(dx_\alpha \otimes dx_\alpha - dy_\alpha \otimes dy_\alpha),$$

where $\alpha = 1, 2, \ldots, n$, then the set $(\tilde{M}, \phi, \xi, \eta, g)$ is an almost paracontact metric manifold.

3 Main Theorems

Definition 3.1. Let $(M, g, S(TM))$ be a lightlike submanifold of a para-Sasakian manifold $\tilde{M}$. Then $M$ is called a GCR-lightlike submanifold if the following conditions are provided,

i) There exist two subbundles $\mu_1$ and $\mu_2$ of $Rad TM$ such that

$$Rad TM = \mu_1 \oplus \mu_2, \quad \phi(\mu_1) = \mu_1, \quad \phi(\mu_2) \subset S(TM),$$

(3.1)

ii) There exist two subbundles $\tilde{D}_1$ and $\tilde{D}_2$ of $S(TM)$ such that

$$S(TM) = \{\tilde{\phi}(\mu_2) \oplus \tilde{D}_2\} \perp \tilde{D}_1 \perp \{\xi\}, \quad \tilde{\phi}(\tilde{D}_2) = D_0 \perp \tilde{D}_0,$$

(3.2)

where $\tilde{D}_1$ is an invariant non-degenerate distribution on $M$, $\{\xi\}$ is a 1-dimensional distribution spanned by $\xi$ and $D_0$, $\tilde{D}_0$ are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$, respectively.

So one has the following:

$$TM = D \oplus \tilde{D}_2 \oplus \{\xi\}, \quad D = Rad TM \oplus \tilde{D}_1 \oplus \tilde{\phi}(\mu_2).$$

(3.3)

Let $Q$, $S_1$, $S_2$ be the projection morphisms on $D$, $\phi\tilde{D}_0$, $\phi\tilde{D}_0$, respectively. So we have

$$X = QX + \xi + S_1X + S_2X,$$

for $X \in \Gamma(TM)$. Applying $\phi$ to the both sides of above equation, we get

$$\phi X = \alpha X + \beta S_1X + \beta S_2X,$$

(3.4)

where $\alpha X \in \Gamma(D)$, $\beta S_1X \in \Gamma(D_0)$ and $\beta S_1X \in \Gamma(D_0)$. Thus, one can write

$$\phi X = \alpha X + \beta X,$$

(3.5)

where $\alpha X$ is the tangential component and $\beta X$ is the transversal component of $X$.

Also, for any $U \in \Gamma(tr(TM))$ we have

$$\phi U = BU + CU,$$

(3.6)

where $BU \in \Gamma(TM)$ and $CU \in \Gamma(tr(TM))$.

Differentiating (3.4) and in view of (2.8), (2.9) and (3.6), we get

$$D^l(X, \beta S_1Y) = -\nabla^l_X \beta S_2Y + \beta S_2 \nabla_X Y$$

(3.7)

$$-h^l(X, \alpha Y) + Ch^l(X, Y),$$

$$D^s(X, \beta S_2Y) = -\nabla^s_X \beta S_1Y + \beta S_1 \nabla_X Y$$

(3.8)

$$-h^s(X, \alpha Y) + Chs(X, Y).$$

By using (2.23) with (2.7) and (2.8), we give the following.
Lemma 3.2. Let $M$ be a GCR-lightlike submanifold of a para-Sasakian manifold. Then, we have
\[
(\nabla_X \alpha)Y = -g(X,Y)\xi + \eta(X)Y + A_{\beta Y}X + Bh(X,Y),
\] (3.9)
and
\[
(\nabla_X \beta)Y = -h(X,\alpha Y) + Ch(X,Y),
\] (3.10)
for all $X,Y \in \Gamma(TM)$, where $(\nabla_X \alpha)Y = \nabla_X \alpha Y - \alpha \nabla_X Y$ and $(\nabla_X \beta)Y = \nabla_X^l \beta Y - \beta \nabla_X Y$.

Lemma 3.3. Let $M$ be a GCR-lightlike submanifold of a para-Sasakian manifold. Then, we have
\[
(\nabla_X B)U = A_{CU}X - \alpha A_{U}X,
\] (3.11)
\[
(\nabla_X C)U = -\beta A_{U}X - h(X, BU),
\] (3.12)
for $X \in \Gamma(TM)$ and $U \in \Gamma(\text{tr}(TM))$, where $(\nabla_X B)U = \nabla_X BU - B\nabla_X U$ and $(\nabla_X C)U = \nabla_X^l CU - C\nabla_X U$.

Theorem 3.4. Let $(\bar{M}, \bar{\phi}, \bar{\xi}, \eta, \bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, we have
\[
g(\nabla_X Y, Z) = -g(\alpha A_{\beta Y}X, Z),
\] (3.13)
for any $Y \in \Gamma(D_2)$ and $Z \in \Gamma(D)$.

Proof. By use of (3.9), we get
\[
-\alpha \nabla_X Y = -g(X,Y)\xi + A_{\beta Y}X + Bh(X,Y),
\] (3.14)
for any $Y \in \Gamma(D_2)$.

If we take $Z \in \Gamma(D)$ then we get $\tilde{\phi}Z \in \Gamma(D)$. Using this result in (3.14), we obtain
\[
g(\alpha \nabla_X Y, \tilde{\phi}Z) = -g(A_{\beta Y}X, \tilde{\phi}Z).
\]

In view of (2.17) in above equation, we get (3.13).

Also taking $Z \in \Gamma(D_0)$, we obtain $\nabla_X Y = -\alpha A_{BY}X$. \qed

Theorem 3.5. Let $(\bar{M}, \bar{\phi}, \bar{\xi}, \eta, \bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, $D \oplus \{\xi\}$ is integrable if and only if
\[
h(X, \alpha Y) = h(\alpha X, Y).
\] (3.15)

Proof. From (3.7) and (3.8), for any $X, Y \in \Gamma(D \oplus \{\xi\})$, we get
\[
\beta S \nabla_X Y = h^l(X, \alpha Y) - Ch^l(X, Y).
\] (3.16)
Replacing $X$ by $Y$ in (3.16), we get
\[
\beta S \nabla_Y X = h^l(Y, \alpha X) - Ch^l(Y, X).
\]

Finally, we arrive at
\[
\beta [X, Y] = h(X, \alpha Y) - h(Y, \alpha X),
\]
which completes the proof. \qed

Theorem 3.6. Let $(\bar{M}, \bar{\phi}, \bar{\xi}, \eta, \bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, $D_2$ is integrable if and only if
\[
A_{\bar{\phi}U}V = A_{\bar{\phi}V}U.
\] (3.17)
Proof. From (3.9), we get
\[-f(\nabla_U V) = -g(U, V)\xi + A_{\beta V} U + Bh(U, V),\]
for all \(U, V \in \Gamma(\tilde{D}_2)\). Replacing \(U\) and \(V\) in (3.18), we have
\[-f(\nabla_V U) = -g(V, U)\xi + A_{\beta U} V + Bh(V, U).\]
Subtracting (3.19) from (3.18), we obtain
\[f[U, V] = A_{\beta V} U - A_{\beta U} V,
which gives the equation (3.17) and completes the proof. \(\Box\)

Theorem 3.7. Let \((\bar{M}, \bar{\delta}, \xi, \eta, \bar{g})\) be a para-Sasakian manifold and \((M, g, S(TM))\) be a GCR-lightlike submanifold of \(\bar{M}\). Then, \(D \oplus \{\xi\}\) defines a totally geodesic foliation if and only if
\[Bh(X, \bar{\delta}Y) = 0.\]

Proof. In view of definition of GCR-lightlike submanifolds and the decomposition (3.3), \(D \oplus \{\xi\}\) defines a totally geodesic foliation if and only if
\[g(\nabla_X Y, \bar{\delta}E) = 0, \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}), \quad E \in \Gamma(\mu_2),\]
and
\[g(\nabla_X Y, \bar{\delta}W) = 0, \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}), \quad W \in \Gamma(\tilde{D}_0).\]
Using (2.7) and (2.23), we have
\[g(\nabla_X Y, \bar{\delta}E) = -g(\bar{\delta}\nabla_X Y, E) = -g(\nabla_X \bar{\delta}Y, E) = -g(b^i(X, \alpha Y), E).\]
Similarly, we get
\[g(\nabla_X Y, \bar{\delta}W) = -g(\bar{\delta}\nabla_X Y, W) = -g(\nabla_X \bar{\delta}Y, W) = -g(h^s(X, \alpha Y), W).\]
So, this completes the proof. \(\Box\)

Theorem 3.8. Let \((\bar{M}, \bar{\delta}, \xi, \eta, \bar{g})\) be a para-Sasakian manifold and \((M, g, S(TM))\) be a GCR-lightlike submanifold of \(\bar{M}\). The distribution \(\tilde{D}_2\) defines a totally geodesic foliation on \(M\) if and only if, for all \(X, Y \in \Gamma(\tilde{D}_2)\) and \(N \in \Gamma(\text{itr}(TM))\),

i) \(A_N X\) has no component on \(\bar{\delta}(\mu_2) \perp \bar{\delta}\tilde{D}_0\),

and

ii) \(A_{\beta Y} X\) has no component on \(\mu_2 \perp \tilde{D}_2\).

Proof. In view of the definition of GCR-lightlike submanifolds and the decomposition (3.3), \(\tilde{D}_2\) defines a totally geodesic foliation if and only if
\[g(\nabla_X Y, N) = 0, \quad N \in \Gamma(\text{itr}(TM)),\]
\[g(\nabla_X Y, \bar{\delta}P_1) = 0, \quad P_1 \in \Gamma(D_0),\]
\[g(\nabla_X Y, \bar{\delta}Z) = 0, \quad Z \in \Gamma(\tilde{D}_1),\]
\[g(\nabla_X Y, \xi) = 0,\]
for all \(X, Y \in \Gamma(\tilde{D}_2)\). Using (2.7) and (2.8), we find
\[g(\nabla_X Y, N) = g(\nabla_X Y, N) = -g(Y, \nabla_X N) = g(Y, A_N X).\]
Similarly using (2.7) and (2.23), we get
\[g(\nabla_X Y, \bar{\delta}P_1) = -g(\bar{\delta}\nabla_X Y, P_1)\]
\[= -g(\nabla_X \bar{\delta}Y, P_1)\]
\[= -g(\nabla_X \beta Y, P_1)\]
\[= g(A_{\beta Y} X, P_1),\]
\[ g(\nabla_X Y, \tilde{\phi} Z) = -g(\tilde{\phi} \nabla_X Y, Z) \]
\[ = -g(\nabla_X \tilde{\phi} Y, Z) \]
\[ = -g(\nabla_X \beta Y, Z) \]
\[ = g(A_{\beta Y} X, Z), \quad (3.22) \]

and
\[ g(\nabla_X Y, \xi) = g(\nabla_X Y, \xi) \]
\[ = -g(Y, \nabla_X \xi) \]
\[ = g(Y, \tilde{\phi} X) \]
\[ = 0. \quad (3.23) \]

So, in view of (3.20)-(3.23), the proof is completed. \( \square \)

**Theorem 3.9.** Let \((\bar{M}, \tilde{\phi}, \xi, \eta, \bar{g})\) be a para-Sasakian manifold and \((M, g, S(TM))\) be a GCR-lightlike submanifold of \(\bar{M}\). Then, the induced connection \(\nabla\) is a metric connection if and only if
\[ -A^*_{\tilde{\phi} E} X + \nabla^*_X \tilde{\phi} E \in \Gamma(\tilde{\phi}(\mu_2) \perp \mu_1), \quad X \in \Gamma(TM), \quad E \in \Gamma(\mu_1), \]
\[ \nabla_X \tilde{\phi} E + h^*(X, \tilde{\phi} E) \in \Gamma(\tilde{\phi}(\mu_2) \perp \mu_1), \quad X \in \Gamma(TM), \quad E \in \Gamma(\mu_2), \]
\[ h(X, \tilde{\phi} E) \in \Gamma(D_0 \perp D_0)^\perp \text{ and } A^*_E X \in \Gamma(D_2 \perp D_1 \perp \tilde{\phi}(\mu_2)). \]

**Proof.** By use of (2.23), for any \(X \in \Gamma(TM), E \in \Gamma(RadTM)\), we get
\[ \nabla_X \tilde{\phi} E = \tilde{\phi} \nabla_X E. \quad (3.24) \]

Now, by considering (2.11) and (2.17), we get
\[ \nabla_X E + h(X, E) = \tilde{\phi}(\nabla_X \tilde{\phi} E + h(X, \tilde{\phi} E)) + g(A^*_E X, E)\xi. \quad (3.25) \]

Suppose that \(E \in \Gamma(\mu_1)\), so \(\tilde{\phi} E \in \Gamma(\mu_1)\). Again in view of (2.11) and (3.25), we find
\[ \nabla_X E + h(X, E) = \tilde{\phi}(-A^*_{\tilde{\phi} E} X + \nabla^*_X \tilde{\phi} E + h(X, \tilde{\phi} E)) \]
\[ + g(A^*_E X, E)\xi, \quad (3.26) \]
which yields
\[ \nabla_X E = -\alpha A^*_E X + \alpha \nabla^*_X \tilde{\phi} E + Bh(X, \tilde{\phi} E) + g(A^*_E X, E)\xi. \quad (3.27) \]

Thus \(\nabla_X E \in \Gamma(RadTM)\) if and only if
\[ Bh(X, \tilde{\phi} E) = 0, \]
\[ -\alpha A^*_{\tilde{\phi} E} X + \alpha \nabla^*_X \tilde{\phi} E \in \Gamma(RadTM), \]
and \(g(A^*_E X, E) = 0\) if and only if
\[ h(X, \tilde{\phi} E) \in \Gamma(D_0 \perp D_0)^\perp, \quad (3.28) \]
\[ A^*_E X \in \Gamma(D_2 \perp D_1 \perp \tilde{\phi}(\mu_2)). \quad (3.29) \]

Now, suppose that \(E \in \Gamma(\mu_2)\). By use of (2.10) with (3.25), we arrive at
\[ \nabla_X E = \alpha \nabla^*_X \tilde{\phi} E + \alpha h^*(X, \tilde{\phi} E) + Bh(X, \tilde{\phi} E) \]
\[ + g(A^*_E X, E)\xi. \quad (3.30) \]

Therefore \(\nabla_X E \in \Gamma(RadTM)\) if and only if
\[ Bh(X, \tilde{\phi} E) = 0, \]
\[ -\alpha A^*_{\tilde{\phi} E} X + \alpha \nabla^*_X \tilde{\phi} E \in \Gamma(RadTM), \]
and \( g(A^*_E X, \xi) = 0 \) if and only if
\[
h(X, \tilde{\phi}E) \in \Gamma(L \perp S)^{\bot},
\]
\[
\nabla_X \tilde{\phi}E + h^*(X, \tilde{\phi}E) \in \Gamma(\tilde{\phi}(\mu_2) \perp \mu_1), \tag{3.31}
\]
\[
A^*_E X \in \Gamma(D_2 \perp D_1 \perp \tilde{\phi}(\mu_2)). \tag{3.32}
\]
Thus, in view of (3.28), (3.29), (3.31) and (3.32), the proof is completed. \( \square \)

References


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