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DETERMINANTS FOR THE CLASS OF BAZILEVIC FUNCTION

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Abstract. In this paper the author invesitigate the coefficient bounds using symmetric Toepliz determinants $T_2(2), T_2(3), T_3(2)$ and $T_3(1)$ for the fiunctions in the class of Bazilevic family denoted by $B_n(\alpha, \beta, g)$.

1 Introduction

Let A be the class of normalied anlytic functions in the open unit disk $U = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let S denotes the subclass of A consisting of all univalent function in U, normalized with $f(0) = 0,\ f'(0) - 1 = 0$.
Recently, in the theory of univalent function much efforts and attention has been concerted on the estimates of bounds of Hankel matrices, this is so because Hankel determinant play a vital role in different branches of academic endeavous with so many useful applications [11].
There are other matrices that are close relation with Hankel determinants such are the Toepliz determinants. A Toepliz determinant can be thought of as upsidedown or inversion of Hankel determinant, in that Hankel determinant have constant entries along the reverse diagonal, whereas Toepliz matrices have constant entries along the diagonal. There are various applications of Toepliz determinant to a wide range of areas of both pure and applied Mathematics; (see [20]).
The Hankel determinant of f for $q \geq 1$, and $k \geq 1$ was studied by Pommoroke [11,12] as

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & \cdots & \cdots & \\ a_{k+q-1} \cdots & \cdots & & a_{k+2q-2} \end{vmatrix},$$

and defined the symmetric Toepliz determinant $T_q(k)$ as follows

$$T_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_k & \cdots & a_{n+q} \\ . & . & . & . \\ a_{k+q-1} & a_{k+q} \cdots & & a_k \end{vmatrix}.$$

In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix},$$

$$T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_4 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

In the present investigation, we consider the symmetric Toepliz determinant and obtain the estimates of that determinants, whose elements are the coefficients of a_k of Bazilevic function.
There is a long standing history as regards the problem of finding best possible bounds $||a_{k+1}| - |a_k||$ for the function $f \in S$ [5].
It is well-known that $||a_{k+1}| - |a_k|| \leq n$, but finding the exact values of the constant A for S and its various subclasses proved difficult. Therefore, finding the estimates for $T_q(k)$ is related to finding bounds for $A(k) = |a_{k+1} - a_k|$. However, the the function $h(z) = \frac{z}{(1+z)^2}$ proved that the best possible upperbound obtainable for $A(k)$ is $2k + 1$, and so obtainning bounds for $A(k)$ is different from finding bounds for $||a_{k+1}| - |a_k||$.
In the present investigation, we obtain the coefficient bounds for symmetric Toepliz determinant $T_2(2),\ \ T_2(3)$; $T_3(2)$ and $T_3(1)$, for $f \in B_n(\alpha, \beta, g)$ the class of Bazelevic functions which shall be discussed in the next section.

2 Class $B(\alpha, \beta, g)$, Definition and Preliminary

The class of Bazilevic functions denoted by $B(\alpha, \beta, g)$ was discovered in 1955 by a Russian Mathematician called Bazilevic [3] and he Bazilevic defined the class as

$$f(z) = \left\{ \frac{\alpha}{(1+\beta^2)} \int_0^z (p(t) - i\beta) t^{-\left(1+\frac{i\alpha\beta}{1+\beta^2}\right)} g(z)^{-\frac{\alpha}{1+\beta^2}} dt \right\}^{\frac{1+i\beta}{\alpha}} \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a Caratheodory function [4] and $g(z)$ is any starlike function, that is $Re\frac{zg'(z)}{g(z)} > 0$. The parameters α and β are real numbers $\alpha > 0$, all powers are meant to be principal determination only. The class of function in (2.1) is denoted by $B(\alpha, \beta, g)$.
The class had been investigated severally by many authors, the likes of Yamaguchi [19], Noonan [7], Oladipo [8], Oladipo and Breaz [9], Thomas [17], Abduhalim [1], Opoola [10], Macgregor [6], Singh [15], Tuan and Anh [18], and many others, but to the best of our knowledge not in the direction of the present investigation. The class of Bazilevic function is almost the biggest family of univalent function and a very useful tool in geometric function theory. Except that, he, Bazilevic showed that each function $f \in B(\alpha, \beta, g)$ is univalent in D, very little is known regarding the family as a whole. However, with some simplifications, it may be possible to understand and investigate the family. Indeed, it is easy to verify that, with special choices of the parameters α and β and the function $g(z)$, the family $B(\alpha, \beta, g)$ crack down to some well-known subclasses of univalent functions.
For instance, if we choose $\beta = 0$ in (2.1), we have

$$f(z) = \left\{ \alpha \int_0^z \frac{p(v)}{v} g(v)^\alpha dv \right\}^{\frac{1}{\alpha}}. \tag{2.2}$$

On differentiating (2.2) we have

$$\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^\alpha} = p(z).$$

Or equivalently

$$Re\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^\alpha} > 0. \tag{2.3}$$

The subclasses of Bazilevic functions satisfying (2.3) are called Bazilevic functions of type α and are denoted by $B(\alpha)$ see Singh [15]. In 1973, Noonman [7] gave a plausible description of functions of the class $B(\alpha)$ as those functions in S for which $r < 1$ and the tangent to the curve $D_\alpha(r) = \left\{\beta f\left(re^{i\theta}\right)^\alpha, 0 \le \theta < 2\pi\right\}$ never turns back on itself as much as π radian. If α is taking as 1, the class $B(\alpha)$ reduces to the family of close-to-convex function. That is,

$$Re\frac{zf'(z)}{g(z)} > 0 \quad z \in D. \tag{2.4}$$

Suppose we replace $g(z)$ by $f(z)$ in (2.5) then we have

$$Re\frac{zf'(z)}{f(z)} > 0 \quad z \in D,$$

which implies that $f(z)$ is starlike.
Furthermore, in 1992, Abdulhalim [1] introduced a generalization of functions satisfying (4) by putting $g(z)^\alpha \equiv z^\alpha$ as

$$Re\frac{D^n f(z)^\alpha}{z^\alpha} > 0 \quad z \in D, \tag{2.5}$$

which are largely non-univalent in the unit disk, but by proving the inclusion

$$B_{n+1}(\alpha) \subset B_n(\alpha),$$

Abdulhalim in [1] was able to show that for all $n \in N$, each function of the $B_1(\alpha)$ is univalent in D.
In 1994, Opoola [10], and also Babalola [2] gave a more generalized form of Abdulhalim's geometric condition (6), with some little modification in [2], by defining a class $T_n^\alpha(\beta)$ whose functions satisfying

$$Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta \quad z \in D$$

where $\alpha > 0$ is real $0 \le \beta < 1$ and D^n is the Salagean derivative operator defined in [14] as follows

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z),$$

$$D^n f(z) = D(D^{n-1}f(z)) = z(D^{n-1}f(z))' = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Furthermore, we wish to quickly say here that from (1.1) we can write that

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha.$$

Using binomial expansion on the above we obtain

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}.$$

Definition 2.1 Let f be analytic in U, and be given by (1.1). Then the function $f \in B_{n+1}(\alpha, \beta, g)$ if and only if

$$Re\frac{z\left(D^n f(z)^\alpha\right)'}{\alpha^{n+1} g(z)^\alpha} > \beta \tag{2.6}$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, $g(z)$ is any starlike function, that is $Re\frac{zg'(z)}{g(z)} > 0$ and D^n is the well known Salgean derivative operator defined earlier [14].
Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 ..... = 1 + \sum_{k=2}^{\infty} c_k z^k \tag{2.7}$$

which are regular in the open unit disk and satisfy $p(0) = 1, Rep(z) > 0$ for any $z \in U$. $p(z)$ is called the Cartheodory function [4].
Lemma 2.1 [13], Let the function $p \in P$ be given by the series in (2.7) then sharp estimate $|c_k| \leq 2, k = 1, 2, 3, ...$ holds, the inequality is sharp for each k.
Lemma 2.2 [13], The power series for $p(z) = 1 + \sum_{k=2}^{\infty} c_k z^k$, for $p \in P$; and let the function $f \in A$ be given by (1.1), then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some $x, |x| \leq 1$ and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\rho$$

for some complex value $\rho, |\rho| \leq 1$.

3 Main Results

Theorem 3.1 Let $\alpha > 0$ be real, $0 \leq \beta < 1, n \in N \cup \{0\}$, and $g(z)$ be any starlike function, for any starlike function $|b_k| \leq 2$. If the function $f(z)$ of the form (1.1) belongs to the class $B_{n+1}(\alpha, \beta, g)$ then

$$T_2(2) = \left|a_3^2(\alpha) - a_2^2(\alpha)\right| \leq \frac{4\alpha^{2(n+1)}(4 - 3\beta)^2}{(\alpha + 2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}(2 - \beta)^2}{(\alpha + 1)^{2(n+1)}}. \tag{3.1}$$

Proof: Let the function $f \in B_{n+1}(\alpha, \beta, g)$, there exists $p \in P$ such that

$$z(D^n f(z)^\alpha)' - \alpha^{n+1}\beta g(z)^\alpha = \alpha^{n+1}(1 - \beta)p(z)g(z)^\alpha. \tag{3.2}$$

Then equating the coefficient in (3.2) to obtain

$$a_2 = \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 1)^{n+1}} c_1 + \frac{\alpha^{n+1}}{(\alpha + 1)^{n+1}} b_2(\alpha),$$

$$a_3 = \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 1)^{n+1}} c_2 + \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 1)^{n+1}} c_1 b_2(\alpha) + \frac{\alpha^{n+1}}{(\alpha + 2)^{n+1}} b_3(\alpha),$$

$$a_4 = \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 3)^{n+1}} c_3 + \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 3)^{n+1}} c_2 b_2 + \frac{\alpha^{n+1}(1 - \beta)}{(\alpha + 3)^{n+1}} c_1 b_3 + \frac{\alpha^{n+1}}{(\alpha + 3)^{n+1}} b_4.$$

Thus we have

$$\left|a_3^2(\alpha) - a_2^2(\alpha)\right| =$$

$$\left|\frac{4\alpha^{2(n+1)}(1 - \beta)^2}{(\alpha + 2)^{2(n+1)}} c_1^2 + \frac{\alpha^{2(n+1)}(1 - \beta)^2}{(\alpha + 2)^{2(n+1)}} c_2^2 + \frac{4\alpha^{2(n+1)}(1 - \beta)^2}{(\alpha + 2)^{2(n+1)}} c_1^2 + \frac{4\alpha^{2(n+1)}(1 - \beta)}{(\alpha + 2)^{2(n+1)}} c_2\right|$$

$$+\left|\frac{8\alpha^{2(n+1)}(1 - \beta)}{(\alpha + 2)^{2(n+1)}} c_1 + \frac{4\alpha^{2(n+1)}}{(\alpha + 2)^{2(n+1)}} - \frac{\alpha^{2(n+1)}(1 - \beta)^2}{(\alpha + 1)^{2(n+1)}} c_1^2 - \frac{4\alpha^{2(n+1)}(1 - \beta)}{(\alpha + 1)^{2(n+1)}} c_1 - \frac{4\alpha^{2(n+1)}}{(\alpha + 1)^{2(n+1)}}\right|$$

Substituting for c_2 using Lemma 2.2 in the above equation we have

$$\left|a_3^2(\alpha) - a_2^2(\alpha)\right| =$$

$$\left|\frac{\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}c_1^4 + \frac{2\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c_1^3 + \frac{4\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c_1^2 - \frac{\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+1)^{2(n+1)}}c_1^2\right| +$$

$$\left|\frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c_1^2 + \frac{8\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c_1 - \frac{4\alpha^{2(n+1)}(1-\beta)}{(\alpha+1)^{2(n+1)}}c_1 + \frac{4\alpha^{2(n+1)}}{(\alpha+2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}}{(\alpha+1)^{2(n+1)}}\right|$$

$$+\left|\frac{\alpha^{2(n+1)}(1-\beta)^2}{2\,(\alpha+2)^{2(n+1)}}c_1^2 xH + \frac{\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}x^2H^2 + \frac{2\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}C_1xH + \frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}xH\right|$$

By Lemma 2.1, we have $|c_1| \leq 2$. Fo convinience of notation, we take $c_1 = c$ and assume without loss of generality that $c \in [0,2]$. applying triangle inequality with $H = 4 - c^2$ we obtain the following by substiting c_2 as defined in Lemma 2.2

$$\left|a_3^2(\alpha) - a_2^2(\alpha)\right| \leq \left|\frac{\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}c^4 + \frac{2\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c^3 + \frac{4\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c^2 - \frac{\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+1)^{2(n+1)}}c^2\right| +$$

$$\left|\frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c^2 + \frac{8\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c - \frac{4\alpha^{2(n+1)}(1-\beta)}{(\alpha+1)^{2(n+1)}}c + \frac{4\alpha^{2(n+1)}}{(\alpha+2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}}{(\alpha+1)^{2(n+1)}}\right|$$

$$+\frac{\alpha^{2(n+1)}(1-\beta)^2}{2\,(\alpha+2)^{2(n+1)}}c^2\,|x|\,H + \frac{\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}\,|x|\,2H^2 + \frac{2\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}c\,|x|\,H + \frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}\,|x|\,H$$

$$= \phi(|x|).$$

Differentiating $\phi(|x|)$ and obviously $\phi'(|x|) > 0$ on [0,1] and so $\phi(|x|) \leq \phi(1)$. Hence

$$\left|a_3^2(\alpha) - a_2^2(\alpha)\right| \leq \left|\frac{\alpha^{2(n+1)}(1-\beta)^2}{4\,(\alpha+2)^{2(n+1)}}c^4 + \frac{2\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c^3 + \frac{4\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+2)^{2(n+1)}}c_1^2 - \frac{\alpha^{2(n+1)}(1-\beta)^2}{(\alpha+1)^{2(n+1)}}c^2\right| +$$

$$\left|\frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c_1^2 + \frac{8\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}c_1 - \frac{4\alpha^{2(n+1)}(1-\beta)}{(\alpha+1)^{2(n+1)}}c_1 + \frac{4\alpha^{2(n+1)}}{(\alpha+2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}}{(\alpha+1)^{2(n+1)}}\right|$$

$$+\frac{\alpha^{2(n+1)}(1-\beta)^2}{2\,(\alpha+2)^{2(n+1)}}c^2H + \frac{\alpha^{2(n+1)}(1-\beta)^2}{2\,(\alpha+2)^{2(n+1)}}H^2 + \frac{\alpha^{2(n+1)}(1-\beta)^2}{2\,(\alpha+2)^{2(n+1)}}cH + \frac{2\alpha^{2(n+1)}(1-\beta)}{(\alpha+2)^{2(n+1)}}H$$

Trivially, we can show that the expression $\phi(|x|)$ has a maximum value

$$\frac{4\alpha^{2(n+1)}(4-3\beta)^2}{(\alpha+2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}(2-\beta)^2}{(\alpha+1)^{2(n+1)}} \quad on\ [0,2], \quad when \ \ c = 2$$

If we set $n = 0$ and $\beta = 0$ in Theorem 3.1 we have

Corollary 3.1 Let $\alpha > 0$ be real and if the function $f(z)$ be of the form (1.1) belongs to $B_0(\alpha, 0, g) \equiv B_0(\alpha, g)$ then

$$T_2(2) = \left|a_3^2(\alpha) - a_2^2(\alpha)\right| \leq 16\alpha^2\left[\frac{4}{(\alpha+2)^2} - \frac{1}{(\alpha+1)^2}\right]$$

Suppose $\beta = 0, n = 0$, and $\alpha = 1$ then we have
Corollary 3.2 Let $\alpha = 1$ be real and if the function $f(z)$ be of the form (1.1) belongs to $B_0(1, g) \equiv B_0(g)$ then

$$T_2(2) = \left|a_3^2 - a_2^2\right| \leq 3.11$$

Theorem 3.2 Let $0 \leq \beta < 1,\ \alpha > 0$ be real $n \in N \cup \{0\}$, if the function $f(z)$ be of the form (1.1) belongs to the class $B_{n+1}(\alpha, \beta, g)$ then

$$T_2(3) = \left|a_4^2(\alpha) - a_3^2(\alpha)\right| \leq \frac{4\alpha^{2(n+1)}(6-5\beta)^2}{(\alpha+3)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}(4-3\beta)^2}{(\alpha+2)^{2(n+1)}}$$

Proof: Following the method of proof of Theorem 3.1 and using Lemma 2.2 to express c_2 and c_3 interms of c_1 and letting $H = 4 - c_1^2$ and $M = (1 - |x|^2)\rho$ the desired results shall be obtained.
If we set $n = 0$ and $\beta = 0$ in Theorem 3.2 we have
Corollary 3.3 Let $\alpha > 0$ be real and if the function $f(z)$ be of the form (1.1) belongs to $B_1(\alpha, 0, g) \equiv B(\alpha, g)$ then

$$T_2(2) = \left|a_4^2(\alpha) - a_3^2(\alpha)\right| \leq 16\alpha^2 \left[\frac{9}{(\alpha+3)^2} - \frac{4}{(\alpha+2)^2}\right]$$

Suppose $\beta = 0, n = 0$, and $\alpha = 1$ then we have
Corollary 3.4 Let $\alpha = 1$ be real and if the function $f(z)$ be of the form (1.1) belongs to $B_1(\alpha, 0, g) \equiv B(\alpha, g)$ then

$$T_2(2) = \left|a_4^2(\alpha) - a_3^2(\alpha)\right| \leq 1.89$$

Theorem 3.3 Let $0 \leq \beta < 1, \alpha > 0$ be real, $n \in N \cup \{0\}$ and if the function $f(z)$ be of the form (1.1) belongs to the class $B_{n+1}(\alpha, \beta, g)$ then

$$T_3(2) = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)\right| \leq$$

$$\left[\frac{2\alpha^{(n+1)}(2-\beta)}{(\alpha+1)^{(n+1)}} - \frac{2\alpha^{(n+1)}(6-5\beta)}{(\alpha+3)^{(n+1)}}\right]\left[\frac{4\alpha^{2(n+1)}(2-\beta)^2}{(\alpha+1)^{2(n+1)}} - \frac{8\alpha^{2(n+1)}(4-3\beta)^2}{(\alpha+2)^{2(n+1)}} + \frac{4\alpha^{2(n+1)}(2-\beta)(6-5\beta)}{(\alpha+2)^{(n+1)}(\alpha+3)^{n+1}}\right]$$

Proof: The proof also follows the same method of Theorem 3.1 with some simple substitutions.
Theorem 3.4 Let $0 \leq \beta < 1,\quad \alpha > 0$ be real $n \in N \cup \{0\}$ and $g(z)$ is any starlike function, and if $f(z)$ is of form (1.1) belong to $B_{n+1}(\alpha, \beta, g)$ then

$$T_3(1) = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \leq 1 + \frac{32\alpha^{3(n+1)}(4-3\beta)(2-\beta)^2}{(\alpha+1)^{2(n+1)}(\alpha+2)^{2(n+1)}} - \frac{4\alpha^{2(n+1)}(4-3\beta)^2}{(\alpha+2)^{2(n+1)}} - \frac{8\alpha^{2(n+1)}(2-\beta)^2}{(\alpha+2)^{2(n+1)}}$$

Proof: The proof follows from the earlier Theorems 3.1 and the Lemmas 2.2.
With various choices of parameters involved α, β, n and g results for many existing classes for functions studied in earlier mentioned literatures could be derived and many new could be arrived at.

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