Palestine Journal of Mathematics

Vol. 8(1)(2019) , 299–306

On a Generalization of δ-Armendariz Rings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 6S36, 16U99, 16W20.

Keywords and phrases: McCoy rings, reversible rings, derivation.

This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research.

Abstract. For a derivation δ of a ring R, we introduce the δ-McCoy rings which are a generalization of the δ-Armendariz rings, and investigate their properties. Some properties of this generalization are established, and connections of properties of a δ-McCoy ring R with $n \times n$ upper triangular $T(R, n, \sigma)$ are investigated. We study relationship between the δ-McCoy property of R and its polynomial ring, $R[x]$. We also prove that every ring isomorphism preserves δ-McCoy structure. As a consequence we extend and unify several known results related to McCoy rings.

1 Introduction

Throughout this paper, all rings are associative with identity. We use $R[x]$ to denote the polynomial ring with indeterminate x over R. Denote E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. Let R be a ring, δ be a derivation of R, that is δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x; \delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = ax + \delta(a)$, for any $a \in R$. Rege and Chhawchharia[13] introduced the notion of an Armendariz ring. They defined a ring R to be an *Armendariz ring* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1 + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. The name "Armendariz ring" was chosen because Armendariz had been showed that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. According to cohn [2], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$. R is called *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = \{0\}$. Semicommutative rings are studied in papers of Du [3], Hirano [7], Huh, Lee and Smoktunowicz [8], and Nielnes [11]. Reduced rings are clearly reversible and reversible rings are semicommutative, but the converse is not true in general [11]. For a derivation δ, Nasr and Moussavi [10], introduced a generalization of reduced rings and Armendariz rings which they called a δ-*Armendariz* ring. They defined a ring R to be a δ-Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1 + \cdots + b_mx^m \in R[x, \delta]$ satisfy $f(x)g(x) = 0$ then $a_ix^ib_jx^j = 0$ for all i, j.

According to Nielson [11], a ring R is called *right McCoy* (resp., *left McCoy*) if for any polynomials $f(x), g(x) \in R[x]\backslash\{0\}$, $f(x)g(x) = 0$ implies $f(x)c = 0$ (resp., $sg(x) = 0$) for some $0 \neq c \in R$ (resp., $0 \neq s \in R$). A ring is called *McCoy* if it is both left and right McCoy. By McCoy [9], commutative rings are McCoy rings. Reduced rings are Armendariz and Armendariz rings are McCoy. Habibi, Moussavi and Alhevaz [4], called a ring R to be δ-*skew McCoy*, if for each polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1 + \cdots + b_mx^m \in R[x, \delta]$ satisfy $f(x)g(x) = 0$ then there exists $0 \neq c \in R$ such that $a_ix^ic = 0$ for all i.

Motivated by the above results, for a derivation δ of a ring R, we investigate a generalization of the δ-skew McCoy and δ-Armendariz rings which we call it δ-*McCoy* ring. We call a ring R δ-McCoy, if for each polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1 + \cdots + b_mx^m \in R[x, \delta]$, $f(x)g(x) = 0$ implies that there exists $0 \neq c \in R$ such that $f(x)c = 0$. Clearly, $a_ix^ic = 0$ for all i, implies $f(x)c = 0$ but the converse is not true. On the other hand, it is obvious that every δ-Armendariz ring is δ-McCoy but Example 2.1, shows that δ-McCoy rings are a proper generalization of δ-Armendariz rings.

2 δ -McCoy rings

We begin this section by the following definition and also we study properties of δ-McCoy rings.

Definition 2.1. Let δ be a derivation of a ring R. The ring R is called *δ-McCoy* if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\delta]$, $f(x)g(x) = 0$, implies that there exists $c \in R - \{0\}$ such that $f(x)c = 0$ i.e., $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c) = 0$ for $k = 0, 1, ..., m$.

It is clear that a ring R is right McCoy if R is 0-McCoy, where 0 is the zero mapping.

Proposition 2.2. *Let δ be a derivation of a ring R. Let S be a ring and $\varphi : R \to S$ be a ring isomorphism. Then R is δ-McCoy if and only if S is $\varphi\delta\varphi^{-1}$-McCoy.*

Proof. Let $\alpha' = \varphi\alpha\varphi^{-1}$ and $\delta' = \varphi\delta\varphi^{-1}$. Since $\delta'(ab)$=$\varphi\delta(\varphi^{-1}(a)\varphi^{-1}(b))$=$\varphi((\delta\varphi^{-1}(a)\varphi^{-1}(b)+ \varphi^{-1}(a)(\delta\varphi^{-1}(b)))$=$\delta'(a)b + a\delta'(b)$, then δ' is a derivation of S. Suppose a'=$\varphi(a)$, for each $a \in R$. Therefore $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x)$=$\sum_{j=0}^{n} b_j x^j$ are nonzero in $R[x;\delta]$ if and only if $p'(x)$=$\sum_{i=0}^{m} a'_i x^i$ and $q'(x)$=$\sum_{j=0}^{n} b'_j x^j$ are nonzero in $S[x;\delta']$. On the other hand, $p(x)q(x)$=0 if and only if $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l} a_i \delta^{i-l}(b_{k-l})$=0 if and only if $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l} a'_i \varphi(\delta^{i-l}(b_{k-l})) = 0$ if and only if $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l} a'_i \varphi(\varphi^{-1}\varphi\delta^{i-l}\varphi^{-1}\varphi(b_{k-l})) = 0$ if and only if $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l} a'_i \delta'^{i-l}(b'_{k-l})$=0 if and only if $p'(x)q'(x)$=0 for $k = 0, 1, ..., m{+}n$. Also $\sum_{l=k}^{m}\binom{l}{k} a_l \delta^{l-k}(c) = 0$, for some nonzero $c \in R$ if and only if $\varphi(\sum_{l=k}^{m}\binom{l}{k} a_l \delta^{l-k}(c))$=0 if and only if $\sum_{l=k}^{m}\binom{l}{k}\varphi(a_l)\varphi\delta^{l-k}\varphi^{-1}\varphi(c) = 0$ if and only if $\sum_{l=k}^{m}\binom{l}{k} a'_l \delta'^{l-k}(c') = 0$, for some nonzero c'=$\varphi(c) \in S$. Thus R is δ-McCoy if and only if S is $\varphi\delta\varphi^{-1}$-McCoy. □

For any derivation δ, R is said to be δ-compatible if for each $a, b \in R$, $ab = 0$ implies that $a\delta(b) = 0$. The following lemma is appeared in [6].

Lemma 2.3. *Let R be a δ-compatible ring. If $ab = 0$, then $a\delta^m(b) = 0 = \delta^m(a)b$, for all positive integer m.*

In the following result we prove that δ-McCoy rings is a fairly big class which includes for instance, reversible δ-compatible rings.

Theorem 2.4. *Every reversible δ-compatible ring is δ-McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials in $R[x;\delta]$ such that $f(x)g(x) = 0$. We can assume $g(x)$ has minimum degree that satisfies $f(x)g(x) = 0$ and $b_1 \neq 0$. As in the proof of [4, Theorem 3.6], we can show that $a_i b_j = 0$, for each i and j, and this implies $\sum_{l=k}^{m}\binom{l}{k} a_l \delta^{l-k}(b_1) = 0$ by Lemma 2.3, and so R is δ-McCoy. Since $f(x)g(x) = 0$ and R is reversible, we have $a_m b_n = 0 = b_n a_m$. So $b_n x^n a_m = 0$, since R is δ-compatible. On the other hand, $f(x)g(x)a_m = f(x)(\sum_{j=0}^{n} b_j x^j)a_m = 0$. Thus $f(x)(b_0 + ... + b_{n-1}x^{n-1})a_m = 0$. Since the degree of $g(x)$ is minimum, we have $(b_0 + ... + b_{n-1}x^{n-1})a_m = 0$. So $b_j a_m = a_m b_j = 0$, for each $0 \leq j \leq n-1$, since R is reversible and δ-compatible. Hence $a_m x^m b_j = 0$, for $0 \leq j \leq n$, since R is δ-compatible. So $(a_0 + ... + a_{m-1}x^{m-1})g(x) = 0$, and hence $a_{m-1}b_n = 0$. Therefore, $a_{m-1}b_n = b_n a_{m-1} = 0$. On the other hand, we have $f(x)g(x)a_{m-1} = 0$. This implies that $f(x)(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since $b_n x^n a_{m-1} = 0$. Thus we have $(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since the degree of $g(x)$ is minimum, and so according to above $a_{m-1}b_j = b_j a_{m-1} = 0$, for each j. Continuing in this way, we get $a_i b_j = 0$, for each i and j, and the result follows. □

If we take $\delta = 0$ in Theorem 2.4, we deduce the following result.

Corollary 2.5. *Reversible rings are McCoy.*

The following result shows that, for any derivation δ of R, δ-McCoy ring R is a generalization of reduced rings.

Corollary 2.6. *Every reduced ring R is δ-McCoy, for any derivation δ of R.*

Now we turn our attention to study some extensions of δ-McCoy rings.
Let R_k be a ring, for each $k \in I$, δ_k a derivation of R_k and $R = \prod_{k\in I} R_k$. Then the map $\delta : R \to R$ defined by $\delta((a_k)) = (\delta_k(a_k))$ is a derivation of R.

Proposition 2.7. *Let R_k be a ring with a derivation δ_k, where $k \in I$. If R_k is δ_k- McCoy, for each $k \in I$ then $R = \prod_{k\in I} R_k$ is δ-McCoy.*

Proof. Let each R_k be a δ_k- McCoy ring, $R = \prod_{k\in I} R_k$ and $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\delta]\backslash\{0\}$ such that $f(x)g(x) = 0$, where $a_i = (a_i^{(k)})$ and $b_j = (b_j^{(k)})$. Consider $f_k(x) = \sum_{i=0}^{m} a_i^{(k)} x^i$ and $g_k(x) = \sum_{j=0}^{n} b_j^{(k)} x^j \in R[x;\delta_k]$. Since $f_k(x)g_k(x) = 0$ and R_k is δ_k-McCoy ring, there exists $s_k \in R_k$ such that $\sum_{l=t}^{m} \binom{l}{t} a_l^{(k)} \delta_k^{l-t}(s_k) = 0$. Thus,

$$\sum_{l=t}^{m} \binom{l}{t} (a_l^{(1)}, \cdots, a_l^{(k)}, \cdots) \delta^{l-t}(0, \cdots, s_k, 0, \cdots) =$$

$$(0, \cdots, \sum_{l=t}^{m} \binom{l}{t} a_l^{(k)} \delta^{l-t}(s_k), 0, \cdots) = 0.$$

Therefore R is δ-McCoy. □

Now we provide several examples of δ-McCoy rings. Let R be a ring and σ denotes an endomorphism of R with $\sigma(1) = 1$. In [1], the authors introduced skew triangular matrix ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to condition $E_{ij} r = \sigma^{j-i}(r) E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii} b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + ... + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \le j$ and denoted it by $T_n(R,\sigma)$. The derivation δ of R is extended to $\overline{\delta} : T_n(R,\sigma) \to T_n(R,\sigma)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R,n,\sigma)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R,n,\sigma)$. We can denote $A = (a_{ij}) \in T(R,n,\sigma)$ by $(a_{11}, ..., a_{1n})$. Then $T(R,n,\sigma)$ is a ring with addition point-wise and multiplication given by,

$$(a_0, ..., a_{n-1})(b_0, ..., b_{n-1}) = (a_0 b_0, a_0 * b_1 + a_1 * b_0, ..., a_0 * b_{n-1} + ... + a_{n-1} * b_0),$$

with $a_i * b_j = a_i \sigma^i(b_j)$, for each i and j. Therefore, clearly one can see that $T(R,\ n,\ \sigma) \cong R[x;\ \sigma]/(x^n)$, where (x^n) is the ideal generated by x^n in $R[x;\ \sigma]$.

We consider the following two subrings of $S(R,n,\sigma)$, as follows (see[5]),

$$A(R,n,\sigma) = \sum_{j=1}^{[\frac{n}{2}]} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=[\frac{n}{2}]+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1}$$

$$B(R,n,\sigma) = \{A + rE_{1k} | A \in A(R,n,\sigma), r \in R\}, n = 2k \ge 4.$$

Let σ be an endomorphism and δ a derivation of a ring R such that $\delta\sigma = \sigma\delta$. One can see that the map $\overline{\sigma} : R[x;\ \delta] \to R[x;\ \delta]$ defined by $\overline{\sigma}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \sigma(a_i) x^i$ is an endomorphism of the polynomial ring $R[x;\ \delta]$.

Theorem 2.8. *Let R be a ring, σ be an endomorphism and δ a derivation of R. Then S is $\overline{\delta}$- McCoy if and only if R is δ-McCoy, where S is one of the rings $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$, or $T(R,n,\sigma)$.*

Proof. We only prove that $S(R,n,\sigma)$ is $\overline{\delta}$-McCoy, and the proof of the other cases are similar. First, consider the map $\phi : S(R,n,\sigma)[x;\overline{\delta}] \to S(R[x;\delta],n,\overline{\sigma})$, given by $\phi(\sum_{k=0}^{r} A_k x^k) = (f_{ij})$, where $A_k = (a_{ij}^{(k)})$ in $S(R,n,\sigma)$ and $f_{ij}(x) = \sum_{k=0}^{r} a_{ij}^{(k)} x^k$ in $R[x;\delta]$, for each $0 \le k \le r$ and $1 \le i,j \le n$. It is easy to see that ϕ is an isomorphism. Suppose R is δ-McCoy. Let $p(x) = \sum_{k=0}^{r} A_k x^k$ and $q(x) = \sum_{t=0}^{s} B_t x^t$ be nonzero polynomials in $S(R,n,\sigma)[x;\overline{\delta}]$ such that

$p(x)q(x) = 0$, where $A_k = (a_{ij}^{(k)})$ and $B_t = (b_{ij}^{(t)})$ in $S(R, n, \sigma)$, for $0 \le k \le r$ and $0 \le t \le s$. Thus $(h_{ij}) = (f_{ij})(g_{ij}) = 0$, where $f_{ij}(x) = \sum_{k=0}^{r} a_{ij}^{(k)} x^k$ and $g_{ij}(x) = \sum_{t=0}^{s} b_{ij}^{(t)} x^l$ in $R[x;\ \delta]$, for $1 \le i, j \le n$. So we have the following equations,

$$h_{11} = f_{11} g_{11} = 0;$$

$$h_{12} = f_{11} g_{12} + f_{12} \overline{\sigma}(g_{11}) = 0;$$
$$h_{23} = f_{11} g_{23} + f_{23} \overline{\sigma}(g_{11}) = 0;$$
$$\vdots$$
$$h_{n-1,n} = f_{11} g_{n-1,n} + f_{n-1,n} \overline{\sigma}(g_{11}) = 0;$$
$$h_{13} = f_{11} g_{13} + f_{12} \overline{\sigma}(g_{23}) + f_{13} \overline{\sigma}^2(g_{33}) = 0;$$
$$\vdots$$

If $f_{11}(x) = 0$, clearly $\sum_{l=k}^{r} \binom{l}{k} A_l \overline{\delta}^{(l-k)}(E_{1n}) = 0$ for $k = 0, 1, ..., r$. Thus $S(R, n, \sigma)$ is $\overline{\delta}$-McCoy. Let $f_{11}(x) \neq 0$. By above equations, there exists a nonzero $g' \in \{g_{ij} | 1 \le i, j \le n\}$ such that $f_{11} g' = 0$. Since R is δ-McCoy, there exists $0 \neq c \in R$ such that $\sum_{l=k}^{r} \binom{l}{k} a_{11}^{(l)} \delta^{(l-k)}(c) = 0$ for $k = 0, 1, ..., r$. Let $C = cE_{1n}$. We have

$$\sum_{l=k}^{r} \binom{l}{k} A_l \overline{\delta}^{(l-k)}(C) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \sum_{l=k}^{r} \binom{l}{k} a_{11}^{(l)} \delta^{(l-k)}(c) \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = 0$$

for $k = 0, 1, ..., r$ and so $S(R, n, \sigma)$ is $\overline{\delta}$-McCoy. Conversely, suppose that $S(R, n, \sigma)$ is $\overline{\delta}$-McCoy. Let $f(x) = \sum_{i=0}^{r} a_i x^i$ and $g(x) = \sum_{j=0}^{s} b_j x^j$ be nonzero polynomials in $R[x; \delta]$ such that $f(x)g(x) = 0$. Let $F(x) = \sum_{i=0}^{r} (a_i I_n) x^i$ and $G(x) = \sum_{j=0}^{s} (b_j I_n) x^j$. Therefore, $F(x)G(x) = 0$. Since $S(R, n, \sigma)$ is $\overline{\delta}$-McCoy, there exists $0 \neq C = (c_{ij}) \in S(R, n, \sigma)$ such that $\sum_{l=k}^{r} \binom{l}{k} a_l I_n \delta^{(l-k)}(C) = 0$ for $k = 0, 1, ..., r$. Since C is nonzero, there exists nonzero C_{uv}, for some $1 \le u, v \le n$, and $\sum_{l=k}^{r} \binom{k}{l} a_l \delta^{(l-k)}(c_{uv}) = 0$, for $k = 0, 1, ..., r$. So R is δ-McCoy, and the result follows. □

Corollary 2.9. *For a ring R and for $n \ge 2$, let*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a \end{pmatrix} | a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} | a_1, a_2, \cdots a_n \in R \right\}.$$

Since $R_n = S(R, n, id_R)$ and $V_n(R) = T(R, n, id_R)$, then R_n (resp., $V_n(R)$) is $\overline{\delta}$-McCoy if and only if R is δ-McCoy by Theorem 2.7.

Note that $V_n(R) \cong R[x]/(x^n)$, where (x^n) is an ideal of $R[x]$ generated by x^n for $n \geq 2$. Hence we have the following corollary.

Corollary 2.10. *Let δ be a derivation of a ring R and $n \geq 2$. Then R is δ-McCoy if and only if the factor ring $R[x]/(x^n)$ is $\overline{\delta}$-McCoy.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the $T(R, M) = R \bigoplus M$ with the usual addition and the multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Let δ be a derivation of a ring R. Then δ is extended to the derivation $\overline{\delta} : T(R, R) \to T(R, R)$ by $\overline{\delta} \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} = \begin{pmatrix} \delta(r) & \delta(m) \\ 0 & \delta(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R, R)$.

Corollary 2.11. *Let δ be a derivation of a ring R. Then R is a δ-McCoy ring if and only if the trivial extension $T(R, R)$ is a $\overline{\delta}$-McCoy ring.*

It is clear that δ-Armendariz rings are δ-McCoy but the converse is not true by the following Example.

Example 2.12. $T(\mathbb{Z}_4, \mathbb{Z}_4)$ is 0-McCoy by corollary 2.5, but since $\mathbb{Z}_4$ is not reduced, it is not 0-Armendariz by [10, corollary 5.6].

Based on Theorem 2.8, one may suspect that $T_n(R)$ over a δ-McCoy ring is still $\overline{\delta}$-McCoy. But the following proposition erases the possibility.

Proposition 2.13. *Let R be a ring and δ a derivation of R. Then $T_n(R)$ is not $\overline{\delta}$-McCoy for any $n > 1$.*

Proof. Let $f(x) = E_{12} + E_{33} + E_{44} + \cdots + E_{nn} + E_{11}x$ and $g(x) = E_{12} - E_{22}x \in T_n(R)[x]$, where E_{ij}'s are the usual matrix units. Thus $f(x)g(x) = 0$, but if $f(x)C = 0$ for some $C = (c_{ij}) \in T_n(R)$ then $A + Bx = 0$ where

$$A = \begin{pmatrix} \delta(c_{11}) & c_{22} + \delta(c_{12}) & c_{23} + \delta(c_{13}) & \cdots & c_{2n} + \delta(c_{1n}) \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_{33} & \cdots & c_{3n} \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & c_{nn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and so $C = 0$. Therefore $T_n(R)$ is not $\overline{\delta}$-McCoy. □

Let I be an ideal and δ be a derivation of R. If $\delta(I) \subseteq I$, then $\delta' : R/I \to R/I$ defined by $\delta'(a + I) = \delta(I) + I$ for $a \in R$, is a derivation of the factor ring R/I. Now it is natural to ask whether R is a δ-McCoy ring if for any nonzero proper ideal I of R, R/I is $\overline{\delta}$-McCoy and I is δ-McCoy, where I considered as a δ-McCoy ring without identity. However, we have a negative answer to this question by the following example.

Example 2.14. Let F be a field and δ be a derivation of F. Consider $R = T_2(F)$, which is not $\overline{\delta}$- McCoy by Proposition 2.13. Next we show that R/I is δ'-McCoy and I is $\delta-$McCoy ring for any nonzero proper ideal I of R. Note that the only nonzero ideals of R are $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

First, let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $R/I \cong F$ and so R/I is δ'-McCoy, by Corollary 2.6. Let $f(x) = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix} x^i$ and $g(x) = \sum_{j=0}^{n} \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix} x^j$ be nonzero polynomials of $I[x]$ such that $f(x)g(x) = 0$, implying

$$f_1(x)g_1(x) = f_1(x)g_2(x) = 0, \tag{2.1}$$

where $f_1(x) = \sum_{i=0}^{m} a_i x^i$, $g_1(x) = \sum_{j=0}^{n} d_j x^j$, $g_2(x) = \sum_{j=0}^{n} d_j x^j \in F[x]$. If $f_1(x) = 0$, then $\sum_{l=k}^{m} \binom{l}{k} \begin{pmatrix} a_l & b_l \\ 0 & 0 \end{pmatrix} \overline{\delta}^{(l-k)}(E_{11}) = 0$ for $k = 0, 1, \cdots, m$. Suppose $f_1(x) \neq 0$. Since $g(x) \neq 0$, $g_1(x) \neq 0$. From (2.1) and the condition F is $\delta-$ McCoy, we have $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{(l-k)}(c) = 0$ for some nonzero $c \in F$, whence

$$\sum_{l=k}^{m} \binom{l}{k} \begin{pmatrix} a_l & b_l \\ o & 0 \end{pmatrix} \overline{\delta}^{(l-k)}(ce_{11}) = \begin{pmatrix} \sum_{l=k}^{m} \binom{l}{k} a_l \delta^{(l-k)}(c) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = 0$$

for $k = 0, 1, \cdots, m$. Next let $J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then R/J is δ'-McCoy and J is δ-McCoy by the same method. Finally, let $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Since $R/K \cong F \oplus F$, then R/K is δ'-McCoy by Proposition 2.6. Since for any $f(x) = \sum_{i=0}^{m} \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix} x^i$ $\in K[x]$, $\sum_{l=k}^{m} \binom{l}{k} \begin{pmatrix} 0 & a_l \\ 0 & 0 \end{pmatrix} \overline{\delta}^{(l-k)}(E_{12}) = 0$, K is obviously δ-McCoy.

For a ring R and a derivation δ of R, $\overline{\delta} : R[x] \to R[x]$ defined by $\overline{\delta}(f(x)) = \sum_{i=0}^{m} \delta(a_i)x^i$ for any $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ is a derivation of $R[x]$. Now, we have the following result.

Theorem 2.15. *Let R be a ring and δ a derivation of R. Then R is δ-McCoy if $R[x]$ is $\overline{\delta}$-McCoy.*

Proof. Suppose that $R[x]$ is $\overline{\delta}$-McCoy. Let $f(x)g(x) = 0$ for nonzero polynomials $f(x) = a_0 + a_1 x + \cdots a_m x^m$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$ in $R[x]$. Then let $f(y) = a_0 + a_1 y + \cdots + a_m y^m$, $g(y) = b_0 + b_1 y + \cdots b_n y^n \in (R[x])[y]$, where $(R[x])[y]$ is the polynomial ring with an indeterminate y over $R[x]$. Then $f(y)$ and $g(y)$ are nonzero since $f(x)$ and $g(x)$ are nonzero. Moreover $f(y)g(y) = 0$. So there exists a nonzero $c(x) = c_0 + c_1 x + \cdots + c_t x^t \in R[x]$ such that $f(y)c(x) = 0$, since $R[x]$ is $\overline{\delta}$-McCoy. Then $\sum_{l=k}^{m} \binom{l}{k} a_l \overline{\delta}^{l-k}(c(x)) = 0$ for $k = 0, 1, \cdots, m$. Therefore $\sum_{i=0}^{t} (\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_i)) x^i = 0$. Since $c(x)$ is nonzero, there exists a $c_p \neq 0$, $0 \leq c_p \leq t$. Hence $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_p) = 0$ and so R is δ-McCoy. □

A ring R is called right (resp., left) Ore if, for each $a, b \in R$ with b regular there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$ (resp. $b_1 a = ab_1$). It is well-known that R is a right Ore ring if and only if there exists the classical right quotient ring of R. In the following, we consider the classical quotient rings of δ-McCoy rings. Let R be an Ore ring with a classical right quotient ring Q. Then a derivation δ of R, extends to Q, by setting $\overline{\delta}(rc^{-1}) = (\delta(r) - r\delta(c)c^{-1})c^{-1}$, for each $r, c \in R$.

Theorem 2.16. *Let R be an Ore ring and δ a derivation of R. Then R is δ-McCoy if and only if the classical quotient ring Q of R is $\overline{\delta}$-McCoy.*

Proof. We only prove the sufficient condition. For this, first we show that for each element $f(x) \in Q[x; \overline{\delta}]$ there exists a regular element $c \in R$ such that $f(x) = f'(x)c^{-1}$, for some $f'(x) \in R[x; \delta]$, or equivalently $f(x)c \in R[x; \delta]$. The proof is by induction on $deg(f)$. The case $deg(f) = 0$ is clear. Now, suppose that for all elements $f(x) \in Q[x; \overline{\delta}]$ of degree less than n, the assertion holds, and let $f(x) = \sum_{i=0}^{n} a_i c_i^{-1} x^i \in Q[x; \overline{\delta}]$. Then $f(x)c_n = h(x) + a_n x^n$ with $h(x) \in Q[x; \delta]$ and $deg(h) < n$. By induction hypothesis, there exists some regular element e such that $h(x)e \in R[x; \delta]$. Thus we have $f(x)c_n e = h(x)e + a_n x^n c_n e \in R[x; \delta]$. Also de is a regular element in R, and the result follows. Next suppose that R is δ-McCoy. Let $f(x) = \sum_{i=0}^{m} a_i c_i^{-1} x^i$ and $g(x) = \sum_{j=0}^{n} b_j d_j^{-1} x^j \in Q[x; \overline{\delta}]$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = c^{-1} a'_i$ and $b_i d_i^{-1} = d^{-1} b'_j$ with c , d regular elements in R. Then we have $(\sum_{i=0}^{m} a'_i x^i) d^{-1} (\sum_{j=0}^{n} b'_j x^j) = 0$. By the above argument, there are a regular element $s \in R$ and $p(x) = \sum_{i=0}^{t} b''_i x^i \in R[x; \delta]$ such that $d^{-1}(\sum_{i=0}^{n} b'_i x^i) = (\sum_{i=0}^{t} b''_i x^i) e^{-1}$. Hence $(\sum_{i=0}^{m} a'_i x^i)(\sum_{i=0}^{t} b''_i x^i) = 0$. Since R is δ -McCoy, there exists $0 \neq r \in R$ such that $\sum_{l=k}^{m} \binom{l}{k} a'_l \delta^{l-k}(r) = 0$. Hence $\sum_{l=k}^{m} \binom{l}{k} a_l c_l^{-1} \overline{\delta}^{l-k}(r) = 0$. Therefore Q is $\overline{\delta}$-McCoy. □

Let R be a ring, δ a derivation of R and Δ a multiplicatively closed subset of R consisting of central regular elements. We define $\Delta^{-1}\delta : \Delta^{-1}R \longrightarrow \Delta^{-1}R$ by $\Delta^{-1}\delta(b^{-1}a) = (\delta(b))^{-1}a$ for any $b^{-1}a \in \Delta^{-1}R$. Then $\Delta^{-1}\delta$ is a derivation of $\Delta^{-1}R$.

Proposition 2.17. *Let R be δ-McCoy . Then $\Delta^{-1}R$ is $\Delta^{-1}\delta$-McCoy.*

Proof. Let $S = \Delta^{-1}R$ and $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ be nonzero polynomials in $S[x; \Delta^{-1}\delta]$ such that $f(x)g(x) = 0$. Then we can assume that $a_i = a'_i u^{-1}$ and $b_j = b'_j v^{-1}$ for some $a'_i, b'_j \in R$ and $u, v \in \Delta$ for all i, j. Set $f(x) = \sum_{i=0}^{n} a'_i x^i$, $g(x) = \sum_{j=0}^{m} b'_j x^j$. Thus $f'(x)g'(x) = 0$ in $R[x; \delta]$. Thus there exists $0 \neq c \in R$ such that $\sum_{l=k}^{m} \binom{l}{k} a'_l \delta^{l-k}(c') = 0$. Hence $\sum_{l=k}^{m} \binom{l}{k} a'_l (\Delta^{-1}\delta)^{l-k}(c') = 0$. Therefore S is $\Delta^{-1}\delta$-McCoy ring. □

Corollary 2.18. *Let $R[x, \delta]$ be a δ-McCoy ring. Then $R[x; x^{-1}, \delta]$ is a δ-McCoy ring.*

Proof. It is directly follows from proposition 2.17. Let $\Delta = \{1, x, x^2, \cdots\}$, then clearly Δ is a multiplicatively closed subset of $R[x, \delta]$ and $R[x, x^{-1}, \delta] = \Delta^{-1}R[x, \delta]$. □

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Received: July 23, 2016.
Accepted: April 20, 2017