1 Introduction

Let $A$ denotes the class of all function $f(z)$ which are analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E,$$

(1.1)

Let $S$ be the subclass of $A$, consisting of univalent functions. Let $f \in A$ given by (1.1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in E.$$

We define the convolution product (or Hadamard) of $f$ and $g$ as

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} b_n a_n z^n, \quad z \in E.$$

(1.2)

The Koebe-one quarter theorem [11] shows that the image of $E$ under every univalent function $f \in A$ contains a disk $\{w : |w| < \frac{1}{4}\}$ of radius $\frac{1}{4}$. Every univalent function $f$ has an inverse $f^{-1}$ defined on some disk containing the disk $\{w : |w| < \frac{1}{4}\}$ and satisfying:

$$f^{-1}(f(z)) = z, \quad z \in E,$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4};$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

(1.3)

A function $f \in S$ is said to be bi-univalent on $E$ if $g = f^{-1}$ are both univalent on $E$. Lewin [27] studied the class of bi-univalent functions, obtained the bound $|a_2| \leq 1.51$. Netanyahu [28] showed that Max $|a_2| = \frac{1}{2}$. Brannan and Clunie [10] conjectured that $|a_2| \leq \sqrt{2}$. Ali et al. [1], Altinkaya and Yalcin [6, 7, 8], Frasin and Aouf [13], Hamidi and Jahangiri...
Evidently, Srivastava et al. [29, 30] and Bulut [9] investigate the coefficients bounds for the subclasses of bi-univalent functions. The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory see also [14, 31, 32]. Not much is known about the bounds on general coefficients \(|a_n|\), for \(n \geq 4\) of bi-univalent functions as Ali et al. [1] also declared the bounds for the \(n\)-th \((n \geq 4)\) coefficients of bi-univalent functions an open problem. In the literature only a few work determining the general coefficient \(|a_n|\), for \(n \geq 4\) for the analytic bi-univalent function given by (1.1). For more study see [2, 3, 9, 12, 15, 16, 17, 19, 20, 21, 23, 26, 33].

Using the Faber polynomial expansion of functions \(f\) of the form (1.1), the coefficients of its inverse map \(g = f^{-1}\) are given by,

\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,
\]

where

\[
K_{n-1}^{-n} = \frac{(-n)!}{(2n+1)!(n-5)!} a_2^{n+1} + \frac{(-n)!}{2(-n+1)!(n-3)!} a_2^{n-3} a_3
\]

\[
+ \frac{(-n)!}{(2n+3)!(n-4)!} a_2^{n-4} a_4
\]

\[
+ \frac{(-n)!}{2(-n+2)!(n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2]
\]

\[
+ \frac{(-n)!}{(2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-n+5) a_3 a_4]
\]

\[
+ \sum_{j=7}^{n} a_2^{n-j} V_j,
\]

and \(g = f^{-1}\) given by (1.3), \(V_j\) with \(7 \leq j \leq n\) is a homogeneous polynomial in the variables \(|a_2|, |a_3|, \ldots, |a_n|\) [4]. In particular, the first three terms of \(K_{n-1}^{-n}\) are

\[
\frac{1}{2} K_{1}^{-2} = -a_2,
\]

\[
\frac{1}{3} K_{2}^{-3} = 2a_3^2 - a_3,
\]

\[
\frac{1}{4} K_{3}^{-4} = -(5a_3^3 - 5a_2 a_3 + a_4).
\]

In general, for any \(p \in \mathbb{N}\) and \(n \geq 2\), an expansion of \(K_{n-1}^{p}\) [3] is,

\[
K_{n-1}^{p} = p a_n + \frac{p(p-1)}{2} D_{n-1}^{2} + \frac{p!}{(p-3)!} D_{n-1}^{3} + \ldots + \frac{p!}{(p-n+1)!(n-1)!} D_{n-1}^{n-1},
\]

where \(D_{n-1}^{p} = D_{n-1}^{p}(a_2, a_3, \ldots)\) [5] given by

\[
D_{n-1}^{m}(a_2, \ldots, a_n) = \sum_{\mu_1, \ldots, \mu_n} \frac{m! (a_2)^{\mu_2} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \quad \text{for } m \leq n.
\]

While \(a_1 = 1\), and the sum is taken over all nonnegative integer \(\mu_1, \ldots, \mu_n\) satisfying:

\[
\mu_1 + \mu_2 + \ldots + \mu_n = m,
\]

and

\[
\mu_1 + 2\mu_2 + \ldots + (n-1)\mu_{n-1} = n-1.
\]

Evidently, \(E_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}\), (see [2]), or equivalently,

\[
D_{n-1}^{m}(a_1, a_2, \ldots, a_n) = \sum_{\mu_1, \ldots, \mu_n} \frac{m! (a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \quad \text{for } m \leq n,
\]
again \( a_1 = 1 \), and the taking the sum over all nonnegative integer \( \mu_1, \ldots, \mu_n \) satisfying:
\[
\mu_1 + \mu_2 + \ldots + \mu_n = m, \\
\mu_1 + 2\mu_2 + \ldots + (n)\mu_n = n.
\]
It is clear that
\[
D_n^a(a_1, \ldots, a_n) = D_1^a,
\]
the first and last polynomials are
\[
D_n^a = a_n^a \quad \text{and} \quad D_1^a = a_n.
\]
For \( f(z) \) and \( g(z) \) analytic in \( E \), we say that \( f(z) \) is subordinate to \( g(z) \) (written as \( f \prec g \)) if there exists a Schwarz function
\[
u(z) = \sum_{n=1}^{\infty} u_n z^n,
\]
with \( u(0) = 0 \) and \( |u(z)| < 1 \) in \( E \), such that \( f(z) = g(u(z)) \). For the Schwarz function \( u(z) \), \( |u_n| \leq 1 \), see [11].

For a complex parameters \( a, b, c \), with \( c \neq 0, -1, -2, \ldots \), the generalized Hypergeometric function
\( \, _2F_1(a, b, c, k, z) \) is defined as
\[
\, _2F_1(a, b, c, k, z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b + kn)}{\Gamma(c + kn)n!} z^n
\]
\[
= 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_{n-1} \Gamma(b + k(n-1))z^{n-1}}{\Gamma(c + k(n-1))(n-1)!},
\]
(1.6)
where \( \Re(c - k) > 0, |z| < 1 \), and \( (a)_n \) is the Pochhammer symbol.

By using generalized Hypergeometric function given by (1.6) we define a convolution operator \( \mathcal{J}(a, b, c, k) \) as follows:
\[
\mathcal{J}(a, b, c, k) f(z) = z \, _2F_1(a, b, c, k; z) * f(z) = z + \sum_{n=2}^{\infty} \Upsilon_n a_n z^n,
\]
(1.7)
where
\[
\Upsilon(a, b, c, n) = \frac{\Gamma(c)(a)_{n-1} \Gamma(b + k(n-1))}{\Gamma(b) \Gamma(c + k(n-1))(n-1)!}.
\]
(1.8)

For convenience we write \( \Upsilon(a, b, c, n) = \Upsilon_n \).

Here in this investigation we use the Faber polynomial expansions for the class \( S[A, B, \Upsilon_n] \), to determine a general coefficients bounds \( |a_n| \), for \( n \geq 3 \).

2 Coefficient bounds for the function class \( S[A, B, \Upsilon_n] \)

**Definition 2.1.** A function \( f \) defined by (1.1) is said to be in the class \( S[A, B, \Upsilon_n] \) if the following condition are satisfied:
\[
\left( \frac{z [\mathcal{J}(a, b, c, k) f(z)]'}{\mathcal{J}(a, b, c, k) f(z)} \right) < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \ z \in E,
\]
(2.1)
and
\[
\left( \frac{z [\mathcal{J}(a, b, c, k) g(w)]'}{\mathcal{J}(a, b, c, k) g(w)} \right) < \frac{1 + Aw}{1 + Bw}, \quad -1 \leq B < A \leq 1, \ w \in E,
\]
(2.2)
where the function \( g(z) \) is given by (1.3), that is, the extension of \( f^{-1} \) to \( E \).

**Special Cases:**

i) For \( a = c \) and \( b = 1 \) in (2.1) and (2.2) we have the class \( S[A, B, \Upsilon_n] = S[A, B] \), defined by Hamidi and Jahangiri [17].
Lemma 2.2. [11, 21]. Let \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in A \) be a positive real part functions so that \( \Re(p(z)) > 0 \) for \( |z| < 1 \). If \( \alpha \geq \frac{1}{2} \). Then

\[
|p_2 + \alpha p_1^2| \leq 2 + \alpha |p_1|^2.
\]

Lemma 2.3. [17]. Let \( \varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \in A \) be a Schwarz function so that \( |\varphi(z)| < 1 \) for \( |z| < 1 \). If \( \gamma \geq 0 \). Then

\[
|\varphi_2 + \gamma \varphi_1^2| \leq 1 + (\gamma - 1) |\varphi_1|^2.
\]

3 Main Results

In this section, we will prove our main results.

**Theorem 3.1.** For \(-1 \leq B < A \leq 1\), if both functions \( f \) and \( f^{-1} \) map \( g = f^{-1} \) are in \( S[A, B, Y_n] \), for \( a_k = 0; 2 \leq k \leq n - 1 \), then

\[
|\alpha_n| \leq \frac{(A - B)}{(n - 1) Y_n}, \quad n \geq 3.
\]

**Proof.** For the function \( f \in S[A, B, Y_n] \) of the form (1.1) we have the expansion

\[
\frac{z [J(a, b, c, k)x] f(z)}{J(a, b, c, k)f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, ..., a_n) z^{n-1}, \quad (3.1)
\]

As for the inverse map \( g = f^{-1} \), considering (1.3) we obtain

\[
\frac{z [J(a, b, c, k)x] g(w)}{J(a, b, c, k)g(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(b_2, b_3, ..., b_n) w^{n-1}, \quad (3.2)
\]

where, \( b_n = \frac{1}{n} K_{n-1}^{n-1} (a_2, a_3, ...). \)

\[
F_1 = -Y_2 a_2,
F_2 = Y_2^2 a_2 - 2 Y_3 a_3,
F_2 = -Y_2^2 a_2 + 3 Y_2 Y_3 a_3 - 3 Y_4 a_4.
\]

In general

\[
F_{n-1}(a_2, a_3, ..., a_n) = \left[ \sum_{i_1 + 2i_2 + ... + (n-1)i_n = n-1} \left\{ A(i_1, i_2, i_3, ..., i_{n-1})(Y_2 a_2)^{i_1}(Y_3 a_3)^{i_2}...(Y_n a_n)^{i_n-1} \right\} \right].
\]

\[
A(i_1, i_2, i_3, ..., i_{n-1}) = (-1)^{(n-1) + 2i_1 + ... + n_{n-1}} \frac{(i_1 + i_2 + i_3 + ... + i_{n-1} - 1)! (n - 1)}{(i_1!)^2 (i_3!) (i_{n-1}!)}.
\]

Since, both functions \( f \) and its inverse map \( g = f^{-1} \) are in \( S[A, B, Y_n] \), by the definition of subordination, there exist two Schwarz functions \( p(z) = \sum_{n=1}^{\infty} c_n z^n \), and \( q(w) = \sum_{n=1}^{\infty} d_n w^n \), where \( z, w \in E \). So that we have

\[
\frac{z [J(a, b, c, k)x] f(z)}{J(a, b, c, k)f(z)} = \frac{1 + A(p(z))}{1 + B(p(z))} = 1 - \sum_{n=1}^{\infty} (A - B) K_n^{-1}(c_1, c_2, ..., c_n, B) z^n \quad (3.3)
\]

and

\[
\frac{z [J(a, b, c, k)x] g(w)}{J(a, b, c, k)g(w)} = \frac{1 + A(q(w))}{1 + B(q(w))} = 1 - \sum_{n=1}^{\infty} (A - B) K_n^{-1}(d_1, d_2, ..., d_n, B) w^n. \quad (3.4)
\]
In general [2, 3] for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of $K_n^p(k_1, k_2, ..., k_n, B)$

$$K_n^p(k_1, k_2, ..., k_n, B) = \frac{p!}{(p-n)!n!}k_1^n B^{n-1} + \frac{p!}{(p-n+1)!n!(n-2)!}k_1^{n-2}k_2 B^{n-2}$$

$$+ \frac{p!}{(p-n+2)!(n-3)!}k_1^{n-3}k_3 B^{n-3}$$

$$+ \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}k_3^2 B^{n-4} + \frac{p!}{2(n-p+3)!(n-5)!}k_1^{n-5}k_3k_4 B^{n-5}$$

$$+ \sum_{j \geq 6} k_1^{n-1}X_j,$$

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $k_1, k_2, ..., k_n$.

For the coefficients of the Schwarz functions $p(z)$ and $q(w)$ $|c_n| \leq 1$ and $|d_n| \leq 1$, [11]. Comparing the corresponding coefficients of (3.1) and (3.3) we have

$$F_{n-1}(a_2, a_3, ..., a_n) = (A - B)K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B)$$

(3.5)

which under the assumption $a_m = 0; 2 \leq k \leq n - 1$, we have

$$-(n - 1)Y_n a_n = -(A - B)c_{n-1}.$$  

(3.6)

Similarly corresponding coefficients of (3.2) and (3.4) we have

$$F_{n-1}(b_2, b_3, ..., b_n) = (A - B)K_{n-1}^{-1}(d_1, d_2, ..., d_{n-1}, B),$$

(3.7)

which by hypothesis, we obtain

$$-(n - 1)Y_n b_n = -(A - B)d_{n-1}.$$  

Note that for $a_m = 0; 2 \leq k \leq n - 1$, we have $b_n = -a_n$ and therefore

$$-(n - 1)Y_n a_n = -(A - B)d_{n-1}. $$  

(3.8)

Taking the absolute values of (3.6) and (3.8) we obtain the required result

$$|a_n| \leq \frac{(A - B)}{(n-1)Y_n}.$$  

For $a = c$ and $b = 1$ in Theorem 3.1, we have the following Corollary

**Corollary 3.2.** [17] For $-1 \leq B < A \leq 1$, if both functions $f$ and $f^{-1}$ map $g = f^{-1}$ are in $S[A, B]$, for $a_k = 0; 2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{(A - B)}{n - 1}, \quad n \geq 3.$$  

**Theorem 3.3.** For $-1 \leq B < A \leq 1$, if both functions $f$ and $f^{-1}$ map $g = f^{-1}$ are in $S[A, B, Y_n]$ then

$$|a_2| \leq \begin{cases} \frac{(A-B)}{2\sqrt{(1+A)}} & \text{if } 0 \leq B < A, \\ \frac{(A-B)}{1^2} & \text{otherwise,} \end{cases}$$

and

$$\left|a_3 - \frac{Y_n}{2}a_2^2\right| \leq \begin{cases} \frac{(A-B)}{2A} \left(1 - \frac{(A+1)}{(A-B)} |Y_2a_2|^2 \right)^{\frac{1}{2}}, & \text{if } A \leq 0, \\ \frac{(A-B)}{1^2}, & \text{if } A > 0. \end{cases}$$

(3.9)
Proof. For $n = 2, 3$ in (3.5) and (3.7) we have

$$\Upsilon_2 a_2 = (A - B)c_1,$$

(3.10)

$$\Upsilon_2^2 a_2^2 - 2\Upsilon_3 a_3 = (A - B)(Bc_1^2 - c_2),$$

(3.11)

$$-\Upsilon_2 a_2 = (A - B)d_1,$$

(3.12)

$$-3\Upsilon_2^2 a_2^2 + 2\Upsilon_3 a_3 = (A - B)(Bd_1^2 - d_2).$$

(3.13)

Taking absolute values of both sides of (3.10) and (3.12) we have

$$|a_2| \leq \frac{(A - B)}{\Upsilon_2}.$$

Adding (3.11) and (3.13) yields

$$-2\Upsilon_2^2 a_2^2 = (A - B)\left\{(Bc_1^2 - c_2) + (Bd_1^2 - d_2)\right\}.$$

Taking absolute values of both sides of the above equation, we obtain

$$2\Upsilon_2^2 |a_2|^2 \leq (A - B)\left\{|c_2 + (-B)c_1^2| + |d_2 + (-B)d_1^2|\right\}.$$

If $B \leq 0$, then by Lemma 2.3, we have

$$2\Upsilon_2^2 |a_2|^2 \leq (A - B)\left\{1 + (-B - 1)|c_1|^2 + 1 + (-B - 1)|d_1|^2\right\}.$$

By using

$$\frac{|\Upsilon_2 a_2|^2}{(A - B)} = |c_1|^2 = |d_1|^2,$$

we have

$$|a_2|^2 \leq \frac{(A - B)}{\Upsilon_2^2} - \frac{(1 + B)}{(A - B)} |a_2|^2.$$

After simple algebraic calculation we have

$$|a_2| \leq \frac{(A - B)}{\Upsilon_2 \sqrt{(1 + A)}}.$$

Obviously, for $A > 0$ we have

$$\frac{(A - B)}{\Upsilon_2 \sqrt{(1 + A)}} < \frac{(A - B)}{\Upsilon_2}.$$

Now rewrite equation (3.13) as

$$2\Upsilon_3 (a_3 - \frac{\Upsilon_2^2 a_2^2}{\Upsilon_3}) = (A - B)(Bd_1^2 - d_2) + \Upsilon_2^2 a_2^2.$$

By using $(A - B)^2d_1^2 = \Upsilon_2^2 a_2^2$ we obtain

$$2\Upsilon_3 (a_3 - \frac{\Upsilon_2^2 a_2^2}{\Upsilon_3}) = -(A - B)(d_2 - Ad_1^2).$$

Taking the absolute values of both sides gives

$$2\Upsilon_3 \left|a_3 - \frac{\Upsilon_2^2 a_2^2}{\Upsilon_3}\right| = (A - B) \left|d_2 + (-A)d_1^2\right|.$$

If $A \leq 0$, then by Lemma 2.3, we have

$$\left|a_3 - \frac{\Upsilon_2^2 a_2^2}{\Upsilon_3}\right| = \frac{(A - B)}{2\Upsilon_3} \left(1 + (-A - 1)|d_1^2|\right).$$
by using \( |d_1|^2 = \frac{|\Upsilon_2 a_1|^2}{(A-B)^2} \), we obtain
\[
\left| a_3 - \frac{\Upsilon_2^2}{\Upsilon_3} a_3^2 \right| = \frac{(A-B)}{2\Upsilon_3} \left( 1 - \frac{(A+1)}{(A-B)^2} |\Upsilon_2 a_2|^2 \right).
\]

For \( A > 0 \), we subtract (3.11) from (3.13) to get
\[
4\Upsilon_3 \left( a_3 - \frac{\Upsilon_2^2}{\Upsilon_3} a_3^2 \right) = (A-B) \left[ B(d_1^2 - c_1^2) + (c_2 - d_2) \right].
\]

Using the fact that \( c_1^2 = d_1^2 \) and taking the absolute values of both sides of the above equation, we obtain the desired inequality
\[
\left| a_3 - \frac{\Upsilon_2^2}{\Upsilon_3} a_3^2 \right| \leq \frac{(A-B)}{2\Upsilon_3}.
\]

\[\Box\]

For \( a = c \) and \( b = 1 \), we have following Corollary.

**Corollary 3.4.**[17] For \( -1 \leq B < A \leq 1 \), if both functions \( f \) and \( f^{-1} \) map \( g = f^{-1} \) are in \( S[A,B] \) then
\[
|a_2| \leq \begin{cases} 
\frac{(A-B)}{\sqrt{(1+A)}}, & \text{if } 0 \leq B < A, \\
(A-B), & \text{otherwise}.
\end{cases}
\]

And
\[
\left| a_3 - \frac{\Upsilon_2^2}{\Upsilon_3} a_3^2 \right| \leq \begin{cases} 
\frac{(A-B)}{2} \left( 1 - \frac{(A+1)}{(A-B)^2} |a_2|^2 \right), & \text{if } A \leq 0,
\\(A-B), & \text{if } A > 0.
\end{cases}
\]

**References**


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