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SPECTRAL PROPERTIES of a q-BOUNDARY VALUE PROBLEM with PIECEWISE-CONTINUOUS COEFFICIENT

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Abstract. This work aims to examine a boundary value problem which consists a second order q-differential equation together with a piecewise-continuous function. An inner product is introduced in a suitable Hilbert space. The orthogonality of the eigenfunctions, realness and simplicity of the eigenvalues are investigated. The Green's function is constructed and some of its properties are given.

1 Introduction

In this paper, we study the boundary value problem

$$l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x)y(x) = \lambda r(x)y(x), \quad (x \in [0,\pi],\ \lambda \in \mathbb{C}), \tag{1.1}$$

$$U_1(y) := \alpha_1 y(0) + \alpha_2 D_{q^{-1}} y(0) = 0, \tag{1.2}$$

$$U_2(y) := \beta_1 y(\pi) + \beta_2 D_{q^{-1}} y(\pi) = 0, \tag{1.3}$$

where $q \in (0,1]$ is fixed, $v(\cdot)$ is a real valued function which is continuous at zero, the coefficients in the boundary conditions (1.2), (1.3) are nonzero arbitrary real numbers and the function $r(x)$ is a piecewise continuous function such as

$$r(x) = \begin{cases} r_1, & 0 \le x < a, \\ r_2, & a < x \le \pi. \end{cases}$$

The Sturm-Liouville theory has been the keystone for the development of spectral methods and the theory of self-adjoint operators [5]. For many applications, the Sturm-Liouville problems are studied as boundary value problems [16]. However, to date, mostly classical differential operators in Sturm-Liouville problems have been used, in [6] (see also [7]) Annaby and Mansour served a natural departure from the typical Sturm-Liouville problem by replacing the derivative with Jackson q-derivative and considered the boundary value problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x)y(x) = \lambda y(x), \quad (0 \le x < a < \infty,\ \lambda \in \mathbb{C}), \tag{1.4}$$

$$a_{11} y(0) + a_{12} D_{q^{-1}} y(0) = 0, \tag{1.5}$$

$$a_{21} y(a) + a_{22} D_{q^{-1}} y(a) = 0, \tag{1.6}$$

where $v(\cdot)$ is defined on $[0,a]$ and continuous at zero, the coefficients in the boundary conditions are arbitrary real numbers such as the rank of the matrix (a_{ij}) $(1 \le i,j \le 2)$ is 2.

In these studies, the authors formulated a self-adjoint q-difference operator in the Hilbert space $L_q^2(0,a)$, discussed some of the properties of the eigenvalues and eigenfunctions, constructed the Green's function and showed that the set of eigenfunctions forms a complete orthogonal set in $L_q^2(0,a)$. The results of Annaby and Mansour are applied and developed in different aspects. For instance, in [1, 8], sampling theory associated with q-difference equations of the Sturm-Liouville type is considered. In [4, 15], a regular q-fractional Sturm-Liouville problem which includes the left-sided Riemann-Liouville and right-sided Caputo q-fractional derivatives of the same order is formulated and the properties of eigenvalues and eigenfunctions are investigated.

In [3], a Parseval equality and an expansion formula in eigenfunctions for a singular q-Sturm-Liouville operator on the whole line are established. In [2], the eigenvalues and the spectral singularities of non-selfadjoint q-difference equations of second order are investigated. In [9], a boundary value problem consisting of a second order q-difference equation together with Dirichlet boundary conditions is reduced to an eigenvalue problem for a second order Euler q-difference equation by separation of variables and in [10] a q-Sturm-Liouville boundary value problem with spectral parameter in the boundary condition is considered.

The boundary value problem (1.1)-(1.3) is identical with the boundary value problem (1.4)-(1.6) in all respects except for the appearance of the piecewise continuous function in (1.1). Our primary source of inspiration for this work is the Remark 3.2.1 in [7] (pg. 85) which claims that similar results with the ones obtained in [7] (see also [6]) will occur if the Hilbert space $L_q^2(0,a)$ is replaced by a weighted one. We also refer to the articles [11, 12, 13, 14] in which Sturm-Liouville problems with piecewise-continuous coefficient were examined.

The paper is organized as follows. In the next section, we begin the discussion of the problem with some preliminaries. In Section 3, we investigate the properties of eigenvalues and eigenfunctions. Section 4 is devoted to construct the Green's function and to examine some of its characteristics. And finally, Section 5 is dedicated to conclude the paper by summarizing the results.

2 Preliminaries

Let $\phi_1(\cdot,\lambda)$ and $\phi_2(\cdot,\lambda)$ be the solutions of the equation (1.1) satisfying the initial conditions

$$\phi_1(\cdot,\lambda)=1,\ D_q\phi_1(\cdot,\lambda)=0,\ \phi_2(\cdot,\lambda)=0,\ D_q\phi_2(\cdot,\lambda)=1. \tag{2.1}$$

We define

$$\Delta(\lambda):=U_1(\phi_1)U_2(\phi_2)-U_1(\phi_2)U_2(\phi_1). \tag{2.2}$$

The function $\Delta(\lambda)$ is an entire function of λ and has zeros at the eigenvalues of the boundary value problem (1.1)-(1.3). Thus, the set of eigenvalues consists countable elements with no finite limit points.

In the Hilbert space $L_{q,r}^2(0,\pi)$ let an inner product be defined by

$$\langle f,g\rangle:=\int_0^\pi f(x)\overline{g(x)}r(x)d_qx$$

for $f(\cdot),g(\cdot)\in L_{q,r}^2(0,\pi)$.

The following lemma can be proven similar to [7].

Lemma 2.1. *Let $y(\cdot)$, $z(\cdot)\in L_{q,r}^2(0,\pi)$ be defined on $[0,q^{-1}\pi]$. Then, for $x\in[0,\pi]$ we have*

$$D_qy(xq^{-1})=D_{q^{-1}}y(x)=D_{q,xq^{-1}}y(xq^{-1}),$$

$$\langle D_qy,z\rangle=y(\pi)r(\pi)\overline{z(\pi q^{-1})}-\lim_{n\to\infty}y(\pi q^n)r(\pi q^n)\overline{z(\pi q^{n-1})}+\left\langle y,-\frac{1}{q}D_{q^{-1}}z\right\rangle, \tag{2.3}$$

$$\left\langle -\frac{1}{q}D_{q^{-1}}y, z\right\rangle = \lim_{n\to\infty} y(\pi q^{n-1})r(\pi q^{n-1})\overline{z(\pi q^n)} - y(\pi q^{-1})r(\pi q^{-1})\overline{z(\pi)} + \langle y, D_q z\rangle\,. \quad (2.4)$$

Lemma 2.2. *For every $f(\cdot)$, $g(\cdot) \in L^2_{q,r}(0,\pi)$, we have the q-Lagrange's identity*

$$\langle lf(x), g(x)\rangle - \langle f(x), lg(x)\rangle = [f,g](\pi) - \lim_{n\to\infty}[f,g](\pi q^n), \quad (2.5)$$

where

$$[f,g](x) := f(x)r(x)\overline{D_{q^{-1}}g(x)} - D_{q^{-1}}f(x)r(x)\overline{g(x)}. \quad (2.6)$$

Proof. Our aim is to calculate the difference between the inner products $\langle lf(x), g(x)\rangle$, $\langle f(x), lg(x)\rangle$:

$$\begin{aligned}
\langle lf(x), g(x)\rangle - \langle f(x), lg(x)\rangle &= \int_0^\pi \left(-\frac{1}{q}D_{q^{-1}}D_q f(x) + v(x)f(x)\right) g(x)r(x)d_qx \\
&- \int_0^\pi f(x)\left(-\frac{1}{q}D_{q^{-1}}D_q g(x) + v(x)g(x)\right) r(x)d_qx \\
&= \int_0^\pi \left[\left(-\frac{1}{q}D_{q^{-1}}D_q f(x)\right) g(x) - f(x)\left(-\frac{1}{q}D_{q^{-1}}D_q g(x)\right)\right] r(x)d_qx \\
&= \left\langle -\frac{1}{q}D_{q^{-1}}D_q f(x), g(x)\right\rangle - \left\langle f(x), -\frac{1}{q}D_{q^{-1}}D_q g(x)\right\rangle.
\end{aligned}$$

Applying (2.4) with $y(x) = D_q f(x)$, $z(x) = g(x)$ to the term $\left\langle -\frac{1}{q}D_{q^{-1}}D_q f(x), g(x)\right\rangle$ gives us

$$\begin{aligned}
\langle lf(x), g(x)\rangle - \langle f(x), lg(x)\rangle &= \lim_{n\to\infty} D_q f(\pi q^{n-1})\overline{g(\pi q^n)}r(\pi q^{n-1}) - D_q f(\pi q^{-1})r(\pi q^{-1})\overline{g(\pi)} \\
&+ \langle D_q f(x), D_q g(x)\rangle - \left\langle f(x), -\frac{1}{q}D_{q^{-1}}D_q g(x)\right\rangle.
\end{aligned}$$

Applying (2.5) with $y(x) = f(x)$, $z(x) = D_q g(x)$ to the term $\langle D_q f(x), D_q g(x)\rangle$ and making necessary calculations allow us to write

$$\langle lf(x), g(x)\rangle - \langle f(x), lg(x)\rangle = [f,g](\pi) - \lim_{n\to\infty}[f,g](\pi q^n)$$

which completes the proof. □

We define the operator A

$$A : D_A \to L^2_{q,r}(0,\pi)$$

by $Ay = ly$ for all $y \in D_A$, where D_A is the subspace of $L^2_{q,r}(0,\pi)$ consisting of those complex valued functions y that satisfy the boundary conditions (1.2), (1.3) such that $D_q y(\cdot)$ is q-regular at zero and $D_q^2 y(\cdot)$ lies in $L^2_{q,r}(0,\pi)$. Thus, A is the q-difference operator generated by the q-difference equation (1.1) and the boundary conditions (1.2), (1.3).

Theorem 2.3. *The operator A is symmetric.*

Proof. We are supposed to show that the right hand side of equality (2.5) vanishes. Let the functions $f(\cdot)$ and $g(\cdot) \in C_q^2(0)$ satisfy the boundary conditions (1.2), (1.3):

$$\alpha_1 f(0) + \alpha_2 D_{q^{-1}}f(0) = 0, \quad \alpha_1 g(0) + \alpha_2 D_{q^{-1}}g(0) = 0.$$

The continuity of $f(\cdot)$ and $g(\cdot)$ at zero implies that $\lim_{n\to\infty}[f,g](\pi q^n) = [f,g](0)$. Then, equation (2.5) will have the form

$$\langle lf(x), g(x)\rangle - \langle f(x), lg(x)\rangle = [f,g](\pi) - [f,g](0).$$

With the help of boundary conditions (1.2), (1.3) and relation (2.6), we have $[f,g](0) = 0$ and $[f,g](\pi) = 0$. This completes the proof. □

3 Properties of eigenvalues and eigenfunctions

Definition 3.1. A complex number λ^* is said to be an eigenvalue of the boundary value problem (1.1)-(1.3) if there is a non-trivial solution $\phi^*(\cdot)$ which satisfies the problem at this λ^*. In this case the solution $\phi^*(\cdot)$ is called an eigenfunction of the boundary value problem (1.1)-(1.3).

Lemma 3.2. *The eigenvalues of the boundary value problem (1.1)-(1.3) are real.*

Proof. Let λ_0 be an eigenvalue and $f_0(\cdot)$ be the corresponding eigenfunction, then from Theorem 2.3, we have

$$\langle lf_0(x), f_0(x)\rangle = \langle f_0(x), lf_0(x)\rangle .$$

Since $lf_0 = \lambda_0 r f_0$ then

$$\langle lf_0(x), f_0(x)\rangle - \langle f_0(x), lf_0(x)\rangle = (\lambda_0 - \overline{\lambda_0}) \int_0^{\pi} r(x) |f_0(x)|^2 d_q x = 0$$

holds. Since $f_0(\cdot)$ is an eigenfunction of the boundary value problem (1.1)-(1.3) we have $\lambda_0 = \overline{\lambda_0}$. This completes the proof. □

Lemma 3.3. *Eigenfunctions that belong to different eigenvalues are orthogonal.*

Proof. Let $\lambda \neq \mu$ be two eigenvalues of the boundary value problem (1.1)-(1.3) corresponding to the eigenfunctions $f(\cdot)$ and $g(\cdot)$, respectively. Then, from the previous lemma, it can be easily seen that the equation below holds:

$$(\lambda - \mu) \int_0^{\pi} f(x)g(x)r(x)d_q x = 0.$$

Since $\lambda \neq \mu$, we have $\langle f, g\rangle = 0$ and this completes the proof. □

Before presenting the following lemma, it will be useful to keep in mind that the multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to this eigenvalue.

Lemma 3.4. *All eigenvalues of the boundary value problem (1.1)-(1.3) are simple.*

Proof. Let λ_0 be an eigenvalue of the boundary value problem (1.1)-(1.3) and $f_1(\cdot)$ and $f_2(\cdot)$ be the corresponding eigenfunctions. It can be easily seen that the q-Wronskian of the functions $f_1(\cdot)$ and $f_2(\cdot)$ equals zero. Indeed,

$$\begin{aligned} W_q(f_1, f_2)(0) &= f_1(0)D_q f_2(0) - f_2(0)D_q f_1(0) \\ &= f_1(0)D_{q^{-1}} f_2(0) - f_2(0)D_{q^{-1}} f_1(0) = [f_1, f_2](0) = 0. \end{aligned}$$

From Corollary 2.15 in [7] pg. 65, the above equation gives us the fact that the functions $f_1(\cdot)$ and $f_2(\cdot)$ are linearly dependent. Thus, we have a contradiction and this completes the proof. □

We have just proved in Lemma 3.4 that all eigenvalues of the boundary value problem (1.1)-(1.3) are simple from the geometric point of view. Now, in the following theorem, we will try to show that the eigenvalues are also algebraically simple.

Theorem 3.5. *The eigenvalues of the boundary value problem (1.1)-(1.3) are the simple zeros of the function* $\Delta(\lambda)$.

Proof. Let $\theta_1(\cdot, \lambda)$ and $\theta_2(\cdot, \lambda)$ be the functions defined as

$$\begin{cases} \theta_1(x, \lambda) := U_1(\phi_2)\phi_1(x, \lambda) - U_1(\phi_1)\phi_2(x, \lambda), \\ \theta_2(x, \lambda) := U_2(\phi_2)\phi_1(x, \lambda) - U_2(\phi_1)\phi_2(x, \lambda). \end{cases} \tag{3.1}$$

It can be easily seen that the functions $\theta_1(\cdot, \lambda)$ and $\theta_2(\cdot, \lambda)$ are the solutions of the equation (1.1) satisfying the boundary conditions (1.2), (1.3):

$$\theta_1(0, \lambda) = \alpha_2,\ D_{q^{-1}}\theta_1(0, \lambda) = -\alpha_1;\ \theta_2(\pi, \lambda) = \beta_2,\ D_{q^{-1}}\theta_2(\pi, \lambda) = -\beta_1. \tag{3.2}$$

With the help of (2.1), (2.2) and (3.1), we have

$$W_q(\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda))(x)=\Delta(\lambda)W_q(\phi_1(\cdot,\lambda),\phi_2(\cdot,\lambda))(x)=\Delta(\lambda). \tag{3.3}$$

Now, let λ_0 be an eigenvalue of the boundary value problem (1.1)-(1.3). From (3.3), we have the conclusion that the functions $\theta_1(x,\lambda_0)$ and $\theta_2(x,\lambda_0)$ are linearly dependent eigenfunctions of the boundary value problem (1.1)-(1.3). Thus, the existence of a non-zero constant k_0 such that

$$\theta_1(x,\lambda_0)=k_0\theta_2(x,\lambda_0)$$

is valid. (3.2) and (3.3) gives us the opportunity to write

$$\theta_1(\pi,\lambda_0)=k_0\beta_2=k_0\theta_2(\pi,\lambda),\ D_{q^{-1}}\theta_1(\pi,\lambda_0)=k_0D_{q^{-1}}\theta_2(\pi,\lambda)=-k_0\beta_1.$$

Taking $f(x)=\theta_1(x,\lambda)$ and $g(x)=\theta_1(x,\lambda_0)$ in (2.5) implies

$$\begin{aligned}(\lambda-\lambda_0)\int_0^\pi \theta_1(x,\lambda)\theta_1(x,\lambda_0)r(x)d_qx &= [\theta_1(\cdot,\lambda),\theta_1(\cdot,\lambda_0)](\pi)\\ &= k_0[\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda_0)]=k_0W_q(\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda))(q^{-1}\pi)\\ &= k_0\Delta(\lambda).\end{aligned}$$

Since $\Delta(\lambda)$ is an entire function of λ, we have

$$\frac{d}{d\lambda}\Delta(\lambda):=\lim_{\lambda\to\lambda_0}\frac{\Delta(\lambda)}{\lambda-\lambda_0}=\frac{1}{k_0}\int_0^\pi \theta_1^2(x,\lambda_0)r(x)d_qx\neq 0$$

and hence λ_0 is a simple zero of the function $\Delta(\lambda)$. □

4 Green's Function

Let us consider the boundary value problem

$$-\frac{1}{q}D_{q^{-1}}D_qy(x)+v(x)y(x)=\lambda r(x)y(x)+f(x)r(x),\quad (x\in[0,\pi],\lambda\in\mathbb{C}), \tag{4.1}$$

$$\alpha_1y(0)+\alpha_2D_{q^{-1}}y(0)=0, \tag{4.2}$$

$$\beta_1y(\pi)+\beta_2D_{q^{-1}}y(\pi)=0, \tag{4.3}$$

where $f(\cdot)\in L^2_{q,r}(0,\pi)$ is a given function.

If λ is not an eigenvalue of the non-homogeneous problem (4.1)-(4.3), then it has a unique solution. Indeed, it is obvious that the difference of two solutions of the non-homogeneous problem is an eigenfunction of the homogeneous problem and, by assumption, it must be identically zero.

Theorem 4.1. *Assume that λ is not an eigenvalue of the boundary value problem (4.1)-(4.3). Then, the solution of the boundary value problem (4.1)-(4.3) has the following form*

$$\phi(x,\lambda)=\int_0^\pi G(x,t;\lambda)f(t)r(t)d_qt \tag{4.4}$$

where $G(x,t;\lambda)$ is the Green's function defined as

$$G(x,t;\lambda):=\frac{1}{\Delta(\lambda)}\begin{cases}\theta_2(x,\lambda)\theta_1(t,\lambda), & t\le x,\\ \theta_1(x,\lambda)\theta_2(t,\lambda), & x\le t.\end{cases}$$

Conversely, the function $\phi(x,\lambda)$ defined in (4.4) satisfies (4.1) and (4.2), (4.3). If $f(x)$ is q-regular at zero, then (4.4) holds for all $x\in[0,\pi]$.

Proof. By applying the q-analogue of the method of variation of constants, we seek the solution of the boundary value problem (4.1)-(4.3) in the following form

$$\phi(x,\lambda) = c_1(x)\theta_1(x,\lambda) + c_2(x)\theta_2(x,\lambda),$$

where $c_1(x)$, $c_2(x)$ are solutions of the q-difference equations

$$\begin{cases} D_{q,x}c_1(x) = -\frac{q}{\Delta(\lambda)}\theta_2(qx,\lambda)f(qx)r(qx), \\ D_{q,x}c_2(x) = \frac{q}{\Delta(\lambda)}\theta_1(qx,\lambda)f(qx)r(qx). \end{cases} \tag{4.5}$$

If the functions $D_{q,x}c_1(x)$ and $D_{q,x}c_2(x)$ are q-integrable functions on, $[0,t]$ then

$$\lim_{n\to\infty} tq^n\theta_i(tq^{n+1},\lambda)f(tq^{n+1}) = 0$$

$(i=1,2)$ holds. Since $f \in L^2_{q,r}(0,\pi)$, the set

$$S_f := \left\{x \in [0,\pi] : \lim_{n\to\infty} xq^n r(xq^n)\left|f(xq^n)\right|^2 = 0\right\}$$

becomes a q-geometric set which includes $\{\pi q^m, m \in \mathbf{N_0}\}$. Thus, the functions $D_q c_1(x)$ and $D_q c_2(x)$ are q-integrable functions on $[0,x]$ for all $x \in S_f$ and the solutions of (4.5) are

$$c_1(x) = \tilde{c_1} + \frac{q}{\Delta(\lambda)}\int_0^x \theta_2(qt,\lambda)f(qt)r(qt)d_qt,$$

$$c_2(x) = \tilde{c_2} + \frac{q}{\Delta(\lambda)}\int_x^\pi \theta_1(qt,\lambda)f(qt)r(qt)d_qt$$

as $x \in S_f$ and $\tilde{c_1}$,$\tilde{c_2}$ are arbitrary real numbers. Hence the general solution of (4.1) can be written by

$$\phi(x,\lambda) = \tilde{c_1}\theta_1(x,\lambda) + \tilde{c_2}\theta_2(x,\lambda) + \frac{q}{\Delta(\lambda)}\theta_1(x,\lambda)\int_0^x \theta_2(qt,\lambda)f(qt)r(qt)d_qt$$
$$+\frac{q}{\Delta(\lambda)}\theta_2(x,\lambda)\int_x^\pi \theta_1(qt,\lambda)f(qt)r(qt)d_qt \tag{4.6}$$

where $x \in S_f$. $\tilde{c_1}$ and $\tilde{c_2}$ can be determined with the help of the boundary conditions (4.2), (4.3). Indeed,

$$\phi(0,\lambda) = \tilde{c_1}\theta_1(0,\lambda) + \left(\tilde{c_2} + \frac{q}{\Delta(\lambda)}\int_x^\pi \theta_1(qt,\lambda)f(qt)r(qt)d_qt\right)\theta_2(0,\lambda),$$

$$D_{q^{-1}}\phi(0,\lambda) = \tilde{c_1}D_{q^{-1}}\theta_1(0,\lambda) + \left(\tilde{c_2} + \frac{q}{\Delta(\lambda)}\int_x^\pi \theta_1(qt,\lambda)f(qt)r(qt)d_qt\right)D_{q^{-1}}\theta_2(0,\lambda)$$

hold. From (4.2), we have

$$\tilde{c_2} = -\frac{q}{\Delta(\lambda)}\int_0^\pi \theta_1(qt,\lambda)f(qt)r(qt)d_qt.$$

Similarly, by using (4.3) to obtain $\tilde{c_1}$, we have

$$\tilde{c_1} = -\frac{q}{\Delta(\lambda)}\int_0^\pi \theta_2(qt,\lambda)f(qt)r(qt)d_qt.$$

Substituting these values into (4.6), we reach (4.4). Conversely, if (4.4) is given then it can easily be seen that it is a solution of the boundary value problem (4.1)-(4.3). If $f(x)$ is q-regular at zero, then $S_f \equiv [0,\pi]$ and (4.4) becomes valid for all $x \in [0,\pi]$. □

The theorem which is given below lists a number of properties of the Green's function.

Theorem 4.2. *Green's function has the following properties:*

(i) $G(x,t,\lambda)$ *is continuous at the point* $(0,0)$.

(ii) $G(x,t,\lambda) = G(t,x,\lambda)$.

(iii) For each fixed $t \in (0,q\pi]$, $G(x,t,\lambda)$ *satisfies the* q*-difference equation (4.1) in the intervals* $[0,t)$, $(t,\pi]$ *and it also satisfies the boundary conditions (4.2)-(4.3).*

Proof. The proof can easily be obtained by using a similar procedure to [7]. □

5 Conclusion

The main goal of the paper is to develop an alternative approach to the problem that Annaby and Mansour discussed in their paper [6] (see also [7]). This approach is based on the piecewise-continuous coefficient in the differential equation (1.1). We present a Hilbert space with a suitable inner product and we study some of the spectral properties of the boundary value problem (1.1)-(1.3). We also construct the Green's function and mention some of its properties. In our future research, we will try to extend our results to the case of q- boundary value problems consisting differential equations with piecewise-continuous coefficients and eigenparameter-dependent boundary conditions. It is our hope that this paper will initiate new research in the area of the boundary value problems with Jackson q-derivative.

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