4-Prime cordiality of \( m \) copies of some graphs

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Abstract. Let \( G \) be a \((p,q)\) graph. Let \( f : V(G) \to \{1,2,\ldots,k\} \) be a map. For each edge \( uv \), assign the label \( \gcd(f(u),f(v)) \). \( f \) is called \( k \)-prime cordial labeling of \( G \) if \( |v_f(i)−v_f(j)|≤1 \), \( i,j\in\{1,2,\ldots,k\} \) and \( |e_f(0)−e_f(1)|≤1 \) where \( v_f(x) \) denotes the number of vertices labeled with \( x \), \( e_f(1) \) and \( e_f(0) \) respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with a \( k \)-prime cordial labeling is called a \( k \)-prime cordial graph. In this paper we investigate 4-prime cordial labeling behavior of complete graph, book, flower, \( mC_n \) and some more graphs.

1 Introduction

All graphs in this paper are finite, simple and undirected. Let \( G \) be a \((p,q)\) graph where \( p \) refers the number of vertices of \( G \) and \( q \) refers the number of edge of \( G \). The number of vertices of a graph \( G \) is called order of \( G \), and the number of edges is called size of \( G \). In 1987, Cahit introduced the concept of cordial labeling of graphs \([1]\) Sundaram, Ponraj, Somasundaram \([5]\) have introduced the notion of prime cordial labeling. A prime cordial labeling of a graph \( G \) with vertex set \( V \) is a bijection \( f : V(G) \to \{1,2,\ldots,|V|\} \) such that each edge \( uv \) is assigned the label 1 if \( \gcd(f(u),f(v)) = 1 \) and 0 if \( \gcd(f(u),f(v)) > 1 \), then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. Also they discussed the prime cordial labeling behavior of various graphs. Recently Ponraj et al. \([7]\), introduced \( k \)-prime cordial labeling of graphs. In this paper we investigate 4-prime cordial labeling behavior of complete graph, book, flower, \( mC_n \) and some more graphs. Let \( x \) be any real number. Then \( [x] \) stands for the largest integer less than or equal to \( x \) and \( \lfloor x \rfloor \) stands for smallest integer greater than or equal to \( x \). Terms not defined here follow from Harary \([3]\) and Gallian \([2]\).

2 4-Prime cordial labeling

Remark 2.1. A 2-prime cordial labeling is a product cordial labeling. \([6]\)

Definition 2.2. The Join of two graphs \( G_1+G_2 \) is obtained from \( G_1 \) and \( G_2 \) and whose vertex set is \( V(G_1+G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1+G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\} \).

Definition 2.3. The graph \( W_n = C_n + K_1 \) is called a wheel. In a wheel, a vertex of degree 3 is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes.

Definition 2.4. Let \( G_1, G_2 \) respectively be \((p_1,q_1), (p_2,q_2)\) graphs. The corona of \( G_1 \) with \( G_2 \), \( G_1 \odot G_2 \) is the graph obtained by taking one copy of \( G_1 \) and \( p_1 \) copies of \( G_2 \) and joining the \( i \)th vertex of \( G_1 \) with an edge to every vertex in the \( i \)th copy of \( G_2 \).

Definition 2.5. The graph \( P_n \odot K_1 \) is called a comb.

Definition 2.6. The graph \( C_n \odot K_1 \) is called a crown.

Definition 2.7. The Cartesian product graph \( G_1 \square G_2 \) is defined as follows: Consider any two points \( u = (u_1,u_2) \) and \( v = (v_1,v_2) \) in \( V = V_1 \square V_2 \). Then \( u \) and \( v \) are adjacent in \( G_1 \square G_2 \).
whenever \( [u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)] \) or \( [u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)] \). The graph \( L_n = P_n \Box P_2 \) is called a ladder.

**Theorem 2.8.** \( m \) copies of the complete bipartite graph \( K_{n,n} \) is 4-prime cordial for all even values of \( m \).

**Proof.** It is easy to verify that \( K_1, K_2, K_3 \) are 4-prime cordial graphs. On the other hand suppose there exists a 4-prime cordial labeling \( f \) for \( n \geq 4 \) then we have the following cases.Let \( V(K_{n,n}) = V_1 \cup V_2 \). Consider the first copy. Assign the label 2 to all the vertices of \( V_1 \) and 4 to all the vertices of \( V_2 \). Next consider the second copy. In this copy assign the label 1 to all the vertices of \( V_1 \) and 3 to all the vertices of \( V_2 \). In the third copy assign the label to the vertices as in the first copy and the fourth copy as in the second copy. Proceeding like this assign the labels to the consecutive copies. That is assign the label to the vertices of the \( i^{th} \) copy as in \((i-2)^{th}\) copy (\(3 \leq i \leq m\)). his vertex labels is obviously a 4-prime cordial labeling since \( v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{mn}{2} \) and \( e_f(0) = e_f(1) = \frac{mn^2}{2} \).

\( \square \)

**Theorem 2.9.** (i) \( mL_n \) is 4-prime cordial for all even values of \( m \) and all values of \( n \).

(ii) If \( m \) is odd, then \( mL_n \) is 4-prime for all values of \( n \).

**Proof.** Clearly \( mL_n \) has and edges. The proof is divided into two cases depends on the parity of \( m \).

**Case 1.** \( m \) is even.

In the first copy assign the label 2 to all the left side vertices and 4 to all the right side vertices. We now move to the second copy. In this case assign the label as in previous second copy. Proceeding like this we assign the \( \frac{m}{2} \) copies. Next consider the \( \frac{m}{2} + 1^{th} \) copy. In this copy assign the labels 1,3 to the first two vertices of the left side. Then assign the next two vertices and so on. That is assign the label to the left sides as in the pattern \( 1,3,1,3,1,3,1,...,3 \). Similarly assign the right sides in the way \( 3,1,3,1,3,1,...,3 \). This procedure is continue the remaining \( \frac{m}{2} - 1 \) ladders. If \( f \) is this vertex labeling then \( v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{mn}{2} \) and \( e_f(0) = e_f(1) = \frac{mn^2}{2} \).

**Case 2.** \( m \) is odd.

As in case 1 assign the labels to the all the vertices of the first, second, third, ..., \( \frac{(m-1)}{2}^{th} \) copy. We now consider the last \( m^{th} \) copy. In the first column assign the label 2 to the first \( \frac{n+1}{2} \) vertices and in the second column assign the label 4 to the first \( \frac{n+1}{2} \) vertices. In the \( \frac{n+3}{2}^{th} \) vertices of the first column is labelled by 1 and assign the label 3 to the \( \frac{n+5}{2}^{th} \) vertices of the second copy. Next assign the label 1 to the \( \frac{n+5}{2}^{th} \) vertices and 3 to the \( \frac{n+7}{2}^{th} \) vertices of the first column. That is it is labelled as \( 1,3,1,3,1,3,...,1,3 \). In the second column the vertices \( \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \frac{n+9}{2}, \frac{n+11}{2}, \frac{n+13}{2} \) are labelled in the pattern \( 1,3,1,3,1,3,...,3,1 \). This vertex labeling \( f \) is a 4-prime cordial labeling follows from Table 1, 2.

<table>
<thead>
<tr>
<th>Nature of ( m ) and ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
<th>( v_f(4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) is even</td>
<td>( \frac{mn}{2} )</td>
<td>( \frac{mn}{2} )</td>
<td>( \frac{mn}{2} )</td>
<td>( \frac{mn}{2} )</td>
</tr>
<tr>
<td>( m ) is odd and ( n ) is odd</td>
<td>( \frac{mn-1}{2} )</td>
<td>( \frac{mn+1}{2} )</td>
<td>( \frac{mn-1}{2} )</td>
<td>( \frac{mn+1}{2} )</td>
</tr>
</tbody>
</table>

Table 1.

<table>
<thead>
<tr>
<th>Nature of ( m ) and ( n )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) is even</td>
<td>( \frac{m(n-2)}{2} )</td>
<td>( \frac{m(n-2)}{2} )</td>
</tr>
<tr>
<td>( m ) is odd and ( n ) is odd</td>
<td>( \frac{m(n-2)+1}{2} )</td>
<td>( \frac{m(n-2)-1}{2} )</td>
</tr>
</tbody>
</table>

Table 2.

**Theorem 2.10.** \( m \) copies of the comb \( C_n \odot K_1 \) is 4-prime cordial for all values of \( n \).
Proof. Case 1. $m$ is even.
Consider the first copy. Assign the label 2 to the all the pendent vertices and 4 to other vertices. Next we move the second copy. Assign the label 3 to all its pendent vertices and 1 to remaining vertices of this copy. In a similar way assign the label to the vertices of the third copy as in the first one and the fourth copy as in second copy. In general the vertices of the $i^{th}$ copy are labelled as in $(i-2)^{th}$ copy ($3 \leq i \leq m$).

Case 2. $m$ is odd.
Let $f$ be the 4-prime cordial labeling of $m$-1 copies of the comb $C_n \odot K_1$ as in theorem. Then the vertex labeling $f_{veg}$ is defined by

$$
\begin{align*}
  f_{veg}(u) &= \begin{cases} 
    f(u) & \text{if } u \text{ in m-1 copies of } C_n \odot K_1 \\
    g(u) & \text{if } u \text{ in } m^{th} \text{ copies of } C_n \odot K_1
  \end{cases}
\end{align*}
$$

Obviously $f_{veg}$ is a 4-prime cordial labeling of $(m(C_n \odot K_1))$.

\begin{theorem}
$m(B_{n,n})$ is 4-prime cordial for all values $m$.
\end{theorem}

Proof. Let $V(m(B_{n,n})) = \{u_i, v_i, u_i^1, v_i^1 : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(m(B_{n,n})) = \{u_iu_j, u_iv_i, v_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly the order and size of $m(B_{n,n})$ are $2mn + 2m$ and $2mn + 2m - 1$ respectively.

Case 1. $m$ is even.
Consider the first copy. Assign the label 2,4 respectively to the vertices $u_1$ and $v_1$. Next assign the label 2 to the all the pendent vertices $u^1_i$ ($1 \leq i \leq n$) and assign the label 4 to the pendent vertices $v^1_i$ ($1 \leq j \leq n$). Next we consider the second copy. In this copy assign the label 1,3 to the vertices $u_2$ and $v_2$ respectively. Then assign the label 1 to the pendent vertices $u^2_j$ ($1 \leq j \leq n$) and 1 to the vertices $v^2_i$ ($1 \leq j \leq n$). Next assign the label to the vertices of the third copy as in first copy and fourth copy as in second copy and so on. That is assign the label to the vertices of the $i^{th}$ copy as in $(i-2)^{th}$ copy. It is easy to verify that this vertex labeling is a 4-prime cordial labeling.

Case 2. $m$ is odd.
Assign the label to the vertices of first, second, ..., $(m-1)^{th}$ copy as in case 1. As in theorem assign the $m^{th}$ copy we get a 4-prime cordial labeling of $m(B_{n,n})$.

\begin{theorem}
$mB_n$ is 4-prime cordial labeling for all values of $m$.
\end{theorem}

Proof. Let $V(mB_n) = \{u_i, v_i, u_i^1, v_i^1 : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(mB_n) = \{u_iv_i, u_iu_j, v_iv_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Consider the first copy. Assign the label 2,4 respectively to the vertices $u_1$ and $v_1$. Next assign the label 2 to the vertices $u^1_i$ ($1 \leq i \leq n$) and assign the label 4 to the vertices $v^1_i$ ($1 \leq j \leq n$). We now move to the second copy. Assign the label 1,3 to the vertices $u_2$ and $v_2$ respectively. Next assign the label 1 to the vertices $u^2_i$, $u^2_j$, $v^2_i$, $v^2_j$ and 1 to the vertices $v^2_i$, $v^2_j$, $v^2_i$, $v^2_j$. Next consider the $i^{th}$ copy ($i \geq 3$). Assign the label to the vertices of the $i^{th}$ copy as in the $(i-2)^{th}$ copy ($3 \leq i \leq m$). This vertex labeling $f$ is a 4-prime cordial labeling since $v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{m}{2}(n + 1)$ and $e_f(0) = e_f(1) = \frac{m}{2}(2n + 1)$.

Case 2. $m$ is odd.
The 4-prime cordial labeling $f$ of first $(m-1)$ copies together with the 4-prime cordial labeling of $B_{n,n}$ as in theorem we get a 4-prime cordial labeling in this case.

\begin{theorem}
If $n$ is odd, then $W_n \cup W_n$ is 4-prime cordial.
\end{theorem}

Proof. Clearly $W_n \cup W_n$ has $2n+2$ vertices and $4n$ edges. Consider the first copy. Assinx the label 2 to the central vertex and 4 to the first $\frac{n-1}{2}$ consecutive rim vertices. Next assign 2 to the remaining $\frac{n+3}{2}$ rim vertices of this copy. We now move to the second copy. In this copy assign the label 1 to the central vertex. Assign the label 3,1 to the next two rim vertices. Then assign the label 3,1 to the next two rim vertices and 3,1 to the next two continuous in this process until we reach the last vertex. That is the rim vertices are labelled in the pattern 3,1,3,1,3,1... 3. It is easy to verify that the last vertex reveiced the label 3. For this vertex labeling all the edges in the first copy received the label 0 and second copy received the label 1. Hence $W_n \cup W_n$ is 4-prime cordial for the odd values $n$. 

\end{proof}
Theorem 2.14. Let $P_n$ be the path $v_1v_2...v_n$. Then the graph $Q_n$ with the set $V(Q_n) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n - 1\}$ and $E(Q_n) = E(P_n) \cup \{v_iw_i : 1 \leq i \leq n - 1\} \cup \{u_iv_i : 1 \leq i \leq n - 1\}$ is 4-prime cordial.

Proof. Clearly $Q_n$ has 3n-2 vertices 4n-4 edges.

Case 1. $n$ is odd.

First we consider path. Assign the labels 2,4 respectively to the vertices $v_1$ and $v_2$. Next assign the labels 2,4 to the vertices $v_3, u_4$ and so on until we reach the vertex $u_{n-1}$. That is the vertices $u_1, u_2, ..., u_n$ are labeled as 2,4,2,4,...,2,4,2. Note that $u_{n-1}$ received the label 2. Assign the label 3,1 to the two vertices $v_{n-1}$ and $u_{n-1}$. Then assign the label 3,1 to the next two vertices $u_{n+1}, u_{n+2}$. Proceeding like this until we reach the vertex $u_n$. We now move to the vertices $v_i$ and $w_i$. Assign the label 2 to the vertices $v_1, v_2, ..., v_{n-1}$ and 4 to the vertices $w_1, w_2, ..., w_{n-1}$. Next assign the labels 3,1 to the two vertices $u_{n+1}, u_{n+2}$ and 3,1 to the next two vertices $v_{n+3}, v_{n+4}$ and so on. Proceeding like this until we reach the vertex $u_n$. In this process the vertices $u_{n+1}, u_{n+2}, ..., u_n$ are received the labels 3,1,3,1... Similarly assign the labels to the vertices $u_{n+1}, u_{n+2}...$ in the pattern 1,3,1,3...

Case 2. $n$ is even.

As in case 1 assign the labels to the vertices $v_i$ (1 $\leq i \leq n - 1$), $v_i$ and $w_i$ (1 $\leq i \leq n - 2$). Next assign the labels 3,4 and 1 respectively to the vertices $v_{n-1}, w_{n-1}$ and $u_{n-1}$ respectively. If $n \equiv 0$ (mod 4) then interchange the labels of $v_{n-1}$ and $u_{n-1}$. If $n \equiv 2$ (mod 4) then interchange the labels of $w_{n-2}$ and $v_{n-1}$. The table 3 establish that this vertex labeling is a 4-prime cordial labeling of $Q_n$. □

<table>
<thead>
<tr>
<th>Nature of n</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n $\equiv$ 1 (mod 4)</td>
<td>$\frac{3n-3}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n-3}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>n $\equiv$ 3 (mod 4)</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n-3}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n-3}{4}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>n $\equiv$ 0 (mod 4)</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+3}{4}$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+3}{4}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>n $\equiv$ 2 (mod 4)</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+3}{4}$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+3}{4}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
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Table 3.

References


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