Palestine Journal of Mathematics

Vol. 9(1)(2020) , 126–131

ON COMMUTATIVITY OF PRIME RINGS WITH LEFT GENERALIZED DERIVATIONS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16W25; Secondary 16N60,16U80.

Keywords and phrases: Prime and semiprime rings, derivations, left generalized derivations.

The paper was supported by the Anhui Provincial Natural Science Foundation (1808085MA14) and the Key University Science Research Project of Anhui Province (KJ2018A0433).

Abstract Let R be a prime ring with center $Z(R)$, central closure RC, right Martindale quotient ring Q_r, F a left generalized derivation and I a nonzero right ideal of R which is semiprime as a ring. We proved in this article that if one of the following conditions holds: (i) $F(x \circ y) \pm x \circ y = 0$ (ii) $F(xy) \pm xy \in Z(R)$ (iii) $F(x)F(y) \pm xy \in Z(R)$ (iv) $F([x, y]) = \pm[F(x), y]$ (v) $F(x \circ y) = \pm(F(x) \circ y)$ for all $x, y \in I$, then R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.

1 Introduction

In this article, unless otherwise mentioned, R will be a prime ring with characteristic different from two, $Z(R)$ the center of R, Q_r its right Martindale quotient ring. The center of Q_r, denoted by C, is called the extended centroid of R and RC its central closure (we refer to [5] for these object). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for Lie product $xy - yx$ and Jordan product $xy+yx$, respectively. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. By a derivation on R we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is called an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed elements $a, b \in R$. For such a mapping F, it is clear that $F(xy) = F(x)y + x[y, b] = F(x)y + xd(y)$, where $d(y) = [y, b]$. This observation leads to the following definition given in [6]: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Very recently, Koc and Golbasi in [9] discussed multiplicative generalized derivations, a generalization form of generalized derivations, on Lie ideals in semiprime rings.

Many results in literature indicate that the global structure of a ring R is often lightly connected to the behavior of additive mappings defined on R. The first result in this topic is the classical Posner's second theorem. It is proved in [13] that if a prime ring R admits a nonzero derivation d satisfying $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. As is well known, this theorem was regarded as the starting point of many papers concerning the study of various kinds of additive mappings satisfying appropriate algebraic conditions on some subsets of prime and semiprime rings. Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R (for instance, see [4] and [14], where further references can be found). In the year 2001, Ashraf and Rehman [2] proved that a prime ring R must be commutative if it admits a derivation d satisfying either $d(xy) \pm xy \in Z(R)$ or $d(x)d(y) \pm xy \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R. In [3], the authors extended the above results to generalized derivations. In [1], Ashraf and Rehman proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. Later, in [12] this result was also extended to the case of generalized derivations. More precisely, Rehman proved that if a prime ring R admits a generalized derivation F associated with derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, a nonzero ideal I of R, and if $F = 0$ or $d \neq 0$, then R is commutative. Furthermore, the authors in [11] proved Ashraf and Rehman's result is also true for left generalized derivations.

Now recall that an additive mapping $F : R \longrightarrow R$ is called a left generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = d(x)y + xF(y)$ holds for all $x, y \in R$. Note that if $F = d$ then F is just an ordinary derivation. Hence, one may observe that the concept of left generalized derivations includes the concept of derivations and the right multipliers. The following example demonstrates that left generalized derivations in rings do exist. Let S be any ring and $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in S \right\}$. Define maps $F : R \to R$ by $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & d-a \\ c & -c \end{pmatrix}$ and $d : R \to R$ by $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$. Then it is easy to check that F is a left generalized derivation on R. Therefore, it should be interesting to extend some results concerning derivations to left generalized derivations. Motivated by these observations, we shall extend some commutativity results for derivations to left generalized derivations in prime rings.

2 Some preliminaries

In the present paper, we shall make some extensive use of the following basic identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$
$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$
$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

We begin with several known results which will be used in the sequel to prove our theorems and these lemmas can be found in [7], [8] and [10].

Lemma 2.1. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If $aI = 0$ (or $Ia = 0$) for all $a \in R$, then $a = 0$.*

Lemma 2.2. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a derivation d such that $d(I) = 0$, then $d = 0$.*

Lemma 2.3. *If a prime R contains a nonzero commutative right ideal, then R is commutative.*

Lemma 2.4. *Let $F : R \longrightarrow RC$ be an additive map such that $F(xy) = xF(y)$ for all $x, y \in R$. Then there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

3 Main results

Theorem 3.1. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x \circ y) = x \circ y$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. By the hypothesis we have

$$F(x \circ y) = x \circ y \text{ for all } x, y \in I. \tag{3.1}$$

Replacing x by yx in (3.1), we get $F((yx) \circ y) = (yx) \circ y$ for all $x, y \in I$. Recalling that $(yx) \circ y = y(x \circ y)$, so the above equation becomes $F(y(x \circ y)) = y(x \circ y)$ for all $x, y \in I$. Since F is a left generalized derivation, we find that

$$d(y)(x \circ y) + yF(x \circ y) = y(x \circ y) \text{ for all } x, y \in I. \tag{3.2}$$

Combine (3.1) with (3.2) to get $d(y)(x \circ y) = 0$ for all $x, y \in I$. Replacing x by xr we have

$$0 = d(y)((xr) \circ y) = d(y)x[r, y] \text{ for all } x, y \in I, r \in R. \quad (3.3)$$

This means that $d(y)IR[r, y] = 0$. The primeness of R forces that, for each $y \in I$, either $d(y)I = 0$ or $[r, y] = 0$. Let $I_1 = \{y \in I \mid d(y)I = 0\}$ and $I_2 = \{y \in I \mid [r, y] = 0\}$. Then, I_1 and I_2 are both additive subgroups of I such that $I = I_1 \cup I_2$. By Brauer's trick, we get either $I_1 = I$ or $I_2 = I$. On the one hand, if $I_1 = I$, then $d(y)I = 0$ for all $y \in I$, that is, $d(I)I = 0$. In view of Lemma 2.1, $d(I) = 0$. And hence $d = 0$ by Lemma 2.2. In this case $F(xy) = xF(y)$, then by Lemma 2.4, there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$. On the other hand, if $I_2 = I$, then $[r, y] = 0$ for all $y \in I$ and $r \in R$. In particular, $[x, y] = 0$ for all $x, y \in I$. Therefore, I is a commutative right ideal and so R, proving the theorem. □

Corollary 3.2. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x^2) = x^2$ for all $x \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. By the assumption, we get $F(x^2) = x^2$ for all $x \in I$. Linearization of the above equation gives that $F(x^2) + F(y^2) + F(x \circ y) = x^2 + y^2 + x \circ y$ for all $x, y \in I$. This equation can reduce to $F(x \circ y) = x \circ y$ for all $x, y \in I$. Hence by Theorem 3.1, the proof is complete.□

Theorem 3.3. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x \circ y) + x \circ y = 0$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. If F is a left generalized derivation such that $F(x \circ y) + x \circ y = 0$ for all $x, y \in I$, then the left generalized derivation $-F$ satisfies that $(-F)(x \circ y) - x \circ y = 0$, namely, $(-F)(x \circ y) = x \circ y$ for all $x, y \in I$. Thus, application of Theorem 3.1, we get the required result.□

Corollary 3.4. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x^2) + x^2 = 0$ for all $x \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Theorem 3.5. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(xy) - xy \in Z(R)$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. We are given that $F(xy) - xy \in Z(R)$, which can be rewritten as

$$d(x)y + xF(y) - xy \in Z(R) \text{ for all } x, y \in I. \quad (3.4)$$

Replace x by zx in (3.4) to get $d(z)xy + zd(x)y + zxF(y) - zxy \in Z(R)$ for all $x, y, z \in I$. In particular, $[d(z)xy + z(d(x)y + xF(y) - xy), z] = 0$ for all $x, y, z \in I$. This can reduces to $[d(z)xy, z] = 0$ for all $x, y, z \in I$. Expand the last equation to get

$$[d(z), z]xy + d(z)x[y, z] + d(z)[x, z]y = 0 \text{ for all } x, y, z \in I. \quad (3.5)$$

Replace y by yr in (3.5) to get

$$[d(z), z]xyr + d(z)xy[r, z] + d(z)x[y, z]r + d(z)[x, z]yr = 0 \text{ for all } x, y, z \in I, r \in R. \quad (3.6)$$

Combining (3.5) and (3.6), we arrive at $d(z)xy[r, z] = 0$ for all $x, y, z \in I$ and $r \in R$. This implies that $d(z)IRy[r, z] = 0$ for all $y, z \in I$ and $r \in R$. Since R is prime, the above expression yields that either $d(z)I = 0$ of $y[r, z] = 0$. Consequently, the set of $z \in I$ for which these two properties hold are additive subgroups of I whose union is I. As is well known, a group can not be the set-theoretic union of two proper subgroups and thus $d(I) = 0$ or $I[R, I] = 0$. In the former case, by Lemma 2.1 and Lemma 2.2, we find that $d = 0$, and hence $F(xy) = xF(y)$ for all $x, y \in R$. Again by application of Lemma 2.4, there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$. In the latter case, by Lemma 2.1, we get $[R, I] = 0$ whence it follows that $[I, I] = 0$. Therefore, I is a commutative right ideal and Lemma 2.3 yields that R is commutative.□

Using the same technique with necessary variations, we can prove the following:

Theorem 3.6. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(xy) + xy \in Z(R)$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Theorem 3.7. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. We have $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$. Replacing x by xz in above equation, we see that $d(x)zF(y) + x(F(z)F(y) - zy) \in Z(R)$ for all $x, y, z \in I$. This implies that $[d(x)zF(y), x] = 0$, which can be rewritten as

$$[d(x)z, x]F(y) + d(x)z[F(y), x] = 0 \text{ for all } x, y, z \in I. \tag{3.7}$$

Substituting z by $zF(y)$ in (3.7) and using (3.7), we have $d(x)zF(y)[F(y), x] = 0$ for all $x, y, z \in I$. That is to say, $d(x)IRF(y)[F(y), x] = 0$ for all $x, y \in I$. By the primeness of R, either $d(x)I = 0$ or $F(y)[F(y), x] = 0$. Again by By Brauer's trick, we have $d(I)I = 0$ or $F(y)[F(y), I] = 0$. If $d(I)I = 0$, then by Lemma 2.1 we have $d(I) = 0$ and hence $d = 0$. Thus, there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$ and we are done. If $F(y)[F(y), I] = 0$ then $0 = F(y)[F(y), IR] = F(y)[F(y), I]R + F(y)I[F(y), R] = F(y)I[F(y), R]$. It follows from the last equation that $F(y)IR[F(y), R] = 0$. The primeness of R gives that $F(y)I = 0$, in this case $F(y) = 0$ or $[F(y), R] = 0$ in this case $F(y) \in Z(R)$. Henceforth, in each case we have $F(y) \in Z(R)$. Thus our hypothesis yields that $xy \in Z(R)$ for all $x, y \in I$. In particular, $0 = [xy, y] = [x, y]y$ for all $x, y \in I$. Replace x by zx in last equation to get $0 = [zx, y]y = z[x, y]y + [z, y]xy = [z, y]xy$, namely, $[z, y]Iy = 0$ and so $[z, y]IRy = 0$. The primeness of R forces for each $y \in I$, either $[z, y]I = 0$ or $y = 0$. But $y = 0$ also implies that $[z, y]I = 0$. Hence, in both cases we have $[z, y]I = 0$ for all $y, z \in I$. By virtue of Lemma 2.1, $[z, y] = 0$ for all $y, z \in I$. Thus, I is commutative and so R, and we get the required result.□

Proceeding on the same lines with necessary variations, we can prove the following result:

Theorem 3.8. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Theorem 3.9. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F([x, y]) = \pm[F(x), y]$ for all $x, y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. For the sake of clearness, we only prove the case of $F([x, y]) = [F(x), y]$. By our hypothesis, we have

$$F([x, y]) = [F(x), y] \text{ for all } x, y \in I. \tag{3.8}$$

Writing xy instead of y in (3.8) and using (3.8), we get

$$d(x)[x, y] = [F(x), x]y \text{ for all } x, y \in I. \tag{3.9}$$

Take y by $yd(x)$ in (3.9) to get

$$d(x)[x, y]d(x) + d(x)y[x, d(x)] = [F(x), x]yd(x) \text{ for all } x, y \in I. \tag{3.10}$$

Multiplying (3.9) on the right by $d(x)$, we find that

$$d(x)[x,y]d(x) = [F(x),x]yd(x) \text{ for all } x,y \in I. \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$d(x)y[x,d(x)] = 0 \text{ for all } x,y \in I. \tag{3.12}$$

This implies that $d(x)IR[x,d(x)] = 0$ for all $x \in I$. The primeness of R forces that $d(x)I = 0$ and this case implies that $d(x) = 0$ or $[x,d(x)] = 0$. But $d(x) = 0$ also implies that $[x,d(x)] = 0$. So in each case we have $[x,d(x)] = 0$ for all $x \in I$. The linearization of the last equation yields that $[x,d(x)] + [x,d(y)] + [y,d(x)] + [y,d(y)] = 0$, which can reduces to

$$[x,d(y)] + [y,d(x)] = 0 \text{ for all } x,y \in I. \tag{3.13}$$

Replacing x by yx in (3.13), we have $[yx,d(y)] + [y,d(yx)] = 0$ for all $x,y \in I$. This can be rewritten as $y[x,d(y)] + 2[y,d(y)]x + d(y)[y,x] + y[y,d(x)] = 0$ for all $x,y \in I$. Using (3.13) and the fact that $[y,d(y)] = 0$, we arrive at $d(y)[y,x] = 0$ for all $x,y \in I$. Then, $0 = d(y)[y,IR] = d(y)[y,I]R + d(y)I[y,R] = d(y)I[y,R]$ for all $y \in I$. This equation is same as (3.3) in the proof of Theorem 3.1. Hence, using the same arguments we get the required result. In a similar manner, we can prove that the same conclusion holds for $F([x,y]) = -[F(x),y]$ for all $x,y \in I$. The proof is now completed.□

Theorem 3.10. *Let R be a prime ring and I a nonzero right ideal of R which is semiprime as a ring. If R admits a left generalized derivation of F satisfying $F([x,y]) = \pm[F(x),y]$ for all $x,y \in I$, then either R is commutative or there exists $q \in Q_r(RC)$ such that $F(x) = qx$ for all $x \in R$.*

Proof. We only consider the case

$$F(x \circ y) = F(x) \circ y \text{ for all } x,y \in I. \tag{3.14}$$

Replacing y by xy in (3.14) and using (3.14) , we get $d(x)(x \circ y) = [F(x),x]y$ for all $x,y \in I$. Substituting $yd(x)$ for y in the last equation and using the fact that $d(x)(x \circ y)d(x) = [F(x),x]yd(x)$, we have $d(x)y[x,d(x)] = 0$ for all $x,y \in I$. This equation is same as (3.12) in the proof of Theorem 3.9, and we are done.□

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Received: March 29, 2018.
Accepted: November 3, 2018.