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SOME RESULTS ON $\mathcal{F}-$FUSION BANACH FRAME

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Abstract. An extension of fusion Banach frame called $\mathcal{F}-$fusion Banach frame has been introduced and studied. Illustrative examples are given to show the existence of $\mathcal{F}-$fusion Banach frame. In the sequel, it has been proved that if a Banach space has a Banach frame, then it also has an $\mathcal{F}-$fusion Banach frame. A necessary and sufficient condition for a sequence of projections, associated with an $\mathcal{F}-$fusion Banach frame, to be unique has been given. Further, we consider complete $\mathcal{F}-$fusion Banach frame and prove that every weakly compactly generated Banach space has a complete $\mathcal{F}-$fusion Banach frame. Finally, we obtain a Paley-Wiener type stability result for $\mathcal{F}-$fusion Banach frame.

1 Introduction

In the last two decade, frames have been used in various disciplines of science and technology, for instance, signal processing, image processing, sampling theory, wireless communication, etc. These broad applications of frames attracted researchers from applied mathematics as well as pure mathematics due to the powerful tools from operator theory and Banach spaces are being employed to study frames. Recall that, a frame for the Hilbert space $\mathcal{H}$ is a sequence $\{x_n\} \subset \mathcal{H}$ of vectors satisfying:

$$A_1\|x\|^2 \leq \sum_{i=1}^{\infty} |<x, x_i>|^2 \leq A_2\|x\|^2, \qquad x \in \mathcal{H},$$

where the positive constants $0 < A_1 \leq A_2 < \infty$ are said to be lower and upper frame bounds.

The study of frames were first initiated by Duffin and Schaeffer [8] in 1952. In fact, they abstracted the fundamental notion of Gabor [10] for studying signal processing. The idea of Duffin and Scheaffer did not generate much general interest outside of nonharmonic Fourier series. It took more than 30 years to realize the importance and potential of frames. In 1980, R. Young [17] wrote a book in which he presented frames in the abstract setting and again used them in the context of nonharmonic Fourier series. Daubechies, Grossman and Meyer [9] later found a fundamental new application to wavelet and Gabor transforms in which frames play an important role. For detailed introduction on Hilbert frames one may refer to the book by O. Christensen[6].

Casazza and Kutyniok [2, 3, 4] introduced fusion frames (frames of subspaces) and fusion frames systems in Hilbert spaces to study the relation between a frame and its local components. In 1991, Grochenig [11] extended the notion of frames from Hilbert spaces to the Banach spaces and introduced Banach frames. Further, Jain et al. [13], generalized Banach frames and introduced fusion Banach frames (frames of subspaces) for Banach spaces. For more on Banach frame literature one may refer to [5, 12, 13, 14].

In this paper, we generalize fusion Banach frame (frame of subspaces) for Banach spaces and introduce $\mathcal{F}$-fusion Banach frame for Banach spaces. Examples have been given to show their existence. Further, we obtained some results on $\mathcal{F}-$fusion Banach frames. Our results generalized some results of Jain et al. [13] on fusion Banach frames.

More precisely, it has been proved that if a Banach space has a Banach frame, then it also has an $\mathcal{F}-$fusion Banach frame. Also, a necessary and sufficient condition for a sequence of projections, associated with an $\mathcal{F}-$fusion Banach frame, to be unique has been given. Further, a characterization for the exactness of an $\mathcal{F}-$fusion Banach frame has been given. Also, we consider complete $\mathcal{F}-$fusion Banach frame and prove that every weakly compactly generated Banach space has a complete $\mathcal{F}-$fusion Banach frame. Finally, we obtain a stability result for $\mathcal{F}-$fusion Banach frame.

2 Basic definitions and needed results

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d the associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. An associated Banach space is a linear space of sequences which is complete in the given norm, so that it becomes a Banach space.

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$. A sequence $\{x_n\}$ in E is called an M-basis for E if there exists a sequence $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker delta), for all $i, j \in \mathbb{N}$, $[x_n] = E$ and $\{f_n\}$ is total over E. A sequence of projections $\{v_n\}$ on E is total on E if $\{x \in E : v_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([11]). Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$,

(ii) there exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{f_n(x)\}\|_{E_d} \le B\|x\|_E, \qquad x \in E, \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \qquad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre frame operator*). The inequality (2.1) is called the *frame inequality*.

Definition 2.2 ([13]). Let E be a Banach space. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of nonzero linear projections such that $v_n(E) = G_n, n \in \mathbb{N}$. Let $\mathcal{A}$ be a Banach space associated with E and $S : \mathcal{A} \to E$ be an operator. Then $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (fusion Banach frame) for E with respect to $\mathcal{A}$, if

(i) $\{v_n(x)\} \in \mathcal{A}$, for all $x \in E$,

(ii) there exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{v_n(x)\}\|_{\mathcal{A}} \le B\|x\|_E, \qquad x \in E, \tag{2.2}$$

(iii) S is a bounded linear operator such that

$$S(\{v_n(x)\}) = x, \qquad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the frame of subspaces $(\{G_n, v_n\}, S)$. The operator $S : \mathcal{A} \to E$ is called the *reconstruction operator* (or, the *pre frame operator*). The inequality (2.2) is called the *frame of subspaces inequality*. Also, $(\{G_n, v_n\}, S)$ is called *exact* if there exists no reconstruction operator S_0 such that $(\{G_n, v_n\}_{n\neq i}, S_0)$ $(i \in \mathbb{N})$ is a frame of subspaces for E.

A Banach space E is called weakly compactly generated if there is a weakly compact subset W such that the span of W is dense in E.

We finish this section with the following results which will be used in the subsequent part of the paper.

Lemma 2.3 ([1]). *Let G be a separable subspace of a weakly compactly generated Banach space E. Then E has a separable complemented subspace G' containing G.*

Lemma 2.4 ([12]). *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin [f_i]_{i \neq n}$, for all n.*

Theorem 2.5 ([12]). *Let E be a Banach space having a Banach frame. Then E has a normalized tight Banach frame as well as a normalized tight and exact Banach frame.*

3 $\mathcal{F}-$fusion Banach frame

Let us begin this section with the definition of a new notion called $\mathcal{F}-$fusion Banach frame for a Banach space.

Definition 3.1. Let E be a Banach space and F be a closed linear subspace of E. Let $\{G_n\}$ be a sequence of subspaces of E such that $\cap_{n=1}^{\infty} G_n = F$. Let $\{u_n\}$ be a sequence of nonzero linear projections defined on F and let each v_n be a continuous extension of u_n from F to E such that

$$(u_n, v_n)(F, E) = \{\cup_{x \in F}\{u_n(x)\}\} \cup \{\cup_{x \in E \setminus F}\{v_n(x)\}\} = G_n, \qquad n \in \mathbb{N}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces associated with F and E respectively and $S : \mathcal{A} \to E, T : \mathcal{B} \to E$ are operators. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is called an $\mathcal{F}$ *– fusion Banach frame* for E with respect to $\mathcal{A}$ and $\mathcal{B}$ if

(i) $\{u_n(x)\} \in \mathcal{A}$, $x \in F$ and $\{v_n(x)\} \in \mathcal{B}$, $x \in E$.

(ii) $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F, \qquad x \in F.$
and there exist positive constants A, B $(0 < A \leq B < \infty)$ such that

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{B}} \leq B\|x\|_E, \qquad x \in E. \tag{3.1}$$

(iii) S and T be the bounded linear operators such that
$S(\{u_n(x)\}) = x$, for $x \in F$ and $T(\{v_n(x)\}) = x$, for $x \in E$.

The positive constants A and B, respectively, are called the *lower* and *upper frame bounds* of the $\mathcal{F}-$fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$. The operators $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ are called the *reconstruction operators*. The inequality (3.1) is called the *$\mathcal{F}-$fusion Banach frame inequality*. Further, the $\mathcal{F}-$fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$ is said to be *exact* if there exist no reconstruction operators S_0 and T_0 such that $(\{G_n, (u_n, v_n)\}_{n \neq i}, S_0, T_0)$, $(i \in \mathbb{N})$ is an $\mathcal{F}-$fusion Banach frame for E.

Remark 3.2. The $\mathcal{F}-$fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$ is a natural extension of fusion Banach frame $(\{G_n, v_n\}, T)$ for a Banach space E.

Now, we prove the following lemma which we shall use in the subsequent work.

Lemma 3.3. *Let E be a Banach space and F be a closed linear subspace of E. Let $\{G_n\}$ be a sequence of subspaces of E such that $\cap_{n=1}^{\infty} G_n = F$. Let $\{u_n\}$ be a sequence of nonzero linear projections defined on F and let each v_n be a continuous extension of u_n from F to E such that*

$$(u_n, v_n)(F, E) = \{\cup_{x \in F}\{u_n(x)\}\} \cup \{\cup_{x \in E \setminus F}\{v_n(x)\}\} = G_n, \qquad n \in \mathbb{N}.$$

If $\{u_n\}$ is total over F and $\{v_n\}$ is total over E, then $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ are Banach spaces with norm

$$\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F, \qquad x \in F$$

and

$$\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E, \qquad x \in E.$$

Proof. $\mathcal{A}$ and $\mathcal{B}$ are linear spaces under pointwise addition and scalar multiplication. The given norm on $\mathcal{A}$ and $\mathcal{B}$ are well defined because $\{u_n\}$ is total over F and $\{v_n\}$ is total over E. Finally the completeness of E gives the completeness of $\mathcal{A}$ and $\mathcal{B}$. □

Regarding existence of $\mathcal{F}-$fusion Banach frame we illustrate the following examples.

Example 3.4. Let $E = l^{\infty}(X) = \{\{x_n\} : x_n \in X; \sup_{1\leq n<\infty} \|x_n\| < \infty\}$ be a Banach space with norm given by

$$\|\{x_n\}\| = \sup_{1\leq n<\infty} \|x_n\|, \{x_n\} \in E,$$

where $(X, \|.\|)$ is a Banach space. For each $n \in \mathbb{N}$, define

$$G_n = \{\delta_1^x, \delta_n^x : x \in X\},$$

where $\delta_n^x = (0, 0, ..., 0, \underset{n^{th} place}{\overset{x}{\downarrow}}, 0, 0, ...)$, for all $n \in \mathbb{N}$, $x \in X$. Further, define

$$u_n(x) = \delta_1^{x_1}, \qquad x \in F,$$

$$v_n(x) = \delta_n^{x_n}, \qquad x \in E.$$

Then by Lemma 3.3, there exist associated Banach spaces $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F$, $x \in F$ and $\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E$, $x \in E$, respectively.

Define bounded linear operators $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ by $S(\{u_n(x)\}) = x$, for $x \in F$ and $T(\{v_n(x)\}) = x$, for $x \in E$, respectively. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is an $\mathcal{F}-$fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$.

Example 3.5. Let $E = l^2(X) = \{\{x_n\} : x_n \in X; \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}$ be a Banach space with norm given by

$$\|\{x_n\}\| = (\sum_{n=1}^{\infty} \|x_n\|^2)^{1/2}, \qquad \{x_n\} \in E,$$

where $(X, \|.\|)$ is a Banach space. For each $n \in \mathbb{N}$, define

$$G_n = \{\delta_1^x, \delta_{2n-1}^x + \delta_{2n}^x : x \in X\}$$

where $\delta_n^x = (0, 0, ..., 0, \underset{n^{th} place}{\overset{x}{\downarrow}}, 0, 0, ...)$, for all $n \in \mathbb{N}$, $x \in X$. Further, define

$$u_n(x) = \delta_1^{x_1}, \qquad x \in F,$$
$$v_n(x) = \delta_{2n-1}^{x_{2n-1}} + \delta_{2n}^{x_{2n}}, \ x \in E.$$

Then by Lemma 3.3, there exist associated Banach spaces $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F$, $x \in F$ and $\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E$, $x \in E$, respectively.

Define bounded linear operators $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ by $S(\{u_n(x)\}) = x$, for $x \in F$ and $T(\{v_n(x)\}) = x$, for $x \in E$ respectively. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is an $\mathcal{F}-$fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$.

Example 3.6. Let $E = c_0(X) = \{\{x_n\} : x_n \in X; \lim_{n\to\infty} \|x_n\|_X = 0\}$ be a Banach space with norm given by

$$\|\{x_n\}\| = \sup_{1\leq n<\infty} \|x_n\|_X, \qquad \{x_n\} \in E,$$

where $(X, \|.\|)$ is a Banach space. For each $n \in \mathbb{N}$, define

$$G_{2n-1} = \{\delta_1^x, \delta_{2n-1}^x - 2^{n-1}\delta_{2n}^x : x \in X\},$$
$$G_{2n} = \{\delta_1^x, \delta_{2n}^x : x \in X\}.$$

Also, for each $n \in \mathbb{N}$, define

$$u_n(x) = \delta_1^{x_1}, x \in F$$

$$v_{2n-1}(x) = \delta_{2n-1}^{x_{2n-1}} - 2^{n-1}\delta_{2n}^{x_{2n-1}}, x \in E$$

$$v_{2n}(x) = \delta_{2n}^{2^{n-1}x_{2n-1}+x_{2n}}, x \in E.$$

Then by Lemma 3.3, there exist associated Banach spaces $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F$, $x \in F$ and $\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E$, $x \in E$, respectively.

Define bounded linear operators $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ by $S(\{u_n(x)\}) = x$, for $x \in F$ and $T(\{v_n(x)\}) = x$, for $x \in E$, respectively. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is an $\mathcal{F}-$fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$.

4 Main Results

In this section, we elaborate our main results in the form of theorems. Our first result shows that a Banach space having Banach frame also has an $\mathcal{F}-$fusion Banach frame.

Theorem 4.1. *Let E be a Banach space. If E has a Banach frame, then E also has an $\mathcal{F}-$fusion Banach frame.*

Proof. By Theorem 2.5, E has an exact Banach frame say, $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$. Then, by Lemma 2.4, there exist a sequence $\{x_n\} \subset E$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

Now, define $G_n = [x_k, x_n]$, for some fixed positive integer $k \leq n$, $n \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} G_n = [x_k] = F$. Also, for each $n \in \mathbb{N}$, define

$$\begin{aligned} u_n(x) =& \sum_{i=1}^{n} f_i(x) x_i, \qquad x \in F \\ v_n(x) =& f_n(x) x_n, \qquad x \in E. \end{aligned}$$

Then

$$(u_n, v_n)(F, E) = \{\cup_{x \in F} \{u_n(x)\}\} \cup \{\cup_{x \in E \setminus F} \{v_n(x)\}\} = G_n, \qquad \forall n \in \mathbb{N}.$$

Therefore, by Lemma 3.3, $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ is a Banach space with norm given by $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F$, $x \in F$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm given by $\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E$, $x \in E$. Define $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ by $S(\{u_n(x)\}) = x$, $x \in F$ and $T(\{v_n(x)\}) = x$, $x \in E$. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is an $\mathcal{F}$- fusion Banach frame with respect to $\mathcal{A}$ and $\mathcal{B}$. □

Next, we give a characterization for the exactness of $\mathcal{F}$-fusion Banach frame.

Lemma 4.2. *An $\mathcal{F}$-fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$ $(S : \mathcal{A} \to E, T : \mathcal{B} \to E)$ for E with respect to $\mathcal{A}$ and $\mathcal{B}$, where $G_i \cap G_j = \{0\}$, for all $i \neq j$ is exact if and only if $u_n(x) \notin [u_i(x)]_{i \neq n}$, $n \in \mathbb{N}$, $x \in F$ and $v_n(x) \notin [v_i(x)]_{i \neq n}$, $n \in \mathbb{N}$, $x \in E$.*

Proof. Let for some $n \in \mathbb{N}$,

$$u_n(x) \in [u_i(x)]_{i \neq n}, \qquad \forall x \in F$$

and

$$v_n(x) \in [v_i(x)]_{i \neq n}, \qquad \forall x \in E.$$

Then

$$u_n(x) = \lim_{k \to \infty} \sum_{\substack{i=1 \\ i \neq n}}^{k} \alpha_i^{(k)} u_i(x), \qquad \forall x \in F$$

and

$$v_n(x) = \lim_{l \to \infty} \sum_{\substack{i=1 \\ i \neq n}}^{l} \beta_i^{(l)} v_i(x), \qquad \forall x \in E.$$

Therefore, $\{u_i\}_{i \neq n}$ is total over F, also by $\mathcal{F}$-fusion Banach frame inequality for $(\{G_n, (u_n, v_n)\}, S, T)$, $\{v_i\}_{i \neq n}$ is total over E. So, by Lemma 3.3, there exist associated Banach spaces $\mathcal{A}_1 = \{\{u_i(x)\}_{i \neq n} : x \in F\}$ and $\mathcal{B}_1 = \{\{v_i(x)\}_{i \neq n} : x \in E\}$ with norm given by

$$\|\{u_i(x)\}_{i \neq n}\|_{\mathcal{A}_1} = \|x\|_F, \qquad x \in F$$

and

$$\|\{v_i(x)\}_{i\neq n}\|_{\mathcal{B}_1} = \|x\|_E, \qquad x \in E$$

respectively. Let $S_1 : \mathcal{A}_1 \to E$ and $T_1 : \mathcal{B}_1 \to E$ are given by $S_1(\{u_i(x)\}_{i\neq n}) = x$, $x \in F$ and $T_1(\{v_i(x)\}_{i\neq n}) = x$, $x \in E$, respectively. Then $(\{G_i, (u_i, v_i)\}_{i\neq n}, S_1, T_1)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}_1$ and $\mathcal{B}_1$. Hence, $(\{G_n, (u_n, v_n)\}, S, T)$ is not exact.

Conversely, let $u_n(x) \notin [u_i(x)]_{i\neq n}$, for all $n \in \mathbb{N}$, $x \in F$ and $v_n(x) \notin [v_i(x)]_{i\neq n}$, for all $n \in \mathbb{N}$, $x \in E$. Now, there are nonzero $u_n(x) \in G_n$, for $x \in F$ and a nonzero $v_n(x) \in G_n$ for $x \in E$. So that $u_i(u_n(x)) = 0$ for all $i \neq n$ and $v_i(v_n(x)) = 0$ for all $i \neq n$, which contradict the totality of $\{u_n\}$ and $\{v_n\}$.

Therefore, there exist no associated Banach space $\mathcal{A}_0$ and $\mathcal{B}_0$ such that $(\{G_i, (u_i, v_i)\}_{i\neq n}, S, T)(S : \mathcal{A}_0 \to E, T : \mathcal{B}_0 \to E)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}_0$ and $\mathcal{B}_0$. Hence, $(\{G_n, (u_n, v_n)\}, S, T)$ is exact. □

The following result gives a necessary and sufficient condition for the uniqueness of the pair $(\{u_n\}, \{v_n\})$ of sequences of projections associated with $\mathcal{F}$-fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$ for E.

Theorem 4.3. *Let* $(\{G_n, (u_n, v_n)\}, S, T)$ $(S : \mathcal{A} \to E, T : \mathcal{B} \to E)$ *be an* $\mathcal{F}$*-fusion Banach frame for* E *with respect to* $\mathcal{A}$ *and* $\mathcal{B}$*. Then the pair* $(\{u_n\}, \{v_n\})$ *of associated sequences of projections is unique if and only if* $[\cup_{i=1}^{\infty} G_i] = E$.

Proof. Let $[\cup_{i=1}^{\infty} G_i] \neq E$. Then there exist an $0 \neq x_0 \in E \setminus [\cup_{i=1}^{\infty} G_i]$ and an $f_0 \in E^*$ such that $f_0(x_0) = 1$ and $f_0(y) = 0$, for all $y \in [\cup_{i=1}^{\infty} G_i]$.

Let $0 \neq x' \in F$. Define for each $n \in \mathbb{N}$,

$$p_n(x) = u_n(x + f_0(x)x'), \qquad \text{for } x \in F$$

$$q_n(x) = v_n(x + f_0(x)x'), \qquad \text{for } x \in E.$$

Then p_n and q_n are projections of F and E, respectively, onto G_n. Also, $\{p_n\}$ is total over F and $\{q_n\}$ is total over E. Therefore, By Lemma 3.3, there exists associated Banach spaces $\mathcal{A}_1 = \{\{p_n(x)\} : x \in F\}$ and $\mathcal{B}_1 = \{\{q_n(x)\} : x \in E\}$ with norm defined by $\|\{p_n(x)\}\|_{\mathcal{A}_1} = \|x\|_F$, $x \in F$ and $\|\{q_n(x)\}\|_{\mathcal{B}_1} = \|x\|_E$, $x \in E$, respectively.

Let $S_1 : \mathcal{A}_1 \to E$ and $T_1 : \mathcal{B}_1 \to E$ be given by $S_1(\{p_n(x)\}) = x$, $x \in F$ and $T_1(\{q_n(x)\}) = x$, $x \in E$. Then $(\{G_n, (p_n, q_n)\}, S_1, T_1)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}_1$ and $\mathcal{B}_1$.

Conversely, let $[\cup_{i=1}^{\infty} G_i] = E$ and let $(\{p_n\}, \{q_n\})$ be another pair of associated sequences of projections to the $\mathcal{F}$-fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$. Then for each $x \in E$, there exist vectors $\{x_i^{(n)}\}$ $(1 \leq i \leq m_n : n \in \mathbb{N})$ with $x_i^{(n)} \in G_i$ such that $x = \lim_{n\to\infty} \sum_{i=1}^{m_n} x_i^{(n)}$. Then $u_n(x) = p_n(x)$, for all $x \in F$ and $v_n(x) = q_n(x)$, for all $x \in E$. Hence $u_n = p_n$ and $v_n = q_n$, $n \in \mathbb{N}$. □

Let $(\{G_n, (u_n, v_n)\}, S, T)$ be an $\mathcal{F}$-fusion Banach frame and $(\{p_n\}, \{q_n\})$ be any pair of associated sequences of projections to $\{G_n\}$. In the following result, we obtain a necessary and sufficient condition under which there exist associated Banach spaces $\mathcal{A}_1$ and $\mathcal{B}_1$ with reconstruction operators $S_1 : \mathcal{A}_1 \to E$ and $T_1 : \mathcal{B}_1 \to E$ such that $(\{G_n, (p_n, q_n)\}, S_1, T_1)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}_1$ and $\mathcal{B}_1$.

Theorem 4.4. *Let* $(\{G_n, (u_n, v_n)\}, S, T)$ $(S : \mathcal{A} \to E, T : \mathcal{B} \to E)$ *be an* $\mathcal{F}$*-fusion Banach frame for* E *with respect to* $\mathcal{A}$ *and* $\mathcal{B}$*. Then for every pair* $(\{p_n\}, \{q_n\})$ *of associated sequences of projections to* $\{G_n\}$ *there exists reconstruction operators* $S_1 : \mathcal{A}_1 = \{\{p_n(x)\} : x \in F\} \to E$ *and* $T_1 : \mathcal{B}_1 = \{\{q_n(x)\} : x \in E\} \to E$ *such that* $(\{G_n, (p_n, q_n)\}, S_1, T_1)$ *is an* $\mathcal{F}$*-fusion Banach frame for* E *with respect to* $\mathcal{A}_1$ *and* $\mathcal{B}_1$ *if*

$$\left[\cup_{i=1}^{\infty} G_i\right] = E.$$

Proof. Let $[\cup_{i=1}^{\infty} G_i] \neq E$. Then there exist $0 \neq x_0 \in E \setminus [\cup_{i=1}^{\infty} G_i]$ and an $f_0 \in E^*$ such that

$$f_0(x_0) = 1 \text{ and } f_0(y) = 0, \text{ for all } y \in [\cup_{i=1}^{\infty} G_i].$$

Let $u_n(x_0) = x' \in F = \cap_{n=1}^{\infty} G_n$, $n \in \mathbb{N}$ and $v_n(x_0) = x_n \in G_n$, $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$,

$$p_n(x) = f_0(x)(u_n(x) - x'), \quad x \in F,$$

$$q_n(x) = v_n(x) - f_0(x)x_n, \quad x \in E.$$

Then $p_n(x_0) = 0$ and $q_n(x_0) = 0$, $n \in \mathbb{N}$. Therefore, there exist no associated Banach spaces $\mathcal{A}_1$ and $\mathcal{B}_1$ such that $(\{G_n, (p_n, q_n)\}, S_1, T_1)$ $(S_1 : \mathcal{A}_1 \to E, T_1 : \mathcal{B}_1 \to E)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}_1$ and $\mathcal{B}_1$.

The converse part follows in view of Theorem 4.3. □

An $\mathcal{F}$-fusion Banach frame $(\{G_n, (u_n, v_n)\}, S, T)$ $(S : \mathcal{A} \to E, T : \mathcal{B} \to E)$ for E with respect to $\mathcal{A}$ and $\mathcal{B}$ is called complete if $[\cup_{i=1}^{\infty} G_i] = E$. We now give a sufficient condition for a Banach space to have a complete $\mathcal{F}$-fusion Banach frame.

Theorem 4.5. *Every weakly compactly generated Banach space E has a complete $\mathcal{F}$-fusion Banach frame.*

Proof. In view of Lemma 2.3, Let G be a separable complemented subspace of E. Then $E = G_1 \oplus G$, where G_1 is a closed linear subspace of E. Let (u_1, u) and (v_1, v) be the pairs of continuous linear projections of F onto G and E onto G_1, respectively. Let $\{x_n\}$ be an M-basis of G with associated sequence of functionals $\{f_n\}$.

Define, $G_n = [x_k, x_{n-1}]$, for fix $k < n$, $n = 2, 3....$ Further, we define

$$u_n(x) = \sum_{i=1}^{n-1} f_i(u(x))x_i, \quad x \in F,$$

$$v_n(x) = f_{n-1}(v(x))x_{n-1}, \quad x \in E.$$

Then for each $n \in \mathbb{N}$, $\{u_n\}$ and $\{v_n\}$ are sequences of projections of F and E, respectively, onto G_n. Also $\{u_n\}$ is total over F and $\{v_n\}$ is total over E. Therefore by Lemma 3.3, there exist associated Banach spaces $\mathcal{A} = \{\{u_n(x)\} : x \in F\}$ and $\mathcal{B} = \{\{v_n(x)\} : x \in E\}$ with norm given by $\|\{u_n(x)\}\|_{\mathcal{A}} = \|x\|_F$, $x \in F$ and $\|\{v_n(x)\}\|_{\mathcal{B}} = \|x\|_E$, $x \in E$. Define $S : \mathcal{A} \to E$ and $T : \mathcal{B} \to E$ by $S(\{u_n(x)\}) = x$, for $x \in F$ and $T(\{v_n(x)\}) = x$, for $x \in E$, respectively. Then $(\{G_n, (u_n, v_n)\}, S, T)$ is a complete $\mathcal{F}$-fusion Banach frame E with respect to $\mathcal{A}$ and $\mathcal{B}$. □

In the next result we establish the existence of a complete $\mathcal{F}$-fusion Banach frame in a subspace of a given Banach space with an $\mathcal{F}$-fusion Banach frame.

Theorem 4.6. *Let $(\{G_n, (u_n, v_n)\}, S, T)$ $(S : \mathcal{A} \to E, T : \mathcal{B} \to E)$ be an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$. For each n, let F_n be a closed linear subspace of G_n. Let $E_0 = [\cup_{i=1}^{\infty} F_i]$. Then there exist sequences $\{p_n\}$ and $\{q_n\}$ of projections of F and E_0, respectively, onto F_n such that $(\{F_n, (p_n, q_n)\}, S_1, T_1)$ is a complete $\mathcal{F}$-fusion Banach frame for E_0, where $S_1 : \{\{p_n(x)\} : x \in F\} \to E_0$ and $T_1 : \{\{q_n(x)\} : x \in E_0\} \to E_0$ are the reconstruction operators.*

Proof. Let $x \in E_0$. Then, for some $k \in \mathbb{N}$, there exist $y \in G_k$, $z \in [\cup_{i=1, i \neq k}^{\infty} G_i]$ and $\{x_n\} \subset [\cup_{i=1}^{\infty} F_i]$ such that $x = y + z$, where $x = \lim_{n \to \infty} x_n$. Therefore,

$$u_k(x) = \lim_{n \to \infty} u_k(x_n) = y \in G_k, \quad \text{for } x \in F,$$

$$v_k(x) = \lim_{n \to \infty} v_k(x_n) = y \in G_k, \quad \text{for } x \in E_0.$$

Also, if $x \in G_k$ such that $x \notin F_k$, then $x \notin E_0$. Set $p_n = u_n$ and $q_n = v_n|_{E_0}$, $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $\{p_n\}$ and $\{q_n\}$ are sequences of projections of F and E_0 onto F_n. Also, $\{p_n\}$ is total over F and $\{q_n\}$ is total over E_0. Therefore, $\mathcal{A}_1 = \{\{p_n(x)\} : x \in F\}$ and $\mathcal{B}_1 = \{\{q_n(x)\} : x \in E_0\}$ are associated Banach spaces. Define $S_1 : \mathcal{A}_1 \to E_0$ and $T_1 : \mathcal{B}_1 \to E_0$ by $S_1(\{u_n(x)\}) = x$, $x \in F$ and $T_1(\{v_n(x)\}) = x$, $x \in E_0$, respectively. Then $(\{F_n, (p_n, q_n)\}, S_1, T_1)$ is a complete $\mathcal{F}$-fusion Banach frame for E_0 with respect to $\mathcal{A}_1$ and $\mathcal{B}_1$. □

Paley-Wiener type stability result for fusion Banach frame has been obtained in [16]. In the sequel, to conclude the paper, we establish the stability result for $\mathcal{F}$-fusion Banach frame and the result follows by using similar argument as given in the proof of Theorem 5.2 of [16].

Theorem 4.7. *Let E be a Banach space and F a closed linear subspace of E. Let $(\{G_n, (u_n, v_n)\}, S, T)$ be an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$. Let $\{p_n\}$, $\{q_n\}$ be the sequences of nonzero linear projections such that $(p_n, q_n)(F, E) = G_n$ and $\{p_n(x)\} \in \mathcal{A}$ and $\{q_n(x)\} \in \mathcal{B}$, $n \in \mathbb{N}$. Let $L_1 : \mathcal{A} \to \mathcal{A}$ and $L_2 : \mathcal{B} \to \mathcal{B}$ be bounded linear operators such that*

$$L_1(\{p_n(x)\}) = \{u_n(x)\}, \qquad L_2(\{q_n(x)\}) = \{v_n(x)\}, \qquad x \in E.$$

Then, there exists reconstruction operators $U : \mathcal{A} \to E$ and $V : \mathcal{B} \to E$ such that $(\{G_n, (p_n, q_n)\}, U, V)$ is an $\mathcal{F}$-fusion Banach frame for E with respect to $\mathcal{A}$ and $\mathcal{B}$, if and only if there exists constants M_1 and M_2 such that

$$\|(u_n - p_n)(x)\|_{\mathcal{A}} \leq M_1 \min\{\|u_n(x)\|_{\mathcal{A}}, \|p_n(x)\|_{\mathcal{A}}\}, \qquad x \in E$$

and

$$\|(v_n - q_n)(x)\|_{\mathcal{B}} \leq M_2 \min\{\|v_n(x)\|_{\mathcal{B}}, \|q_n(x)\|_{\mathcal{B}}\}, \qquad x \in E.$$

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