Palestine Journal of Mathematics

Vol. 8(1)(2019) , 52–62

Vertex Irregular Total Labeling Of Grid Graph

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Communicated by Ivan Gotchev

MSC 2010 Classifications: 05C15, 05C30; Secondary 20G15.

Keywords and phrases: vertex irregular lebeling, total vertex irregularity strength, Cartesian product, grid graph.

We are indebted to anonymous referees for many useful remarks which improved the first version of this paper.

Abstract A vertex irregular total k-labeling ϕ of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \ldots, k\}$ in such a way that any two different vertices have distinct weights. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G.
In [7], Bokhary et. al proposed a conjecture that the $tvs(P_m \Box P_n) = \lceil \frac{mn+2}{5} \rceil$ for $m, n \geq 2$ and $m, n \in \mathbb{N}$. In this paper we prove this conjecture for $5 \leq m \leq 10$ and $n \geq 1$.

1 Introduction

The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. Bloom and Golomb [5, 6] studied applications of graph labeling to other branches of science.

As a standard notation, assume that $G(V, E)$ is a finite, simple and undirected graph with vertex set V and edge set E. A *total* labeling is defined as a labeling in which all the vertices and edges are labeled. For a graph G, we define a labeling $\phi : V \cup E \to \{1, 2, \ldots, k\}$ to be a vertex irregular total k-labeling of the graph G if for any two distinct vertices $x, y \in G$ $wt(x) \neq wt(y)$, and the weight of a vertex x in the labeling ϕ is

$$wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy),$$

where $N(x)$ is the set of neighbors of x.
In [4] Bača, Jendrol', Miller and Ryan defined new graph invariants, called the *total vertex and edge irregularity strength* of G, denoted by $tvs(G)$ and $tes(G)$, respectively being the minimum value of k for which the graph G has a vertex or edge irregular total k-labeling.

The original motivation for the definition of the total vertex irregularity strength came from irregular assignments and the irregularity strength of graphs introduced in [10] by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba, and studied by numerous authors [9, 12, 13, 14, 15].

An *irregular assignment* is a k-labeling of the edges

$$f : E \to \{1, 2, \ldots, k\}$$

such that the vertex weights

$$w(x) = \sum_{y \in N(x)} f(xy)$$

are different for all vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$. The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G. If an edge labeling $f : E \to \{1, 2, \ldots, s(G)\}$ provides the irregularity strength $s(G)$, then we extend this labeling to total labeling ϕ in such a way

$$\phi(xy) = f(xy) \quad \text{for every } xy \in E(G),$$
$$\phi(x) = 1 \quad \text{for every } x \in V(G).$$

Thus, the total labeling ϕ is a vertex irregular total labeling and for graphs with no component of order ≤ 2, $tvs(G) \leq s(G)$. Nierhoff [16] proved that for all (p, q)-graphs G with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G) \leq p - 1$. From this result it follows that

$$tvs(G) \leq p - 1.$$

In [4] several bounds and exact values of $tvs(G)$ and $tes(G)$ were determined for different types of graphs (in particular for stars, cliques and prisms). Among others, the authors proved the following theorem:

Theorem 1.1. *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{p + \delta}{\Delta + 1} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1.$$

For graphs with no component of order ≤ 2, Bača *et al.* in [4] strengthened these upper bounds by proving that

$$tvs(G) \leq p - 1 - \left\lceil \frac{p - 2}{\Delta + 1} \right\rceil.$$

These results were then improved by Przybyło in [17] for sparse graphs and for graphs with large minimum degree. In the latter case the bounds

$$tvs(G) < 32 \, \frac{p}{\delta} + 8$$

in general and

$$tvs(G) < 8 \, \frac{p}{r} + 3$$

for r-regular (p, q)-graphs were proved to hold.

In [3] Anholcer, Kalkowski and Przybyło established a new upper bound of the form

$$tvs(G) \leq 3\frac{p}{\delta} + 1.$$

Wijaya and Slamin [18] found the exact values of the total vertex irregularity strength of wheels, fans, suns and friendship graphs. Wijaya, Slamin, Surahmat and Jendrol' [19] determined an exact value for complete bipartite graphs. The total vertex irregularity strengths of cubic graphs, wheel related graphs, Jahangir graphs, circulant graphs and certain classes of unicyclic graphs have been determined by Ahmad et al. in [1, 2].
The main aim of this paper is to determine the exact values for the total vertex irregularity strength of the grid graph $P_m \Box P_n$.

2 Vertex irregular total labeling of grid graph

A *Cartesian product* of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$. If we consider graph H as the path graph P_n with $V(P_n) = \{x_p : p = 1, 2, ..., n\}$, $E(P_n) = \{x_p x_{p+1} : p = 1, 2, ..., n - 1\}$ and graph G as the path graph P_m with $V(P_m) = \{x_q : q = 1, 2, ..., m\}$, $E(P_m) = \{x_q x_{q+1} : q = 1, 2, ..., m - 1\}$ then

$V(P_m \square P_n) = \{(x_p, x_q) = x_{p,q} : p = 1, 2, ..., m,\ q = 1, 2, ..., n\}$ is the vertex set of $P_m \square P_n$ and $E(P_m \square P_n) = \{x_{p,q}x_{p,q+1} : 1 \le p \le m,\ 1 \le q \le n-1\} \cup \{x_{p,q}x_{p+1,q} :\ 1 \le p \le m-1,\ 1 \le q \le n\}$ is the edge set of $P_m \square P_n$. So, $P_m \square P_n$ is the graph of order mn and size $2mn - m - n$. The graph $P_m \square P_n$ is known as grid graph.
Chunling, Xiaohui1, Yuansheng and Liping [11] found the total vertex irregularity strength of $P_2 \square P_n$. Later, Bokhary et al. [7] determined the exact values of the total vertex irregularity strength for the graphs $P_3 \square P_n$ and $P_4 \square P_n$. In this paper we have determined the total vertex irregularity strength of the grid graph $P_m \square P_n$ for $5 \le m \le 10$.

Theorem 2.1. *For* $5 \le m \le 10$ *and* $n \ge m$*,*

$$tvs(P_m \square P_n) = \lceil \frac{mn+2}{5} \rceil$$

Proof. From Theorem 1.1, it implies that

$$tvs(P_m \square P_n) \ge \lceil \frac{mn+2}{5} \rceil. \tag{2.1}$$

Let $\lceil \frac{mn+2}{5} \rceil = k_m$. In order to prove that k_m is the upper bound for $tvs(P_m \square P_n)$, we define a total k_m-labeling as follows:
For $5 \le m \le 10$,

$$\phi(x_{1,p}) = \begin{cases} p, & p = 1, 2, ..., n-1 \\ 2, & p = n \end{cases}$$

$$\phi(x_{2,p}) = \begin{cases} n, & p = 1, n \\ p+1, & p = 2, 3, ..., n-2 \\ 2m-8, & p = n-1 \ (n \text{ is odd}) \\ n, & p = n-1 \ (n \text{ is even}) \end{cases}$$

$$\phi(x_{m-1,p}) = \begin{cases} k_m, & p = 1 \ (m \ne 10) \\ k_m - 2, & p = 1 \ (m = 10) \\ p, & p = 2, ..., n-2 \\ 2m-5, & p = n-1 \quad (n \text{ is odd and } m \ne 10) \\ 2m-7, & p = n-1 \quad (n \text{ is odd and } m = 10) \\ n-1, & p = n-1 \quad (n \text{ is even}) \\ n+1, & p = n \quad (n \text{ is odd}) \\ k_m, & p = n \ (n \text{ is even and } m \ne 10) \\ k_m - 2, & p = n \ (n \text{ is even and } m = 10) \end{cases}$$

$$\phi(x_{m,p}) = \begin{cases} 3, & p = 1 \\ p-1, & p = 2, ..., n-1 \\ 2, & p = n \end{cases}$$

$$\phi(x_{1,p}x_{1,p+1}) = 1, \qquad p = 1, 2, ..., n-1$$

$$\phi(x_{2,p}x_{2,p+1}) = \begin{cases} n, & 1 \le p \le n-2 \ (p \text{ is odd}) \\ 2m-8, & 1 \le p \le n-2 \ (p \text{ is even}) \\ n, & p = n-1 \end{cases}$$

$$\phi(x_{m-1,p}x_{m-1,p+1}) = \begin{cases} \lfloor\frac{(10-m)n-2}{5}\rfloor, & 1 \le p \le n-2 \text{ (}p\text{ is odd and } m \neq 10) \\ 1, & 1 \le p \le n-2 \text{ (}p\text{ is odd and } m = 10) \\ 2m-5, & 1 \le p \le n-2 \text{ (}p\text{ is even and } m \neq 10) \\ 2m-7, & 1 \le p \le n-2 \text{ (}p\text{ is even and } m = 10) \\ n-1, & p = n-1 \text{ (}n\text{ is odd)} \\ \lfloor\frac{(10-m)n-2}{5}\rfloor, & p = n-1 \text{ (}n\text{ is even and } m \neq 10) \\ 1, & p = n-1 \text{ (}n\text{ is even and } m = 10) \end{cases}$$

$$\phi(x_{m,p}x_{m,p+1}) = 2, \quad p = 1, 2, ..., n-1$$

$$\phi(x_{1,p}x_{2,p}) = \begin{cases} 1, & p = 1, n \\ 3, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{2,p}x_{3,p}) = \begin{cases} 2, & p = 1 \\ n+1, & p = 2, ..., n-1 \\ 3, & p = n \end{cases}$$

$$\phi(x_{m-2,p}x_{m-1,p}) = \begin{cases} 4, & p = 1 \\ k_m, & p = 2, ..., n-1 \\ 5, & p = n \end{cases}$$

$$\phi(x_{m-1,p}x_{m,p}) = \begin{cases} 1, & p = 1, n \\ n, & p = 2, ..., n-1 \end{cases}$$

For $m = 5$,

$$\phi(x_{3,p}) = \begin{cases} n, & p = 1 \\ p-1, & p = 2, 3, ..., n-1 \\ n-1, & p = n \end{cases}$$

$$\phi(x_{3,p}x_{3,p+1}) = \begin{cases} k_5, & p = 1, 2, ..., n-1 \end{cases}$$

For $6 \le m \le 10$,

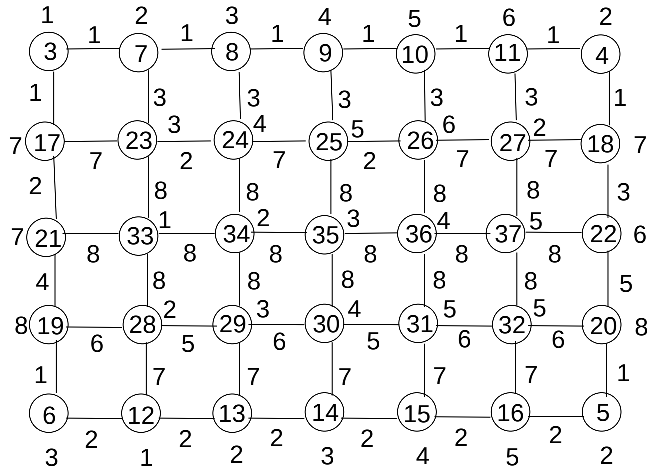

Figure 1. The vertex irregular total labeling of graph $P_5 \square P_7$

$$\phi(x_{3,p}) = \begin{cases} k_m, & p = 1 \ (m \neq 10) \\ k_m - 2, & p = 1 \ (m = 10) \\ p + 3, & p = 2, ..., n-2 \\ 2m + n - 11, & p = n - 1 \ (n \text{ is odd and } m \neq 10) \\ 2m + n - 13, & p = n - 1 \ (n \text{ is odd and } m = 10) \\ n + 2, & p = n - 1 \quad (n \text{ is even}) \\ n - 2, & p = n \qquad (n \text{ is odd}) \\ k_m, & p = n \qquad (n \text{ is even and } m \neq 10) \\ k_m - 2, & p = n \qquad (n \text{ is even and } m = 10) \end{cases}$$

$$\phi(x_{3,p}x_{3,p+1}) = \begin{cases} \lfloor \frac{(10-m)n-2}{5} \rfloor, & 1 \leq p \leq n-2 \ (p \text{ is odd and } m \neq 10) \\ 1, & 1 \leq p \leq n-2 \ (p \text{ is odd and } m = 10) \\ 2m + n - 11, & 1 \leq p \leq n-2 \ (p \text{ is even and } m \neq 10) \\ 2m + n - 13, & 1 \leq p \leq n-2 \ (p \text{ is even and } m = 10) \\ n + 2, & p = n - 1 \qquad (n \text{ is odd}) \\ \lfloor \frac{(10-m)n-2}{5} \rfloor, & p = n - 1 \qquad (n \text{ is even and } m \neq 10) \\ 1, & p = n - 1 \qquad (n \text{ is even and } m = 10) \end{cases}$$

$$\phi(x_{3,p}x_{4,p}) = \begin{cases} 5, & p = 1, n \\ k_m, & p = 2, ..., n-1 \end{cases}$$

For $m = 6$,

$$\phi(x_{4,p}) = \begin{cases} \lfloor \frac{4n-2}{5} \rfloor, & p = 1, n \\ \lceil \frac{n-5}{5} \rceil + p, & n \equiv 0, 1 (\text{mod } 5) \quad 2 \leq p \leq n-1 \\ \lceil \frac{n-5}{5} \rceil + (p+1), & n \equiv 2 (\text{mod } 5) \quad 2 \leq p \leq n-1 \\ \lceil \frac{n-5}{5} \rceil + (p+2), & n \equiv 3 (\text{mod } 5) \quad 2 \leq p \leq n-1 \\ \lceil \frac{n-5}{5} \rceil + (p-1), & n \equiv 4 (\text{mod } 5) \quad 2 \leq p \leq n-1 \end{cases}$$

$$\phi(x_{4,p}x_{4,p+1}) = k_6, \quad p = 1, 2, ..., n-1$$

For $m = 7$,

$$\phi(x_{4,p}) = \begin{cases} \lfloor \frac{3n-2}{5} \rfloor + 1, & p = 1 \\ 2\lceil \frac{n-2}{5} \rceil + p - 1, & n \equiv 0, 2, 3 (\text{mod } 5) \quad 2 \leq p \leq n-1 \\ 2\lceil \frac{n-2}{5} \rceil + (p+1), & n \equiv 1, 4 (\text{mod } 5) \quad 2 \leq p \leq n-1 \\ \lfloor \frac{3n-2}{5} \rfloor + 2, & p = n \end{cases}$$

$$\phi(x_{4,p}x_{4,p+1}) = \begin{cases} k_7, & p = 1, 2, ..., n-1 \end{cases}$$

$$\phi(x_{4,p}x_{5,p}) = \begin{cases} 5, & p = 1, n \\ k_7, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{5,p}) = \begin{cases} n, & p = 1, n \\ p + 5, & p = 2, ..., n-2 \\ 2(\lfloor \frac{3n-2}{5} \rfloor), & p = n-1 \ (n \text{ is odd}) \\ n + 4, & p = n-1 \ (n \text{ is even}) \\ n - 4, & p = n \quad (n \text{ is odd}) \\ n, & p = n \quad (n \text{ is even}) \end{cases}$$

$$\phi(x_{5,p}x_{5,p+1}) = \begin{cases} n, & 1 \leq p \leq n-2 \ (p \text{ is odd}) \\ 2\lfloor \frac{3n-2}{5} \rfloor, & 1 \leq p \leq n-2 \ (p \text{ is even}) \\ n + 4, & p = n-1 \quad (n \text{ is odd}) \\ n, & p = n-1 \quad (n \text{ is even}) \end{cases}$$

For $8 \leq m \leq 10$,

$$\phi(x_{4,p}) = \begin{cases} \lceil \frac{(m-5)n+2}{5} \rceil, & p = 1, n \\ p + 2m - 11, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{4,p}x_{4,p+1}) = \begin{cases} \lfloor \frac{(15-m)n-2}{5} \rfloor, & p = 1, 2, ..., n-1 \end{cases}$$

$$\phi(x_{4,p}x_{5,p}) = \begin{cases} 6, & p = 1 \\ k_m, & p = 2, ..., n-1 \\ 7, & p = n \end{cases}$$

For $m = 8$,

$$\phi(x_{5,p}) = \begin{cases} \lfloor \frac{2n-2}{5} \rfloor + 2, & p = 1, n \\ n - 4\lceil \frac{n-6}{10} \rceil + (p-1), & p = 1, ..., n-1 \quad (n \text{ is even}) \\ n - 4\lceil \frac{n-1}{10} \rceil + (p+1), & p = 1, ..., n-1 \quad (n \text{ is odd}) \end{cases}$$

$$\phi(x_{5,p}x_{5,p+1}) = k_8, \quad p = 1, 2, ..., n-1$$

$$\phi(x_{5,p}x_{6,p}) = \begin{cases} 5, & p = 1, n \\ k_8, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{6,p}) = \begin{cases} n-6, & p = 1 \\ p+1, & p = 2, ..., n-2 \\ 2(\lfloor \frac{2n-2}{5} \rfloor), & p = n-1 \;\; (n \text{ is odd}) \\ n, & p = n-1 \;\; (n \text{ is even}) \\ n, & p = n \quad (n \text{ is odd}) \\ n-6, & p = n \quad (n \text{ is even}) \end{cases}$$

$$\phi(x_{6,p}x_{6,p+1}) = \begin{cases} n+6, & 1 \le p \le n-2 \;\; (p \text{ is odd}) \\ 2(\lfloor \frac{2n-2}{5} \rfloor), & 1 \le p \le n-2 \;\; (p \text{ is even}) \\ n, & p = n-1 \quad (n \text{ is odd}) \\ n+6, & p = n-1 \quad (n \text{ is even}) \end{cases}$$

For $m = 9$,

$$\phi(x_{5,p}) = \begin{cases} \lfloor \frac{n-2}{5} \rfloor + 1, & p = 1 \\ 4(\lfloor \frac{n-2}{5} \lfloor) + (p+3), & p = 2, ..., n-1 \\ \lfloor \frac{n-2}{5} \rfloor, & p = n \end{cases}$$

$$\phi(x_{5,p}x_{5,p+1}) = k_9, \quad p = 1, 2, ..., n-1$$

$$\phi(x_{5,p}x_{6,p}) = \begin{cases} 8, & p = 1 \\ k_9, & p = 2, ..., n-1 \\ 9, & p = n \end{cases}$$

$$\phi(x_{6,p}) = \begin{cases} \lceil \frac{4n+2}{5} \rceil - \lfloor \frac{3n}{5} \rfloor, & p = 1 \\ p+5, & p = 2, ..., n-2 \\ \lfloor \frac{6n-2}{5} \rfloor + \lceil \frac{2n}{5} \rceil, & p = n-1 \;\; (n \text{ is odd}) \\ n+4, & p = n-1 \;\; (n \text{ is even}) \\ n-4, & p = n \;\; (n \text{ is odd}) \\ \lceil \frac{4n+2}{5} \rceil - \lfloor \frac{3n}{5} \rfloor, & p = n \;\; (n \text{ is even}) \end{cases}$$

$$\phi(x_{6,p}x_{6,p+1}) = \begin{cases} \lfloor \frac{6n-2}{5} \rfloor + \lfloor \frac{3n}{5} \rfloor, & 1 \leq p \leq n-2 \text{ (p is odd)} \\ \lfloor \frac{6n-2}{5} \rfloor + \lceil \frac{2n}{5} \rceil, & 1 \leq p \leq n-2 \text{ (p is even)} \\ n+4, & p = n-1 \text{ (n is odd)} \\ \lfloor \frac{6n-2}{5} \rfloor + \lfloor \frac{3n}{5} \rfloor, & p = n-1 \text{ (n is even)} \end{cases}$$

$$\phi(x_{6,p}x_{7,p}) = \begin{cases} 5, & p = 1, n \\ k_9, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{7,p}) = \begin{cases} n, & p = 1 \\ p+9, & p = 2, ..., n-2 \\ 2(\lfloor \frac{n-2}{5} \rfloor), & p = n-1 \text{ (n is odd)} \\ n+8, & p = n-1 \text{ (n is even)} \\ n-8, & p = n \text{ (n is odd)} \\ n, & p = n \text{ (n is even)} \end{cases}$$

$$\phi(x_{7,p}x_{7,p+1}) = \begin{cases} n, & 1 \leq p \leq n-2 \text{ (p is odd)} \\ 2(\lfloor \frac{n-2}{5} \rfloor), & 1 \leq p \leq n-2 \text{ (p is even)} \\ n+8, & p = n-1 \text{ (n is odd)} \\ n, & p = n-1 \text{ (n is even)} \end{cases}$$

For $m = 10$,

$$\phi(x_{5,p}) = \begin{cases} 3, & p = 1, n \\ p+3, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{5,p}x_{5,p+1}) = \begin{cases} 2n, & p = 1, 2, ..., n-1 \end{cases}$$

$$\phi(x_{5,p}x_{6,p}) = \begin{cases} 6, & p = 1, n \\ k_{10}, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{6,p}) = \begin{cases} 2, & p = 1, n \\ n+(p-1), & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{6,p}x_{6,p+1}) = k_{10}, \quad p = 1, 2, ..., n-1$$

$$\phi(x_{6,p}x_{7,p}) = \begin{cases} 8, & p = 1 \\ k_{10}, & p = 2, ..., n-1 \\ 9, & p = n \end{cases}$$

$$\phi(x_{7,p}) = \begin{cases} n, & p = 1 \\ p+4, & p = 2, ..., n-2 \\ k_{10}, & p = n-1 \ \ (n \text{ is odd}) \\ n+3, & p = n-1 \ \ (n \text{ is even}) \\ n-3, & p = n \quad\quad (n \text{ is odd}) \\ n, & p = n \quad\quad (n \text{ is even}) \end{cases}$$

$$\phi(x_{7,p}x_{7,p+1}) = \begin{cases} n, & 1 \leq p \leq n-2 \ (p \text{ is odd}) \\ k_{10}, & 1 \leq p \leq n-2 \ (p \text{ is even}) \\ n+3, & p = n-1 \quad\quad (n \text{ is odd}) \\ n, & p = n-1 \quad\quad (n \text{ is even}) \end{cases}$$

$$\phi(x_{7,p}x_{8,p}) = \begin{cases} 5, & p = 1, n \\ 2n+1, & p = 2, ..., n-1 \end{cases}$$

$$\phi(x_{8,p}) = \begin{cases} n, & p = 1 \\ p, & p = 2, ..., n-2 \\ 9, & p = n-1 \quad\quad (n \text{ is odd}) \\ n-1, & p = n-1 \quad\quad (n \text{ is even}) \\ n+1, & p = n \quad\quad (n \text{ is odd}) \\ n, & p = n \quad\quad (n \text{ is even}) \end{cases}$$

$$\phi(x_{8,p}x_{8,p+1}) = \begin{cases} n, & 1 \leq p \leq n-2 \quad\quad (p \text{ is odd}) \\ 9, & 1 \leq p \leq n-2 \quad\quad (p \text{ is even}) \\ n-1, & p = n-1 \quad\quad (n \text{ is odd}) \\ n, & p = n-1 \quad\quad (n \text{ is even}) \end{cases}$$

From above labeling, the weights of the vertices of the graph $P_m \square P_n$, for $5 \leq m \leq 10$ and $n \geq m$ are calculated as follows:

$wt(x_{1,1}) = 3,\ wt(x_{1,n}) = 4,\ wt(x_{m,n}) = 5,\ wt(x_{m,1}) = 6,$

$$wt(x_{i,1}) = \begin{cases} 2n+4i-5, & \text{if } 2 \leq i \leq \lceil \frac{m}{2} \rceil \\ 2n+4(m-i)+1, & \text{if } \lceil \frac{m}{2} \rceil + 1 \leq i \leq m-1 \end{cases}$$

$$wt(x_{i,n}) = \begin{cases} 2n+4i-4, & \text{if } 2 \leq i \leq \lceil \frac{m}{2} \rceil \\ 2n+4(m-i)+2, & \text{if } \lceil \frac{m}{2} \rceil + 1 \leq i \leq m-1 \end{cases}$$

For $2 \leq p \leq n-1$,

$$wt(x_{i,p}) = \begin{cases} p+5, & \text{if } i=1 \\ 2(i-1)n+2m-(4i-5)+p, & \text{if } 2 \le i \le \lceil \frac{m}{2} \rceil \\ (2(m-i)+1)n+4i-2m-1+p, & \text{if } \lceil \frac{m}{2} \rceil + 1 \le i \le m-1 \\ n+3+p, & \text{if } i=m \end{cases}$$

It can be easily checked that all the vertices of $P_m \square P_n$ have distinct weights for $5 \le m \le 10$ and $n \ge m$. Hence ϕ is a total vertex irregular labeling. Therefore

$$tvs(P_m \square P_n) \le \lceil \frac{mn+2}{5} \rceil. \tag{2.2}$$

From Equation (2.1) and (2.2) it is concluded that, for $5 \le m \le 10$ and $n \ge m$,

$$tvs(P_m \square P_n) = \lceil \frac{mn+2}{5} \rceil.$$

This completes the proof. □

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Received: April 29, 2017.
Accepted: December 20, 2017