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FABER POLYNOMIAL COEFFICIENTS ESTIMATES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we introduce certain new subclass of bi-univalent functions in an open unit disk associated with generalized Hypergeometric function. By using Faber polynomial expansions to find a general coefficient bounds $|a_n|$, for $n \geq 3$, of class of bi-subordinate functions subject to a gap series condition, also find initial coefficients bounds.

1 Introduction

Let $\mathcal{A}$ denotes the class of all function $f(z)$ which are analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in E, \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of univalent functions. Let $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad z \in E.$$

We define the convolution product (or Hadamard) of f and g as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} b_n a_n z^n, \qquad z \in E. \tag{1.2}$$

The Koebe-one quarter theorem [11] shows that the image of E under every univalent function $f \in \mathcal{A}$ contains a disk $\{w : |w| < \frac{1}{4}\}$ of radius $\frac{1}{4}$. Every univalent function f has an inverse f^{-1} defined on some disk containing the disk $\{w : |w| < \frac{1}{4}\}$ and satisfying:

$$f^{-1}(f(z)) = z, \ z \in E,$$

and

$$f(f^{-1}(w)) = w, \ |w| < r_0(f), \ \ r_0(f) \geq \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{1.3}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent on E if $g = f^{-1}$ are both univalent on E.
Lewin [27] studied the class of bi-univalent functions, obtained the bound $|a_2| \leq 1.51$. Netanyahu [28] showed that Max $|a_2| = \frac{4}{3}$. Brannan and Clunie [10] conjectured that $|a_2| \leq \sqrt{2}$. Ali et al. [1], Altinkaya and Yalcin [6, 7, 8], Frasin and Aouf [13], Hamidi and Jahangiri

[15, 16, 22, 23], Srivastava et al. [29, 30] and Bulut [9] investigate the coefficients bounds for the subclasses of bi-univalent functions.
The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory see also [14, 31, 32]. Not much is known about the bounds on general coefficients $|a_n|$, for $n \geq 4$ of bi-univalent functions as Ali et al. [1] also declared the bounds for the n-th $(n \geq 4)$ coefficients of bi-univalent functions an open problem. In the literature only a few work determining the general coefficient $|a_n|$, for $n \geq 4$ for the analytic bi-univalent function given by (1.1). For more study see [2, 3, 9, 12, 15, 16, 17, 19, 20, 21, 23, 26, 33].

Using the Faber polynomial expansion of functions f of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ are given by,

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-5)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 \\ &+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ &+\frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] \\ &+\sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned}$$

and $g = f^{-1}$ given by (1.3), V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, ... |a_n|$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -(5a_2^3 - 5a_2a_3 + a_4). \end{aligned} \tag{1.4}$$

In general, for any $p \in N$ and $n \geq 2$, an expansion of K_{n-1}^p [3] is,

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1}, \tag{1.5}$$

where $D_{n-1}^p = D_{n-1}^p(a_2, a_3....)$ [5] given by

$$D_{n-1}^m(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_1!, ..., \mu_{n-1}!}, \quad for\ m \leq n.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

and

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n - 1.$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$,(see [2]), or equivalently,

$$D_n^m(a_1, a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_1!, ..., \mu_n!}, \qquad \text{for } m \leq n,$$

again $a_1 = 1$, and the taking the sum over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\begin{aligned} \mu_1 + \mu_2 + ... + \mu_n &= m, \\ \mu_1 + 2\mu_2 + ... + (n)\mu_n &= n. \end{aligned}$$

It is clear that

$$D_n^n(a_1, ..., a_n) = D_1^n,$$

the first and last polynomials are

$$D_n^n = a_1^n \quad \text{and} \quad D_n^1 = a_n.$$

For $f(z)$ and $g(z)$ analytic in E, we say that $f(z)$ is subordinate to $g(z)$ (written as $f \prec g$) if there exists a Schwarz function

$$u(z) = \sum_{n=1}^{\infty} u_n z^n,$$

with $u(0) = 0$ and $|u(z)| < 1$ in E, such that $f(z) = g(u(z))$. For the Schwarz function $u(z)$, $|u_n| \leq 1$, see [11].

For a complex parameters a, b, c, with $c \neq 0, -1, -2...$,the generalized Hypergeometric function ${}_2F_1(a, b, c, k, z)$ is defined as"

$$\begin{aligned} {}_2F_1(a, b, c, k, z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b + kn)}{\Gamma(c + kn) n!} z^n \\ &= 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_{n-1} \Gamma(b + k(n-1)) z^{n-1}}{\Gamma(c + k(n-1))(n-1)!}, \end{aligned} \tag{1.6}$$

where $\Re(c - 1 - b) > 0$, $|z| < 1$, and $(a)_n$ is the Pochhammer symbol.
By using generalized Hypergeometric function given by (1.6) we define a convolution operator $\mathcal{J}(a, b, c, k)$ as follows:

$$\mathcal{J}(a, b, c, k) f(z) = z\, {}_2F_1(a, b, c, k; z) * f(z) = z + \sum_{n=2}^{\infty} \Upsilon_n a_n z^n, \tag{1.7}$$

where

$$\Upsilon(a, b, c, n) = \frac{\Gamma(c)(a)_{n-1} \Gamma(b + k(n-1))}{\Gamma(b) \Gamma(c + k(n-1))(n-1)!}. \tag{1.8}$$

For convenience we write $\Upsilon(a, b, c, n) = \Upsilon_n$.
Here in this investigation we use the Faber polynomial expansions for the class $S[A, B, \Upsilon_n]$, to determine a general coefficients bounds $|a_n|$, for $(n \geq 3)$.

2 Coefficient bounds for the function class $S[A, B, \Upsilon_n]$

Definition 2.1. A function f defined by (1.1) is said to be in the class $S[A, B, \Upsilon_n]$ if the following condition are satisfied:

$$\left(\frac{z \left[\mathcal{J}(a, b, c, k) f(z) \right]'}{\mathcal{J}(a, b, c, k) f(z)} \right) \prec \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1, \ z \in E, \tag{2.1}$$

and

$$\left(\frac{z \left[\mathcal{J}(a, b, c, k) g(w) \right]'}{\mathcal{J}(a, b, c, k) g(w)} \right) \prec \frac{1 + Aw}{1 + Bw}, \ -1 \leq B < A \leq 1, \ w \in E, \tag{2.2}$$

where the function $g(z)$ is given by (1.3), that is, the extension of f^{-1} to E.

Special Cases:
i) For $a = c$ and $b = 1$ in (2.1) and (2.2) we have the class $S[A, B, \Upsilon_n] = S[A, B]$, defined by Hamidi and Jahangiri [17].

Lemma 2.2. *[11, 21]. Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in A$ be a positive real part functions so that $\Re(p(z)) > 0$ for $|z| < 1$. If $\alpha \geq \frac{-1}{2}$. Then*

$$\left|p_2 + \alpha p_1^2\right| \leq 2 + \alpha \left|p_1\right|^2.$$

Lemma 2.3. *[17]. Let $\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \in A$ be a Schwarz function so that $|\varphi(z)| < 1$ for $|z| < 1$. If $\gamma \geq 0$. Then*

$$\left|\varphi_2 + \gamma\varphi_1^2\right| \leq 1 + (\gamma - 1)\left|\varphi_1\right|^2.$$

3 Main Results

In this section, we will prove our main results.

Theorem 3.1. *For $-1 \leq B < A \leq 1$, if both functions f and f^{-1} map $g = f^{-1}$ are in $S[A, B, \Upsilon_n]$, for $a_k = 0$; $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{(A - B)}{(n-1)\Upsilon_n}, \qquad n \geq 3.$$

Proof. For the function $f \in S[A, B, \Upsilon_n]$ of the form (1.1) we have the expansion

$$\frac{z\left[\mathcal{J}(a,b,c,k)f(z)\right]'}{\mathcal{J}(a,b,c,k)f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3...., a_n)z^{n-1}, \tag{3.1}$$

As for the inverse map $g = f^{-1}$, considering (1.3) we obtain

$$\frac{z\left[\mathcal{J}(a,b,c,k)g(w)\right]'}{\mathcal{J}(a,b,c,k)g(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(b_2, b_3...., b_n)w^{n-1}, \tag{3.2}$$

where, $b_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)$.

$$\begin{aligned} F_1 &= -\Upsilon_2 a_2, \\ F_2 &= \Upsilon_2^2 a_2 - 2\Upsilon_3 a_3, \\ F_2 &= -\Upsilon_2^3 a_2^3 + 3\Upsilon_2\Upsilon_3 a_2 a_3 - 3\Upsilon_4 a_4. \end{aligned}$$

In general

$$F_{n-1}(a_2, a_3...., a_n) = \left[\sum_{i_1+2i_2+...+(n-1)i_{n-1}=n-1} \left\{A(i_1, i_2, i_2, ..., i_{n-1})(\Upsilon_2 a_2)^{i_1}(\Upsilon_3 a_3)^{i_2}...(\Upsilon_n a_n)^{i_{n-1}}\right\}\right].$$

$$A(i_1, i_2, i_2, ..., i_{n-1}) = (-1)^{(n-1)+2i_1+...+ni_{n-1}} \frac{(i_1 + i_2 + i_2, ... + i_{n-1} - 1)!\,(n-1)}{(i_1!)(i_2!)...(i_{n-1}!)}.$$

Since, both functions f and its inverse map $g = f^{-1}$ are in $S[A, B, \Upsilon_n]$, by the definition of subordination, there exist two Schwarz functions $p(z) = \sum_{n=1}^{\infty} c_n z^n$, and $q(w) = \sum_{n=1}^{\infty} d_n w^n$, where z, $w \in E$. So that we have

$$\frac{z\left[\mathcal{J}(a,b,c,k)f(z)\right]'}{\mathcal{J}(a,b,c,k)f(z)} = \frac{1 + A(p(z))}{1 + B(p(z))} = 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(c_1, c_2, ..., c_n, B)z^n \tag{3.3}$$

and

$$\frac{z\left[\mathcal{J}(a,b,c,k)g(w)\right]'}{\mathcal{J}(a,b,c,k)g(w)} = \frac{1 + A(q(w))}{1 + B(q(w))} = 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(d_1, d_2, ..., d_n, B)w^n. \tag{3.4}$$

In general [2, 3] for any $p \in N$ and $n \geq 2$, an expansion of $K_n^p(k_1, k_2, ..., k_n, B)$

$$\begin{aligned} K_n^p(k_1, k_2, ..., k_n, B) &= \frac{p!}{(p-n)!n!} k_1^n B^{n-1} + \frac{p!}{(p-n+1)!(n-2)!} k_1^{n-2} k_2 B^{n-2} \\ &+ \frac{p!}{(p-n+2)!(n-3)!} \times k_1^{n-3} k_3 B^{n-3} \\ &+ \frac{p!}{(p-n+3)!(n-4)!} k_1^{n-4} \left[k_4 B^{n-4} + \frac{p-n+3}{2} k_3^2 B \right] \\ &+ \frac{p!}{(p-n+4)!(n-5)!} k_1^{n-5} \left[k_5 B^{n-5} + (p-n+4) k_3 k_4 B \right] \\ &+ \sum_{j \geq 6} k_1^{n-1} X_j, \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, ..., k_n$.
For the coefficients of the Schwarz functions $p(z)$ and $q(w)$ $|c_n| \leq 1$ and $|d_n| \leq 1$, [11].
Comparing the corresponding coefficients of (3.1) and (3.3) we have

$$F_{n-1}(a_2, a_3...., a_n) = (A-B) K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B) \tag{3.5}$$

which under the assumption $a_m = 0;\ 2 \leq k \leq n-1$, we have

$$-(n-1)\Upsilon_n a_n = -(A-B) c_{n-1}. \tag{3.6}$$

Similarly corresponding coefficients of (3.2) and (3.4) we have

$$F_{n-1}(b_2, b_3...., b_n) = (A-B) K_{n-1}^{-1}(d_1, d_2, ..., d_{n-1}, B), \tag{3.7}$$

which by hypothesis, we obtain

$$-(n-1)\Upsilon_n b_n = -(A-B) d_{n-1}.$$

Note that for $a_m = 0;\ 2 \leq k \leq n-1$, we have $b_n = -a_n$ and therefore

$$(n-1)\Upsilon_n a_n = -(A-B) d_{n-1}. \tag{3.8}$$

Taking the absolute values of (3.6) and (3.8) we obtain the required result

$$|a_n| \leq \frac{(A-B)}{(n-1)\Upsilon_n}.$$

□

For $a = c$ and $b = 1$ in Theorem 3.1, we have the following Corollary

Corollary 3.2. *[17] For $-1 \leq B < A \leq 1$, if both functions f and f^{-1} map $g = f^{-1}$ are in $S[A, B]$, for $a_k = 0;\ 2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{(A-B)}{n-1}, \qquad n \geq 3.$$

Theorem 3.3. *For $-1 \leq B < A \leq 1$, if both functions f and f^{-1} map $g = f^{-1}$ are in $S[A, B, \Upsilon_n]$ then*

$$|a_2| \leq \begin{cases} \frac{(A-B)}{\Upsilon_2 \sqrt{(1+A)}}, & \text{if } 0 \leq B < A, \\ \frac{(A-B)}{\Upsilon_2}, & \text{otherwise}, \end{cases}$$

and

$$\left| a_3 - \frac{\Upsilon_2^2}{\Upsilon_3} a_2^2 \right| \leq \begin{cases} \frac{(A-B)}{2\Upsilon_3}\left(1 - \frac{(A+1)}{(A-B)^2} |\Upsilon_2 a_2|^2\right), & \text{if } A \leq 0, \\ \frac{(A-B)}{\Upsilon_2}, & \text{if } A > 0. \end{cases} \tag{3.9}$$

Proof. For $n = 2$, 3 in (3.5) and (3.7) we have

$$\Upsilon_2 a_2 = (A-B)c_1, \tag{3.10}$$

$$\Upsilon_2^2 a_2^2 - 2\Upsilon_3 a_3 = (A-B)(Bc_1^2 - c_2), \tag{3.11}$$

$$-\Upsilon_2 a_2 = (A-B)d_1, \tag{3.12}$$

$$-3\Upsilon_2^2 a_2^2 + 2\Upsilon_3 a_3 = (A-B)(Bd_1^2 - d_2). \tag{3.13}$$

Taking absolute values of both sides of (3.10) and (3.12) we have

$$|a_2| \leq \frac{(A-B)}{\Upsilon_2}.$$

Adding (3.11) and (3.13) yields

$$-2\Upsilon_2^2 a_2^2 = (A-B)\left\{(Bc_1^2 - c_2) + (Bd_1^2 - d_2)\right\}.$$

Taking absolute values of both sides of the above equation, we obtain

$$2\Upsilon_2^2 |a_2|^2 \leq (A-B)\left\{\left|c_2 + (-B)c_1^2\right| + \left|d_2 + (-B)\, d_1^2\right|\right\}.$$

If $B \leq 0$, then by lemma 2.3, we have

$$2\Upsilon_2^2 \left|a_2^2\right| \leq (A-B)\left\{1 + (-B-1)\,|c_1|^2 + 1 + (-B-1)\,|d_1|^2\right\}.$$

By using $\frac{|\Upsilon_2 a_2|^2}{(A-B)^2} = |c_1|^2 = |d_1|^2$, we have

$$|a_2|^2 \leq \frac{(A-B)}{\Upsilon_2^2} - \frac{(1+B)}{(A-B)}\,|a_2|^2.$$

After simple algebraic calculation we have

$$|a_2| \leq \frac{(A-B)}{\Upsilon_2\sqrt{(1+A)}}.$$

Obviously, for $A > 0$ we have

$$\frac{(A-B)}{\Upsilon_2\sqrt{(1+A)}} < \frac{(A-B)}{\Upsilon_2}.$$

Now rewrite equation (3.13) as

$$2\Upsilon_3(a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2) = (A-B)(Bd_1^2 - d_2) + \Upsilon_2^2 a_2^2.$$

By using $(A-B)^2 d_1^2 = \Upsilon_2^2 a_2^2$ we obtain

$$2\Upsilon_3(a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2) = -(A-B)(d_2 - Ad_1^2).$$

Taking the absolute values of both sides gives

$$2\Upsilon_3\left|a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right| = (A-B)\left|d_2 + (-A)\, d_1^2\right|.$$

If $A \leq 0$, then by Lemma 2.3, we have

$$\left|a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right| = \frac{(A-B)}{2\Upsilon_3}(1 + (-A-1)\left|d_1^2\right|),$$

by using $|d_1|^2 = \frac{|\Upsilon_2 a_2|^2}{(A-B)^2}$, we obtain

$$\left|a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right| = \frac{(A-B)}{2\Upsilon_3}(1 - \frac{(A+1)}{(A-B)^2}|\Upsilon_2 a_2|^2).$$

For $A > 0$, we subtract (3.11) from (3.13) to get

$$4\Upsilon_3\left(a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right) = (A-B)\left[B(d_1^2 - c_1^2) + (c_2 - d_2)\right].$$

Using the fact that $c_1^2 = d_1^2$ and taking the absolute values of both sides of the above equation, we obtain the desired inequality

$$\left|a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right| \leq \frac{(A-B)}{2\Upsilon_3}.$$

□

For $a = c$ and $b = 1$, we have following Corollary.

Corollary 3.4. *[17] For $-1 \leq B < A \leq 1$, if both functions f and f^{-1} map $g = f^{-1}$ are in $S[A,B]$ then*

$$|a_2| \leq \begin{cases} \frac{(A-B)}{\sqrt{(1+A)}}, & \text{if } 0 \leq B < A, \\ (A-B), & \text{otherwise}. \end{cases}$$

And

$$\left|a_3 - \frac{\Upsilon_2^2}{\Upsilon_3}a_2^2\right| \leq \begin{cases} \frac{(A-B)}{2}(1 - \frac{(A+1)}{(A-B)^2}|a_2|^2), & \text{if } A \leq 0, \\ (A-B), & \text{if } A > 0. \end{cases}$$

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