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KOROVKIN THEORY FOR EXTRAORDINARY TEST FUNCTIONS BY A-STATISTICAL CONVERGENCE

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Abstract. The classical Korovkin theorem states the uniform convergence of positive linear operators in $C[a, b]$ by providing the convergence only on three test functions $\{e_0, e_1, e_2\}$ where $e_k(x) = x^k,\ k = 0, 1, 2$ and this theorem has been extended in several directions with the aim of finding other subsets of functions, called Korovkin subsets, i.e., satisfying the same property as $\{e_0, e_1, e_2\}$; establishing the same results in abstract Banach spaces. Another direction is to consider more general type of convergences such as convergence generated by a regular summability matrix method, statistical and filter convergence. In this paper we introduce $A-statistical$ Korovkin subset for T and characterize that a subset of $C_0(X)$ is an A-statistical Korovkin subset for T. We also give examples of $A-statistical$ Korovkin subsets for the identity operator.

1 Introduction

Korovkin-type theorems provide simple and useful tools for determining whether a given sequence of positive linear operators, acting on some function spaces, converges to the identity operator. In 1953, Korovkin [8] proved a well known approximation theorem: if $\{L_j\}$ is a sequence of positive linear operators on $C[0, 1]$ such that $\left\|L_j e_k - e_k\right\| \to 0$ as $j \to \infty$ for $k = 0, 1, 2$ where $e_k(x) = x^k$, then L_j converges strongly to the identity operator. This theory has deep connections with real analysis, functional analysis and summability theory. Especially classical Korovkin theory has been generalized using different convergences in summability theory [5, 11, 14]. Besides this Korovkin-type theorems have also been extended in several directions with the aim of such as finding other subsets of functions, called Korovkin subsets, satisfying the same property as $\{e_0, e_1, e_2\}$; establishing the same results in other function spaces or in abstract Banach spaces; establishing the same results for other classes of linear operators [1, 3, 6, 9, 12, 13]. In the present paper we define the concept of $A-statistical$ Korovkin subset and give the characterization for a subset of $C_0(X)$ to be an $A-statistical$ Korovkin subset.

Now we pause to collect some notations. Let $A = \{a_{nj}\}$ be a nonnegative regular matrix. The $A-density$ of $K \subseteq \mathbb{N}$ is given by

$$\delta_A(K) := \lim_n \sum_{j \in K} a_{nj}$$

provided that the limit exists. A sequence $x = (x_j)$ is called $A-statistically$ convergent to a number L if for every $\varepsilon > 0$,

$$\delta_A\left(\{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\}\right) = 0. \tag{1.1}$$

It is not difficult to see that (1.1) is equivalent to

$$\lim_{n \to \infty} \sum_{j : |x_j - L| \geq \varepsilon} a_{nj} = 0, \text{ for every } \varepsilon > 0.$$

This limit expression is denoted by $st_A - \lim_j x_j = L$ [7, 10]. It is known that x is $A-statistically$ convergent to a number L if and only if there exists a subset K of $\mathbb{N}$ such that $\delta_A(K) = 1$

and $\lim_{j \in K} x_j = L$. The cases in which $A = C_1$, the Cesàro matrix and $A = I$, the identity matrix, $A-statistical$ convergence reduces to statistical convergence and ordinary convergence, respectively [4].

Definition 1.1. x is said to be A-statistically bounded if there is a number d such that $\delta_A(\{j : \ |x_j| > d\}) = 0$.

It is easy to see that every A-statistically convergent sequence is also A-statistically bounded.

Definition 1.2. Let $f : E \to \mathbb{R}$ be a real function on a topological space E. The set

$$supp(f) := \overline{\{x : \ f(x) \neq 0\}}$$

is called the support of f.

Let $C(E)$ be the set of all continuous functions on E. If E is locally compact, we will denote by $C_c(E)$ the set of all $f \in C(E)$ with compact support $supp(f)$. A function $f \in C(E)$ lies in $C_c(E)$ just if there is some compact subset of E in the complement of which f is identically zero. We denote by $C_b(E)$ and $C_0(E)$ all bounded, continuous real functions on E and the closure of $C_c(E)$ with respect to the usual sup-norm.

Clearly

$$C_c(E) \subset C_0(E) \subset C_b(E) \subset C(E)$$

since an $f \in C_c(E)$ is bounded on its compact support, hence throughout E.

Recall that positive bounded Radon measure is a positive linear functional on $C_0(E)$. The set of all of positive bounded Radon measures is denoted by M_b^+. It is obvious that every $\mu \in M_b^+$, that is, every positive linear functional $\mu : C_0(E) \to \mathbb{R}$ is continuous with respect to the norm given by

$$\|\mu\| := \sup\left\{|\mu(f)| : \ f \in C_0(E), \ |f| \leq 1\right\}.$$

The following result is known as Urysohn's lemma.

Proposition 1.3. *Let E be a locally compact space and U be an open neighbourhood of the compact subset B. Then $C_c(E)$ contains a function φ which satisfies*

$$0 \leq \varphi \leq 1, \ \ \varphi(B) = \{1\} \ \ and \ \ supp(\varphi) \subset U.$$

Definition 1.4. Let $A = \{a_{nj}\}$ be a nonnegative regular summability method, also E and F be Banach lattices and consider a positive linear operator $T : E \to F$. A subset M of E is said to be an $A - statistical$ Korovkin subset of E for T if for every sequence $\{L_j\}$ of positive linear operators from E into F satisfying

(i) there exists a subset $K \subseteq \mathbb{N}$ such that $\delta_A(K) = 1$ and $\sup_{j \in K} \|L_j\| < \infty$

(ii) $st_A - \lim_{j \to \infty} \|L_j(g) - T(g)\| = 0 \ for \ every \ g \in M,$

then

$$st_A - \lim_{j \to \infty} \|L_j(f) - T(f)\| = 0 \ for \ every \ f \in E.$$

Theorem 1.5. *Let $A = \{a_{nj}\}$ be a nonnegative regular summability method. Let X and Y be locally compact Hausdorff spaces. Further, assume that X has a countable base and Y is metrizable. Given a positive linear operator $T : C_0(X) \to C_0(Y)$ and a subset M of $C_0(X)$, the following statements are equivalent:*

(a) M is an $A - statistical$ Korovkin subset of $C_0(X)$ for T.

(b) If $\mu \in M_b^+(X)$ and $y \in Y$ satisfying $\mu(g) = T(g)(y)$ for every $g \in M$, then $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$.

Proof. Assume that $\mu \in M_b^+(X)$ and $y \in Y$ satisfying $\mu(g) = T(g)(y)$ for every $g \in M$. Let us take a decreasing countable base (U_j) of open neighbourhoods of y in Y. From Proposition 1.3

if we consider the compact set $\{y\}$, we choose $\varphi_j \in C_c(Y)$ such that: $0 \leq \varphi_j \leq 1$, $\varphi_j(y) = 1$ and also $supp(\varphi_j) \subset U_j$. Let us define $L_j : C_0(X) \to C_0(Y)$ by

$$L_j(f) := \mu(f)\varphi_j + v_j T(f)(1 - \varphi_j) \; for \; every \; f \in C_0(X)$$

where the sequence $\{v_j\}$ is nonnegative and $A - statistically$ convergent to 1 but not ordinary convergent. Observe that $\{L_j\}$ is a sequence of positive linear operators and also

$$\|L_j\| \leq \|\mu\| + |v_j|\|T\|$$

so $\sup_{j \in K} \|L_j\| \leq H$ where $K \subset \mathbb{N}$ such that $\delta_A(K) = 1$ since the sequence $\{v_j\}$ is also $A - statistically$ bounded. On the other hand, since $T(g) \in C_0(Y)$ for every $g \in M$, for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\left|T(g)(z) - T(g)(y)\right| \leq \varepsilon \; for \; every \; z \in U_m.$$

So one can get

$$\begin{aligned} |T(g)(y) - v_j T(g)(z)| &= |T(g)(y) - v_j T(g)(z) - T(g)(z) + T(g)(z)| \\ &\leq |T(g)(z) - T(g)(y)| + |T(g)(z)||v_j - 1| \\ &\leq \varepsilon + |T(g)(z)||v_j - 1|, \; for \; every \; z \in U_m. \end{aligned}$$

Moreover for every $j \geq m$ and for every $z \in Y$, we have

$$\begin{aligned} \Big|L_j(g)(z) - T(g)(z)\Big| &= \Big|\mu(g)\varphi_j(z) + v_j T(g)(z) - v_j T(g)(z)\varphi_j(z) - T(g)(z)\Big| \\ &\leq \varphi_j(z)\big|\mu(g) - v_j T(g)(z)\big| + |v_j - 1||T(g)(z)| \\ &\leq \varphi_j(z)\big|T(g)(y) - v_j T(g)(z)\big| + |v_j - 1||T(g)(z)| \\ &\leq \big|T(g)(y) - v_j T(g)(z)\big| + |v_j - 1||T(g)(z)|. \end{aligned}$$

Hence using the last inequality we get for $j \geq m$,

$$\Big|L_j(g)(z) - T(g)(z)\Big| \leq \begin{cases} |v_j - 1||T(g)(z)| & , z \notin U_j \\ \varepsilon + 2|T(g)(z)||v_j - 1| & , z \in U_j \end{cases}$$

and since $\{v_j\}$ is A-statistically convergent to 1, we obtain

$$st_A - \lim_j \Big\|L_j(g) - T(g)\Big\| = 0.$$

Furthermore M is an $A - statistical$ Korovkin subset for T, so it is obtained that for every $f \in C_0(X)$, that $st_A - \lim_j \Big\|L_j(f) - T(f)\Big\| = 0$. But for every $j \geq 1, \; L_j(f)(y) = \mu(f)$, then we obtain $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$. This completes the proof of (b).

Conversely assume that if $\mu \in M_b^+(X)$ and $y \in Y$ satisfy $\mu(g) = T(g)(y)$ for every $g \in M$, then $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$. Observe that

$$if \; \mu \in M_b^+(X) \; and \; \mu(g) = 0 \; for \; every \; g \in M, \; then \; \mu = 0. \tag{1.2}$$

Since X has a countable base, every bounded sequence in $M_b^+(X)$ has a vaguely convergent subsequence (See [2]). Consider now a sequence $\{L_j\}$ of positive linear operators from $C_0(X)$ into $C_0(Y)$ satisfying properties (i) and (ii) of Definition 1.4 and suppose that for some $f_0 \in C_0(X)$ $st_A - \lim_j \|L_j(f_0) - T(f_0)\| \neq 0$, i.e., there exists $K \subset \mathbb{N}$ such that $\delta_A(K) \neq 0$ and $\|L_j(f_0) - T(f_0)\|_{j \in K}$ does not converge to 0. So there exist $\varepsilon_0 > 0$ and $\{y_j\} \subset Y$ such that $\delta_A(\{n(j) \in K : \; j = 1, 2, 3, ...\}) \neq 0$ and

$$\Big|L_{n(j)}(f_0)(y_j) - T(f_0)(y_j)\Big| \geq \varepsilon_0. \tag{1.3}$$

We have two cases: (y_j) is converging to the point at infinity of Y or not. In the first case, since (y_j) converges to the point at infinity of Y we get $\lim_{j\to\infty} h(y_j) = 0$ for every $h \in C_0(Y)$. For every $j \geq 1$, define $\mu_j \in M_b^+(X)$ by

$$\mu_j(f) := L_{n(j)}(f)(y_j) \quad (f \in C_0(X)).$$

From hypothesis, we have $\sup_j \|\mu_j\| \leq \sup_j \|L_{n(j)}\| \leq H$. Since (μ_j) is bounded, we may assume that there exists $\mu \in M_b^+(X)$ such that $\mu_j \to \mu$ vaguely (If necessary the sequence μ_j is replaced with a suitable subsequence). On the other hand if $g \in M$, then

$$\begin{aligned} |\mu_j(g)| &\leq \Big|L_{n(j)}(g)(y_j) - T(g)(y_j)\Big| + |T(g)(y_j)| \\ &\leq \Big\|L_{n(j)}(g) - T(g)\Big\| + |T(g)(y_j)| \end{aligned}$$

which implies $\mu(g) = \lim_j \mu_j(g) = 0$. From (1.2) we obtain $\mu(f_0) = 0$ as well and hence

$$\Big|L_{n(j)}(f_0)(y_j) - T(f_0)(y_j)\Big| = \big|\mu_j(f_0) - T(f_0)(y_j)\big| \to 0.$$

This contradicts (1.3). In the second case the sequence (y_j) does not converge to the point at infinity of Y. By replacing it with a suitable subsequence, we may assume that it converges to some $y \in Y$. Let us consider

$$\mu_j(f) := L_{n(j)}(f)(y_j) \quad (f \in C_0(X)).$$

As in the first case the same reasoning we may assume that there exists $\mu \in M_b^+(X)$ such that $\mu_j \to \mu$ vaguely. Moreover since for every $g \in M$,

$$\Big|\mu_j(g) - T(g)(y_j)\Big| = \big|L_{n(j)}(g)(y_j) - T(g)(y_j)\big| \leq \big\|L_{n(j)}(g) - T(g)\big\| \to 0,$$

we have $\mu(g) = T(g)(y)$. So (b) implies $\mu(f_0) = T(f_0)(y_j)$, i.e.,

$$\lim_{j\to\infty} \Big[L_{n(j)}(f_0)(y_j) - T(f_0)(y_j)\Big] = 0$$

which contradicts (1.3). □

If we replace $T : X \to Y$ with the identity operator $I_X : X \to X$, one can immediately get the following

Theorem 1.6. $A = \{a_{nj}\}$ *be a nonnegative regular summability method. Let* X *be a locally compact Hausdorff space with a countable base, which is then metrizable as well. Given a subset* M *of* $C_0(X)$*, the following statements are equivalent:*
(i) M *is an* $A - statistical$ *Korovkin subset of* $C_0(X)$ *for identity operator* I_X*.*
(ii) If $\mu \in M_b^+(X)$ *and* $x \in X$ *satisfy* $\mu(g) = g(x)$ *for every* $g \in M$*, then* $\mu(f) = f(x)$ *every* $f \in C_0(X)$ *i.e.* $\mu = I_X$*.*

Corollary 1.7. *Under the assumptions of Theorem 1.5, the following statements are equivalent:*
(i) M *is a Korovkin subset of* $C_0(X)$ *for T.*
(ii) M *is an* $A - statistical$ *Korovkin subset of* $C_0(X)$ *for T.*

By using Corollary 1.7 we obtain all results given in Chapter 6 of [2] for $A - statistical$ Korovkin subset.
We first recall that a mapping $\varphi : Y \to X$ is said to be proper if for every compact subset $K \in X$, the inverse image $\varphi^{-1}(K) := \{y \in Y : \varphi(y) \in K\}$ is compact in Y where X and Y are locally compact Hausdorff spaces. In this case, $fo\varphi \in C_0(Y)$ for every $f \in C_0(X)$.
Now we provide an application of our main theorem.

Corollary 1.8. *Let Y be metrizable locally compact Hausdorff space. If M is an $A-statistical$ Korovkin subset of $C_0(X)$ for I_X, then M is an $A-statistical$ Korovkin subset for any positive linear operator $T : C_0(X) \to C_0(Y)$ of the form*

$$T(f) := \lambda(fo\varphi), \quad (f \in C_0(X))$$

where $\lambda \in C_b(Y)$, $\lambda \geq 0$ and $\varphi : Y \to X$ is a proper mapping.

Now we can give these examples of $A-statistical$ Korovkin subsets for identity operator under the light of our Corollary 1.7 and Corollary 6.7 and Proposition 6.8 of [2].

Given $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},\ 0 < \lambda_1 < \lambda_2 < \lambda_3$ then

- $\{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$ is an $A-statistical$ Korovkin subset of $C_0(X)$ where $e_{\lambda_k}(x) := x^{\lambda_k}$ for every $x \in\ X := (0, 1]$ and $k = 1, 2, 3$.
- $\{e_{-\lambda_1}, e_{-\lambda_2}, e_{-\lambda_3}\}$ is an $A-statistical$ Korovkin subset of $C_0(X)$ where $e_{-\lambda_k}(x) := x^{-\lambda_k}$ for every $x \in\ X := [1, +\infty)$ and $k = 1, 2, 3$.
- $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ is an $A-statistical$ Korovkin subset of $C_0(X)$ where $f_{\lambda_k}(x) := exp(-\lambda_k x)$ for every $x \in\ X := [0, +\infty)$ and $k = 1, 2, 3$.

References

[1] F. Altomare and S. Diomede, Contractive Korovkin subsets in weighted spaces of continuous functions. *Rend. Circ. Mat. Palermo* **50**, 547–568 (2001).

[2] F. Altomare, *Korovkin-type theorems and approximation by positive linear operators.* Surveys in Approximation Theory 5.13 (2010).

[3] V. A. Baskakov, On various convergence criteria for linear positive operators (Russian). *Usp. Mat. Nauk.* **16**, 131–134 (1961).

[4] J. A. Fridy, On statistical convergence. *Analysis* **5**, 301–313 (1985).

[5] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence. *Rocky Mountain J. Math.* **32**, 129–138 (2002).

[6] K. Izuchi, H. Takagi and S. Watanabe, Sequential BKW-operators and function algebras. *J. Approx. Theory* **85**, 185–200 (1996).

[7] E. Kolk, Matrix summability of statistically convergent sequences. *Analysis* **13**, 77–83 (1993).

[8] P. P. Korovkin, On convergence of linear operators in the space of continuous functions. *Dokl. Akad. Nauk SSSR* **90**, 961–964 (1953).

[9] C. A. Micchelli, Convergence of positive linear operators on $C(X)$. *J. Approx. Theory* **13**, 305–315 (1975).

[10] H. I. Miller, A measure theoretical subsequences characterization of statistical convergence. *Trans. Amer. Math. Soc.* **347**, 1811–1819 (1995).

[11] I. Sakaoğlu and C. Orhan, Strong summation process in L_p spaces. *Nonlinear Analysis* **86**, 89–94 (2013).

[12] S. E. Takahasi, Bohman-Korovkin-Wulbert operators on $C[0, 1]$ for $\{1, x, x^2, x^3, x^4\}$. *Nihonkai Math. J.* **1**, 155–159 (1990).

[13] S. E. Takahasi, Bohman-Korovkin-Wulbert operators on normed spaces. *J. Approx. Theory* **72**, 174–184 (1993).

[14] M. Ünver, Abel transforms of positive linear operators. *AIP Conference Proceedings* **1558**, 1148–1151 (2013).

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