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Orthogonal Polynomials Connected to Stern-Stolz Series

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Abstract. A typical case study of orthogonal polynomials related to a divergent and convergent S-fractions connected to the Stern-Stolz series, namely,

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

and

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty$$

are presented.

1 Introduction

The convergent S-fraction

$$\frac{1}{1}_{+}\frac{x}{1}_{+}\frac{2x}{1}_{+}\frac{3x}{1}_{+\cdots+}\frac{(n+1)x}{1}_{+\cdots} = 1 + \sum_{n=1}^{\infty}(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)x^n$$

connected to Stern-Stolz series

$$\frac{1}{1} + \frac{1}{2} + \frac{2}{3} + \frac{1\cdot 3}{2\cdot 4} + \frac{2\cdot 4}{3\cdot 5} + \cdots$$

was taken up as a case study in [8]. Four orthogonal polynomials were constructed and only one pair showed classical nature. A similar case study was taken up for the convergent S-fraction which has asymptotic expansion,namely, the Euler's divergent series in the confluent hypergeometric family [9].

Four powerful results available in the literature, namely, the main theorem on convergence and divergence of S-fractions connected to Stern-Stolz series [6], Ramanujan's entry 17 in his second note book [3]on expanding a regular C-fraction into a power series expansion, Favard's theorem on orthogonality of polynomials [4] described by the three term recurrence relations and the theorem which gives useful criteria to describe orthogonal polynomials as classical or not [1], are applied to the typical case study of orthogonal polynomials related to a divergent and convergent S-fractions connected to the Stern-Stolz series, namely,

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

and

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty.$$

All the four orthogonal polynomials are nonclassical. The present investigation is aimed at getting an intuitive picture of effects of convergence and divergence of S-fraction on classical nature of orthogonal polynomials. Still one has to find answer to the question " Is there an S-fraction corresponding to a divergent power series from which one can construct four orthogonal polynomials such that all four polynomials are classical orthogonal polynomials?"

Motivated strongly by the above works, four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergents of a divergent S-fraction connected to Pade approximants for power series expansion. In Section two, we construct the power series using divergent S-fraction and compute four sequences of polynomials. In the third Section, we describe the orthogonality of the two polynomials extracted from denominators and two polynomials extracted from numerators. In the last Section, we shown that they are non classical orthogonal polynomials.

2 S-fraction and its power series expansion

In the literature [3, 10], each continued fraction can be converted into a power series and vice versa. Making use of this we construct the power series from the known divergent and convergent S-fractions. Following the literature, the divergent and convergent regular S - fractions are given by the following theorem [6, 7] :

Theorem 2.1. *The S-fraction $\mathbf{K}(a_n z/1)$ where all $a_n > 0$, has the following properties:*

(i) Its even and odd parts converge locally uniformly in $D = \{z \in C; |\arg(z)| < \pi\}$ to holomorphic functions.

(ii) It converges to a holomorphic function in D if and only if the Stern-Stolz series

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}}$$

of $\mathbf{K}(a_n/1)$ diverges to ∞.

(iii) It diverges for all $z \in D$ if the Stern-Stolz series converges.

Let us consider the divergent S-fraction

$$zD(z) = \frac{1^2 z}{1} {}_{+} \frac{1^2 \cdot 2^2 z}{1} {}_{+} \frac{2^2 \cdot 3^2 z}{1} {}_{+} \frac{3^2 \cdot 4^2 z}{1} {}_{+ \cdots +} \frac{(n-1)^2 \cdot n^2 z}{1} {}_{+ \cdots},$$

connected to Stern-Stolz series

$$\begin{aligned} \frac{1}{a_1} \quad &+ \quad a_1 \frac{1}{a_2} + \frac{1}{a_1} a_2 \frac{1}{a_3} + a_1 \frac{1}{a_2} a_3 \frac{1}{a_4} + \cdots + \prod_{k=1}^{n} (a_k)^{(-1)^{n-k+1}} + \cdots \\ &= \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6} < \infty \end{aligned}$$

and also consider the convergent S-fraction

$$zD_c(z) = \frac{1z}{1} {}_{+} \frac{1!z}{1} {}_{+} \frac{2!z}{1} {}_{+} \frac{3!z}{1} {}_{+ \cdots +} \frac{n!z}{1} {}_{+ \cdots},$$

connected to Stern-Stolz series

$$\begin{aligned} \frac{1}{a_1} \quad &+ \quad a_1 \frac{1}{a_2} + \frac{1}{a_1} a_2 \frac{1}{a_3} + a_1 \frac{1}{a_2} a_3 \frac{1}{a_4} + \cdots + \prod_{k=1}^{n} (a_k)^{(-1)^{n-k+1}} + \cdots \\ &= \quad \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty. \end{aligned}$$

Following result of Ranamujan in his Notebook II, Entry 17 [3] guide us to compute co-efficients of the power series starting from the regular C-fraction.

Entry 17 : Write

$$\frac{1}{1} {}_{+} \frac{a_1 x}{1} {}_{+} \frac{a_2 x}{1} {}_{+} \frac{a_3 x}{1} {}_{+ \cdots} = \sum_{k=0}^{\infty} A_k (-x)^k,$$

where $A_0 = 1$. Let

$$P_n = a_1 a_2 \cdots a_{n-1}(a_1 + a_2 + a_3 + \cdots + a_n).$$

Then

$$\begin{aligned} P_1 &= A_1, \\ P_2 &= A_2, \\ P_3 &= A_3 - a_1 A_2 \\ P_4 &= A_4 - (a_1 + a_2)A_3, \\ P_5 &= A_5 - (a_1 + a_2 + a_3)A_4 + a_1 a_3 A_3, \\ P_6 &= A_6 - (a_1 + a_2 + a_3 + a_4)A_5 + (a_1 a_3 + a_2 a_4 + a_1 a_4)A_4. \end{aligned}$$

In general, for all $n \geq 1$,

$$P_n = \sum_{0 \leq k < n/2} (-1)^k \phi_k(n) A_{n-k},$$

where $\phi_0(n) \equiv 1$ and $\phi_r(n), r \geq 1$, is defined recursively by

$$\phi_r(n+1) - \phi_r(n) = a_{n-1}\phi_{r-1}(n-1).$$

Applying the above result, we obtain the following power series of the desired S-fractions

$$\begin{aligned} D(x) &= \frac{1^2}{1} {}_{+} \frac{1^2 \cdot 2^2 x}{1} {}_{+} \frac{2^2 \cdot 3^2 x}{1} {}_{+} \frac{3^2 \cdot 4^2 x}{1} {}_{+ \cdots +} \frac{(n-1)^2 \cdot n^2 x}{1} {}_{+ \cdots} \\ &= 1 - C_1 x + C_2 x^2 - C_3 x^3 + \cdots \\ &= 1 - 4x + 160x^2 - 27136x^3 + 13195264x^4 - + \cdots \quad . \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} D_c(x) &= \frac{1}{1} {}_{+} \frac{1!x}{1} {}_{+} \frac{2!x}{1} {}_{+} \frac{3!x}{1} {}_{+ \cdots +} \frac{n!x}{1} {}_{+ \cdots} \\ &= 1 - c_1 x + c_2 x^2 - c_3 x^3 + \cdots \\ &= 1 - x + 3x^2 - 21x^3 + 459x^4 - 48069x^5 + - \cdots \quad . \end{aligned} \tag{2.2}$$

2.1 A divergent S-fraction and its power series expansion

In the context of Pade table [2], the divergent S-fraction provides a staircase sequence of Pade approximants

$$[0/0]_{D(x)}, [0/1]_{D(x)}, [1/1]_{D(x)}, [1/2]_{D(x)}, [2/2]_{D(x)}, \ldots, [n-1/n]_{D(x)}, [n/n]_{D(x)}, \ldots,$$

which are given by

$$\frac{A_1(x)}{B_1(x)} = \frac{1}{1} = \frac{P_0^{(0,0)}(x)}{Q_0^{(0,0)}(x)}, \qquad \frac{A_3(x)}{B_3(x)} = \frac{1+36x}{1+40x} = \frac{P_1^{(1,1)}(x)}{Q_1^{(1,1)}(x)}, \ \ldots, \frac{A_{2n+1}(x)}{B_{2n+1}(x)} = \frac{P_n^{(n,n)}(x)}{Q_n^{(n,n)}(x)}$$

and

$$\begin{aligned} \frac{A_2(x)}{B_2(x)} &= \frac{1}{1+4x} = \frac{P_0^{(0,1)}(x)}{Q_0^{(0,1)}(x)}, \ \frac{A_4(x)}{B_4(x)} = \frac{1+180x}{1+184x+576x^2} = \frac{P_1^{(1,2)}(x)}{Q_1^{(1,2)}(x)}, \\ &\ldots, \frac{A_{2n+2}(x)}{B_{2n+2}(x)} = \frac{P_n^{(n-1,n)}(x)}{Q_n^{(n-1,n)}(x)}. \end{aligned}$$

The even order convergents of divergent S-fraction:

Let us make use of definitions of even parts of S-fraction as given in [10]. $[n-1/n]_{D(x)}$ Pade approximants can be computed using the even part of S-fraction (2.1):

$$\frac{1}{1+(1\cdot 2^2)x}\,_-\,\frac{(1\cdot 2^2)(2^2\cdot 3^3)x^2}{1+\ (2^2+4^2)\ 3^2x}\,_-\ldots_-\,\frac{(2n-1)^2(2n)^4(2n+1)^2\ x^2}{1+((2n)^2+(2n+2)^2)(2n+1)^2x}\,_-\ldots.$$

The n^{th} convergent $\dfrac{A_{2n+2}(x)}{B_{2n+2}(x)}$ is given by

$$\frac{\left(1+(2n+1)^2((2n)^2+(2n+2)^2)x\right)A_{2n}(x)-(2n-1)^2(2n)^4(2n+1)^2x^2A_{2n-2}(x)}{\left(1+(2n+1)^2((2n)^2+(2n+2)^2)x\right)B_{2n}(x)-(2n-1)^2(2n)^4(2n+1)^2x^2B_{2n-2}(x)}$$

with

$$\frac{A_2(x)}{B_2(x)}=\frac{1}{1+4x},\qquad \frac{A_4(x)}{B_4(x)}=\frac{1+180x}{1+184x+576x^2},\quad n=2,3,\ldots\ .$$

The odd order convergents of divergent S-fraction:

Let us make use of definitions of odd parts of S-fraction as given in [10]. $[n/n]_{D(x)}$ Pade approximants can be computed using the odd part of S-fraction (2.1):

$$1-\frac{(1\cdot 2^2)x}{1+(1^2+3^2)2^2x}\,_-\,\frac{(2^2\cdot 3^2)(3^2\cdot 4^2)x^2}{1+(3^2+5^2)4^2x}\,_-\ldots_-\,\frac{(2n)^2(2n+1)^4(2n+2)^2x^2}{1+((2n+1)^2+(2n+3)^2)(2n+2)^2x}\,_-\ldots.$$

The n^{th} convergent $\dfrac{A_{2n+1}(x)}{B_{2n+1}(x)}$ is given by

$$\frac{\left(1+(2n+2)^2((2n+1)^2+(2n+3)^2)x\right)A_{2n-1}(x)-(2n)^2(2n+1)^4(2n+2)^2x^2A_{2n-3}(x)}{\left(1+(2n+2)^2((2n+1)^2+(2n+3)^2)x\right)B_{2n-1}(x)-(2n)^2(2n+1)^4(2n+2)^2x^2B_{2n-3}(x)}$$

with

$$\frac{A_1(x)}{B_1(x)}=\frac{1}{1},\qquad \frac{A_3(x)}{B_3(x)}=\frac{1+36x}{1+40x},\quad n=2,3,\ldots\ .$$

2.2 A convergent S-fraction and its power series expansion

In the context of Pade table [2], the convergent S-fraction provides a staircase sequence of Pade approximants

$$[0/0]_{D_c(x)},[0/1]_{D_c(x)},[1/1]_{D_c(x)},[1/2]_{D_c(x)},[2/2]_{D_c(x)},\ldots,[n-1/n]_{D_c(x)},[n/n]_{D_c(x)},\ldots,$$

which are given by

$$\frac{A_1(x)}{B_1(x)}=\frac{1}{1}=\frac{P_0^{(0,0)}(x)}{Q_0^{(0,0)}(x)},\qquad \frac{A_3(x)}{B_3(x)}=\frac{1+2x}{1+3x}=\frac{P_1^{(1,1)}(x)}{Q_1^{(1,1)}(x)},\ \ldots,\frac{A_{2n+1}(x)}{B_{2n+1}(x)}=\frac{P_n^{(n,n)}(x)}{Q_n^{(n,n)}(x)}$$

and

$$\frac{A_2(x)}{B_2(x)}=\frac{1}{1+x}=\frac{P_0^{(0,1)}(x)}{Q_0^{(0,1)}(x)},\ \frac{A_4(x)}{B_4(x)}=\frac{1+8x}{1+9x+6x^2}=\frac{P_1^{(1,2)}(x)}{Q_1^{(1,2)}(x)},$$
$$\ldots,\frac{A_{2n+2}(x)}{B_{2n+2}(x)}=\frac{P_n^{(n-1,n)}(x)}{Q_n^{(n-1,n)}(x)}.$$

The even order convergents of convergent S-fraction:

Let us make use of definitions of even parts of S-fraction as given in [10]. $[n-1/n]_{D_c(x)}$ Pade approximants can be computed using the even part of S-fraction (2.2):

$$\frac{1}{1+x}\,_-\,\frac{1!\cdot 2!x^2}{1+\ (2!+3!)\ x}\,_-\ldots_-\,\frac{(2n!)(2n-1)!x^2}{1+((2n!)+(2n+1)!)x}\,_-\ldots.$$

The n^{th} convergent $\dfrac{A_{2n+2}(x)}{B_{2n+2}(x)}$ is given by

$$\frac{(1+((2n)!+(2n+1)!)x)\,A_{2n}(x)-(2n-1)!(2n)!\,x^2A_{2n-2}(x)}{(1+((2n)!+(2n+1)!)x)\,B_{2n}(x)-(2n-1)!(2n)!\,x^2B_{2n-2}(x)}$$

with

$$\frac{A_2(x)}{B_2(x)}=\frac{1}{1+x},\qquad \frac{A_4(x)}{B_4(x)}=\frac{1+8x}{1+9x+6x^2},\quad n=1,2,3,\dots\ .$$

The odd order convergents of convergent S-fraction:

Let us make use of definitions of odd parts of S-fraction as given in [10]. $[n/n]_{D_c(x)}$ Pade approximants can be computed using the odd part of S-fraction (2.2):

$$1-\frac{1!x}{1+(1!+2!)x\,-}\,\frac{(2!\cdot 3!)x^2}{1+(3!+4!)x\,-}\dots\frac{(2n)!\cdot(2n+1)!x^2}{1+((2n+1)!+(2n+2)!)x\,-}\dots.$$

The n^{th} convergent $\dfrac{A_{2n+3}(x)}{B_{2n+3}(x)}$ is given by

$$\frac{(1+((2n+1)!+(2n+2)!)x)\,A_{2n+1}(x)-(2n)!(2n+1)!x^2A_{2n-1}(x)}{(1+((2n+1)!+(2n+2)!)x)\,B_{2n+1}(x)-(2n)!(2n+1)!x^2B_{2n-1}(x)}$$

with

$$\frac{A_1(x)}{B_1(x)}=\frac{1}{1},\qquad \frac{A_3(x)}{B_3(x)}=\frac{1+2x}{1+3x},\quad n=1,2,3,\dots\ .$$

3 Orthogonal polynomials extracted from S- fraction

In this Section, we describe the orthogonal polynomials thus extracted from S-fraction.

The desired orthogonal polynomials:

$$\begin{aligned}
p_n(x) &= x^nA_{2n+2}\left(\frac{1}{x}\right), \qquad q_n(x)=x^nB_{2n}\left(\frac{1}{x}\right),\\
r_n(x) &= x^nA_{2n+1}\left(\frac{1}{x}\right), \qquad s_n(x)=x^nB_{2n+1}\left(\frac{1}{x}\right),\\
n &= 0,1,2,\dots,\ \text{ where } B_0\left(\frac{1}{x}\right):=1.
\end{aligned}$$

3.1 Orthogonal polynomials extracted from divergent S- fraction

Orthogonality of $q_n(x)$ of divergent S- fraction :

Consider the series

$$D(x)=1-C_1x+C_2x^2-C_3x^3+C_4x^4-C_5x^5+\cdots+(-1)^nC_nx^n+\cdots\ ,$$

where D indicates that the power series is divergent. The linear moment generating function with respect to $D(x)$ denoted by L_D has n^{th} moment,

$$L_D\{x^n\}=(-1)^nC_n.$$

The three term recurrence relation of $q_n(x)$ is

$$\begin{aligned}
q_{n+1}(x) &= \left(x+(2n+1)^2((2n)^2+(2n+2)^2)\right)q_n(x)\\
&\quad -(2n-1)^2(2n)^4(2n+1)^2q_{n-1}(x),\\
q_0(x) &= 1,\qquad q_1(x)=x+4,\qquad n=2,3,\dots\ .
\end{aligned}\tag{3.1}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $q_n(x)$ is

$$L_D\{q_m(x)q_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$, and $\lambda_k = (2k-2)^2(2k-3)^2(2k-1)^2 \quad k = 2, 3, \ldots, n+1$.

Orthogonality of $s_n(x)$ of divergent S- fraction:
Following the literature [2], we obtain the series

$$D_1(x) = \frac{1 - D(x)}{x} = C_1 - C_2x + C_3x^2 - C_4x^3 + \cdots + (-1)^n C_{n+1}x^n + \cdots$$

The linear moment generating function with respect to $D_1(x)$ denoted by L_{D_1} has n^{th} moment

$$L_{D_1}\{x^n\} = (-1)^n C_{n+1}.$$

The three term recurrence relation of $s_n(x)$ is

$$\begin{aligned} s_{n+1}(x) &= (x + (2n+2)^2((2n+1)^2 + (2n+3)^2))s_n(x) \\ &\quad - (2n)^2(2n+1)^4(2n+2)^2 s_{n-1}(x), \\ s_0(x) &= 1, \qquad s_1(x) = x + 40, \quad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.2}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $s_n(x)$ is

$$L_{D_1}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (2n-2)^2(2n-1)^2(2n)^2, \quad k = 2, 3, \ldots, n+1$.

Orthogonality of $r_n(x)$ of divergent S- fraction :
Following the literature [2], we obtain the series

$$\frac{1}{D(x)} = 1 - E_1x + E_2x^2 - E_3x^3 + \cdots + (-1)^n E_nx^n + \cdots$$

and

$$D_2(x) = \frac{\frac{1}{D(x)} - 1}{x} = E_1 - E_2x + E_3x^2 - E_4x^3 + \cdots + (-1)^n E_{n+1}x^n + \cdots \ .$$

The linear moment generating function with respect to $D_2(x)$ denoted by L_{D_2} has n^{th} moment

$$L_{D_2}\{x^n\} = (-1)^n E_{n+1}.$$

The three term recurrence relation of $r_n(x)$ is

$$\begin{aligned} r_{n+1}(x) &= (x + (2n+2)^2((2n+1)^2 + (2n+3)^2))r_n(x) \\ &\quad - (2n)^2(2n+1)^2(2n+2)^2 r_{n-1}(x), \\ r_0(x) &= 1, \qquad r_1(x) = x + 36, \quad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.3}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $r_n(x)$ is

$$L_{D_2}\{r_m(x)r_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (2k-2)^2(2k-1)^2(2n)^2, \quad k = 2, 3, \ldots, n+1$.

Suppose $r_n(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0$. Since $L_{D_2}\{r_0(x)r_n(x)\} = 0$, we can compute E_n using

$$E_n = -[r_{n-1}E_{n-1} + \cdots + r_1E_1 + r_0], \quad E_0 = 1, \quad n = 1, 2, \ldots \ .$$

Orthogonality of $p_n(x)$ of divergent S- fraction:
Following the literature [2], we obtain the series

$$D_3(x) = \frac{\frac{1}{D(x)} - 1 - x}{x^2} = 1 - F_1x + F_2x^2 - F_3x^3 + \cdots + (-1)^nF_nx^n + \cdots \ .$$

The linear moment generating function with respect to $D_3(x)$ denoted by L_{D_3} has n^{th} moment

$$L_{D_3}\{x^n\} = (-1)^nF_n.$$

The three term recurrence relation of $p_n(x)$ is

$$\begin{aligned} p_{n+1}(x) &= \left(x + (2n+3)^2((2n+2)^2 + (2n+4)^2)\right)p_n(x) \\ &\quad - (2n+1)^2(2n+2)^2(2n+3)^2p_{n-1}(x), \\ p_0(x) &= 1, \qquad p_1(x) = x + 180, \quad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.4}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $p_n(x)$ is

$$L_{D_3}\{p_m(x)p_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (2k-1)^2(2k)^2(2k+1)^2, \quad k = 2, 3, \ldots, n+1$.

Suppose $p_n(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0$. Since $L_{D_3}\{p_0(x)p_n(x)\} = 0$, we can compute F_n using

$$F_n = -[p_{n-1}F_{n-1} + \cdots + p_1F_1 + p_0], \quad F_0 = 1, \quad n = 1, 2, \ldots \ .$$

3.2 Orthogonal polynomials extracted from convergent S- fraction

Orthogonality of $q_n(x)$ of convergent S- fraction :
Consider the series

$$D_c(x) = 1 - c_1x + c_2x^2 - c_3x^3 + c_4x^4 - c_5x^5 + \cdots + (-1)^nc_nx^n + \cdots \ ,$$

where D_c indicates that the power series is divergent and the continued fraction is convergent. The linear moment generating function with respect to $D_c(x)$ denoted by L_{D_c} has n^{th} moment,

$$L_{D_c}\{x^n\} = (-1)^nc_n.$$

The three term recurrence relation of $q_n(x)$ is

$$\begin{aligned} q_{n+1}(x) &= \left(x + ((2n)! + (2n+1)!)\right)q_n(x) - (2n-1)!(2n)!q_{n-1}(x), \\ q_0(x) &= 1, \qquad q_1(x) = x + 1, \qquad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.5}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $q_n(x)$ is

$$L_{D_c}\{q_m(x)q_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$, and $\lambda_k = (2k-2)!(2k-3)! \quad k = 2, 3, \ldots, n+1$.

Orthogonality of $s_n(x)$ of convergent S- fraction:
Following the literature [2], we obtain the series

$$D_{1_c}(x) = \frac{1 - D_c(x)}{x} = c_1 - c_2x + c_3x^2 - c_4x^3 + \cdots + (-1)^n c_{n+1}x^n + \cdots$$

The linear moment generating function with respect to $D_{1_c}(x)$ denoted by $L_{D_{1_c}}$ has n^{th} moment

$$L_{D_{1_c}}\{x^n\} = (-1)^n c_{n+1}.$$

The three term recurrence relation of $s_n(x)$ is

$$\begin{aligned} s_{n+1}(x) &= (x + ((2n+1)! + (2n+2)!))s_n(x) - (2n)!(2n+1)!s_{n-1}(x), \\ s_0(x) &= 1, \qquad s_1(x) = x + 3, \quad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.6}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $s_n(x)$ is

$$L_{D_{1_c}}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (2n-2)!(2n-1)!, \quad k = 2, 3, \ldots, n+1$.

Orthogonality of $r_n(x)$ of convergent S- fraction :
Following the literature [2], we obtain the series

$$\frac{1}{D_c(x)} = 1 - e_1x + e_2x^2 - e_3x^3 + \cdots + (-1)^n e_nx^n + \cdots$$

and

$$D_{2_c}(x) = \frac{\frac{1}{D_c(x)} - 1}{x} = e_1 - e_2x + e_3x^2 - e_4x^3 + \cdots + (-1)^n e_{n+1}x^n + \cdots \ .$$

The linear moment generating function with respect to $D_{2_c}(x)$ denoted by $L_{D_{2_c}}$ has n^{th} moment

$$L_{D_{2_c}}\{x^n\} = (-1)^n e_{n+1}.$$

The three term recurrence relation of $r_n(x)$ is

$$\begin{aligned} r_{n+1}(x) &= (x + ((2n+1)! + (2n+2)!))r_n(x) - (2n)!(2n+1)!r_{n-1}(x), \\ r_0(x) &= 1, \qquad r_1(x) = x + 2, \quad n = 1, 2, 3, \ldots \ . \end{aligned} \tag{3.7}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $r_n(x)$ is

$$L_{D_{2_c}}\{r_m(x)r_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = (2k-2)!(2k-1)!, \quad k = 2, 3, \ldots, n+1$.

Suppose $r_n(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0$. Since $L_{D_{2_c}}\{r_0(x)r_n(x)\} = 0$, we can compute e_n using

$$e_n = -[r_{n-1}e_{n-1} + \cdots + r_1e_1 + r_0], \quad e_0 = 1, \quad n = 1, 2, \ldots \ .$$

Orthogonality of $p_n(x)$ of convergent S- fraction:
Following the literature [2], we obtain the series

$$D_{3_c}(x) = \frac{\frac{1}{D_c(x)} - 1 - x}{x^2} = 1 - f_1x + f_2x^2 - f_3x^3 + \cdots + (-1)^n f_nx^n + \cdots \ .$$

The linear moment generating function with respect to $D_{3_c}(x)$ denoted by $L_{D_{3_c}}$ has n^{th} moment

$$L_{D_{3_c}}\{x^n\} = (-1)^n f_n.$$

The three term recurrence relation of $p_n(x)$ is

$$\begin{aligned} p_{n+1}(x) &= (x+((2n+2)!+(2n+3)!))\,p_n(x)-(2n+1)!(2n+2)!p_{n-1}(x), \\ p_0(x) &= 1, \qquad p_1(x)=x+8, \quad n=1,2,3,\dots \;. \end{aligned} \tag{3.8}$$

As a result of applying Favard's theorem, we obtain the orthogonality of $p_n(x)$ is

$$L_{D_{3_c}}\{p_m(x)p_n(x)\}=\begin{cases} 0, & m\neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m=n, \end{cases}$$

where $\lambda_1=1$ and $\lambda_k=(2k)!(2k+1)!, \quad k=2,3,\dots,n+1$.

Suppose $p_n(x)=x^n+p_{n-1}x^{n-1}+\cdots+p_1x+p_0$. Since $L_{D_{3_c}}\{p_0(x)p_n(x)\}=0$, we can compute f_n using

$$f_n=-[p_{n-1}f_{n-1}+\cdots+p_1f_1+p_0], \quad f_0=1, \quad n=1,2,\dots \;.$$

4 Nature of orthogonal polynomials

The following theorem [1], gives necessary and sufficient conditions for classical orthogonality of polynomials:

Theorem 4.1. *The pair* $\left\{P_n(x),\frac{d}{dx}\left(\frac{P_{n+1}(x)}{n+1}\right)\right\}$ *is a classical orthogonal polynomials if and only if*

A. $P_n(x)$ *form orthogonal polynomials with respect to* L.

B. $P_n(x)=\frac{d}{dx}\left(\frac{P_{n+1}(x)}{n+1}\right)-\alpha_n\frac{d}{dx}\left(\frac{P_n(x)}{n}\right)-\alpha_{n-1}\frac{d}{dx}\left(\frac{P_{n-1}(x)}{n-1}\right)$,
$n=2,3,\dots$, *where* α_n *and* α_{n-1} *are non-zero numbers.*

Let us reconsider the divergent S- fraction (2.1) and derive the following result.

Theorem 4.2. *The polynomials* $q_n(x)$, $s_n(x)$, $r_n(x)$ *and* $p_n(x)$ *of the divergent S- fraction are non-classical orthogonal polynomials.*

Proof. Using (3.1), (3.2), (3.3) and (3.4), we directly obtain the result that $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ of divergent S- fraction are orthogonal polynomials with respect to L_D, L_{D_1}, L_{D_2} and L_{D_3} respectively. Now, we observe that $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ do not satisfy the condition B of Theorem 4.1, because

$$\begin{aligned} q_3(x) &= \frac{q_4'(x)}{4}-3304\,\frac{q_3'(x)}{3}+\frac{1553536}{3}\,\frac{q_2'(x)}{2}+\frac{9742592}{3}\,\frac{q_1'(x)}{1}. \\ s_3(x) &= \frac{s_4'(x)}{4}-5428\,\frac{s_3'(x)}{3}+\frac{4841632}{3}\,\frac{s_2'(x)}{2}+30581376\,\frac{s_1'(x)}{1} \\ r_3(x) &= \frac{r_4'(x)}{4}-5429\,\frac{r_3'(x)}{3}+\frac{4835368}{3}\,\frac{r_2'(x)}{2}+\frac{105060112}{3}\,\frac{r_1'(x)}{1} \\ p_3(x) &= \frac{p_4'(x)}{4}-8368\,\frac{p_3'(x)}{3}+\frac{12794536}{3}\,\frac{p_2'(x)}{2}+\frac{563096000}{3}\,\frac{p_1'(x)}{1}. \end{aligned}$$

Hence $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ of divergent S- fraction are non-classical orthogonal polynomials. □

Let us reconsider the convergent S- fraction (2.2) and derive the following result.

Theorem 4.3. *The polynomials* $q_n(x)$, $s_n(x)$, $r_n(x)$ *and* $p_n(x)$ *of the convergent S- fraction are non-classical orthogonal polynomials.*

Proof. Using (3.5), (3.6), (3.7) and (3.8), we directly obtain the result that $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ of convergent S- fraction are orthogonal polynomials with respect to L_{D_c}, $L_{D_{1_c}}$, $L_{D_{2_c}}$ and $L_{D_{3_c}}$ respectively. Now, we observe that $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ do not satisfy the condition B of Theorem 4.1, because

$$\begin{aligned}
q_3(x) &= \frac{q_4'(x)}{4} - \frac{17127}{4}\frac{q_3'(x)}{3} + \frac{79755}{2}\frac{q_2'(x)}{2} + \frac{2907}{4}\frac{q_1'(x)}{1}. \\
s_3(x) &= \frac{s_4'(x)}{4} - \frac{135207}{4}\frac{s_3'(x)}{3} + \frac{6799350}{4}\frac{s_2'(x)}{2} + \frac{480627}{4}\frac{s_1'(x)}{1} \\
r_3(x) &= \frac{r_4'(x)}{4} - 33802\,\frac{r_3'(x)}{3} + \frac{5099080}{3}\frac{r_2'(x)}{2} + \frac{341216}{3}\frac{r_1'(x)}{1} \\
p_3(x) &= \frac{p_4'(x)}{4} - 300922\,\frac{p_3'(x)}{3} + 288528520\,\frac{p_2'(x)}{2} + 14591514688\,\frac{p_1'(x)}{1}.
\end{aligned}$$

Hence $q_n(x)$, $s_n(x)$, $r_n(x)$ and $p_n(x)$ of convergent S- fraction are non-classical orthogonal polynomials. □

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