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Numerical solution for singularly perturbed differential equation via operational matrix based on Genocchi polynomials

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Abstract In this paper, one may present an effective algorithm for treating a singularly perturbed differential equation (SPDE) using Genocchi polynomials (GPs) analytically. Some basic preliminaries for GPs are first presented. Then, a collocation method based on these polynomials is explored for the problem under consideration. The suggested technique is tested for two numerical examples where the first ten terms of the obtained semi-analytical solutions are evaluated at different values of the variable x and the numerical results are compared with two different methods.

1 Introduction

In the field of SPDE, the computation of its solution has been a great challenge and is great importance due to the versatility of such equations in various application field such as fluid mechanics, fluid dynamics, elasticity, aero dynamics. These problems depend on a small positive parameter (ϵ) in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain.
The authors in [2] considered a semi linear ordinary differential equation which was integrated to achieve a first order ordinary differential equation (ODE), and considered both the inner and outer solutions. And these equations as well as numerical methods have been studied by several authors (see [1, 6]).
In this sequel, a new operational matrix of fractional order derivative based on Genocchi polynomials is introduced to provide approximate solutions of SPDE. The outline of this sequel is as follow: In Section 2, Some basic preliminaries is stated. Explanation of the problem is explained in Section 3. Some numerical results are provided in Section 4. Finally, Section 5 will give a conclusion briefly.

2 Some basic preliminaries

Genocchi numbers and polynomials have been extensively studied in various papers (see [5]). The classical Genocchi polynomials $G_n(x)$ is usually defined by the following form

$$\frac{2te^{xt}}{e^t+1} = \sum_{i=0}^{\infty} G_n(x)\frac{t^n}{n!},\ (|t| < \pi), \tag{2.1}$$

where $G_n(x)$ is the GPs of degree n which is given as:

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k},$$

and G_n is the Gennochi number which is given as:

$$G_1 = 1, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 0, G_6 = -3, G_7 = 0,$$
$$G_8 = 17, G_9 = 0, G_{10} = -155,$$
$$G_{2n+1} = 0, n \in N, G_{12} = 2073,$$

hence

$$\begin{aligned}
&G_1(x) = 1,\\
&G_2(x) = 2x - 1,\\
&G_3(x) = 3x^2 - 3x,\\
&G_4(x) = 4x^3 - 6x^2 + 1,\\
&G_5(x) = 5x^4 - 10x^3 + 5x,\\
&G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3,\\
&G_n(x+1) + G_n(x) = 2nx^{n-1}, n \geq 0,\\
&\frac{dG_n(x)}{dx} = nG_{n-1}(x), n \geq 1,\\
&\int_a^b G_n(x)dx = \frac{G_{n+1}(b) - G_{n+1}(a)}{n+1},\\
&\int_0^1 G_n(x)G_m(x)\, dx = \frac{2(-1)^n n!m!}{(n+m)!}G_{n+m},\ m, n \geq 1,
\end{aligned}$$

3 Explanation of the problem

Firstly, SPDE is considered

$$\begin{aligned}
&\epsilon y''(x) + p(x)y'(x) + q(x)y(x) - g(x) = 0,\ 0 \leq x \leq 1, \qquad (3.1)\\
&y(0) = \alpha,\ y(1) = \beta
\end{aligned}$$

where $p(x)$, $q(x)$ and $g(x)$ are sufficiently smooth functions, α and β are arbitrary constants, ϵ is a small positive parameter, and $y(x)$ is unknown function.
Now, the collocation method based on Genocchi operational matrix of derivatives to solve numerically SPDE is presented.
Our strategy is utilizing GPs to approximate the solution $y(x)$ by $y_N(x)$ is as given below.

$$y(x) \approx y_N(x) = \sum_{n=1}^{N} c_n G_n(x) = G(x)C,$$

where

$$\begin{aligned}
&C^T = [c_1, c_2, \ldots, c_n],\\
&G(x) = [G_1(x), G_2(x), \ldots, G_N(x)],
\end{aligned}$$

also

$$\begin{aligned}
&G'(x)^T = MG^T(x), \Rightarrow G'(x) = G(x)M^T,\\
&\vdots\\
&G^{(k)}(x) = G(x)(M^T)^k,
\end{aligned}$$

where M is $N \times N$ operational matrix of derivative which is given as:

$$M = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0,\\
2 & 0 & 0 & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \ldots & N-1 & 0 & 0\\
0 & 0 & 0 & \ldots & 0 & N & 0
\end{bmatrix}$$

then, the k-th derivative of $y_N(x)$ can be stated as

$$y_N^{(k)}(x) = G^{(k)}(x)C = G(x)(M^T)^k C, \tag{3.2}$$

by Eqs. (3.1) and (3.2), we have

$$\epsilon G(x)(M^T)^2 C + p(x)G(x)M^T C + q(x)G(x)C - g(x) = 0, \tag{3.3}$$

to obtain $y_N(x)$, one may use the collocation points $x_j = \frac{j-1}{N}$, $j = 1, 2, \ldots, N-1$. These equations can be solved by Maple 15 software.

Lemma 3.1. *If $y(x) \in C^{n+1}[0,1]$ and $U = Span\{G_1(x), G_2(x), \ldots, G_N(x)\}$, then $G(x)C$ is the best approximation of $y(x)$ out of U when*

$$\|y(x) - G(x)C\| \leq \frac{h^{\frac{2n+3}{2}} R}{(n+1)!\sqrt{2n+3}}, \; x \in [x_i, x_{i+1}] \subset [0,1],$$

where $R = \max_{x \in [x_i, x_{i+1}]} |y^{(n+1)}(x)|$ and $h = x_{i+1} - x_i$.

Proof. See [4]. □

4 Numerical applications

In this section, some results are given to demonstrate the quality of the sated technique in approximating the solution of SPDE.

Example 4.1. First, the following SPDE is considered (see [3])

$$\epsilon y'' + y = 0,$$
$$y(0) = 0, \quad y(1) = 1,$$
$$y_{exact}(x) = \frac{sin(\frac{x}{\sqrt{\epsilon}})}{sin(\frac{1}{\sqrt{\epsilon}})},$$

One may achieve

$$\begin{aligned} y_{approx}(x) &= -20.570113x + 55.29527944x^2 - 250.849768x^3 + 2977.28137x^4 \\ &- 10817.41278x^5 + 17013.66227x^6 - 12817.21305x^7 \\ &+ 4249.369047x^8 - 388.5622535x^9, \end{aligned}$$

with this technique by $n = 10$ and $\epsilon = 10^{-2}$. The approximate and exact solution for $y(x)$ are shown in figure 1. Table 1 demonstrates the absolute error of the this technique.

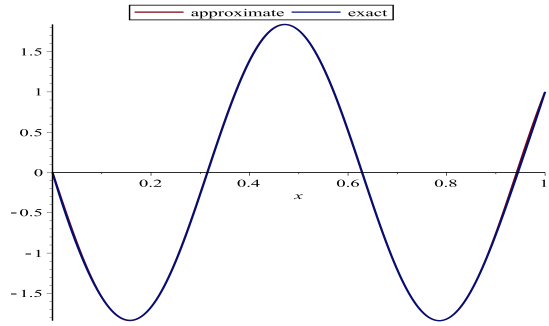

Figure 1. The approximate and exact solution of $y(x)$ for Example 4.1

Table 1. The absolute error of the this method for Example 4.1 with $n = 10$ and $\epsilon = 10^{-2}$

x	error of y
0.1	0.002818580693
0.2	$2.049552816 \times 10^{-7}$
0.3	0.007346408557
0.4	0.006970506336
0.5	0.000001746071946
0.6	0.007481043140
0.7	$9.185305312 \times 10^{-7}$
0.8	0.000003522841231
0.9	0.01414795299
1.0	0.000001500000000

Example 4.2. Second, the following SPDE is considered (see [3])

$$\begin{aligned} &\epsilon y'' + (1 - \frac{x}{2})y' - \frac{1}{2}y = 0, \\ &y(0) = 0, \quad y(1) = 1 \\ &y_{exact}(x) = \frac{1}{2-x} - \frac{1}{2}e^{-\frac{x}{4}+\frac{x^2}{16}}, \end{aligned}$$

One may obtain

$$\begin{aligned} y_{approx}(x) &= 19.882x - 4357.499925x^11 + 668.9680910x^12 - 273.9026x^2 \\ &+ 1962.721x^3 - 8262.3343x^4 + 21640.389x^5 - 35510.57001x^6 \\ &+ 34377.2155x^7 - 14428.05702x^8 - 5597.78005x^9 + 9761.968971x^{10} \end{aligned}$$

with this method by $n = 13$ and $\epsilon = 10^{-2}$. Table 2 demonstrates the absolute error of the this technique. Also, we compare the numerical results of B-splines method (BM) [3], Chebyshev-Gauss grid method (ChgM) and presented method in Table 3.

Table 2. The absolute error of the this method for Example 4.2 with $n = 13$ and $\epsilon = 10^{-2}$

x	error of y
0.1	0.04355702423
0.2	0.0001611588014
0.3	0.004400500700
0.4	0.0007657160000
0.5	0.0004581663000
0.6	0.002234155700
0.7	0.0008172308000
0.8	0.0006226667000
0.9	0.0004170909000
1.0	0.0006610000000

Table 3. Comparison of the absolute error between this method ($n = 13$) and others for Example 4.2 with $\epsilon = 2^{-4}$

x	error of y	error of ChgM ($N = 100$) in [3]	error of BM ($N = 200$) in [3]
0.1	0.006341928300	0.029	0.0354
0.2	0.000001981250000	0.035	0.0325
0.3	$0.6500879490 \times 10^{-3}$	$--$	$--$
0.4	$0.4936620000 \times 10^{-3}$	$--$	$--$
0.5	0.000005197721500	0.0338	0.0234
0.6	$0.1820443981 \times 10^{-2}$	$--$	$--$
0.7	$0.5499775547 \times 10^{-4}$	$--$	$--$
0.8	$0.2315202100 \times 10^{-5}$	$--$	$--$
0.9	$0.2487923486 \times 10^{-4}$	0.0134	0.0084
1.0	$1.873220000 \times 10^{-10}$	$--$	$--$

5 Conclusions

In this paper, GPs stated for solving the SPDE. The stated technique is computationally attractive. Some results are included to explain the validity of this technique. The presented approximate solutions are more accurate compared to the references as it is shown in the tables. By stated technique, the high orders of convergence obtained when it achieved accurate solutions even for small values of n.

References

[1] P.P. Chakravarthy, K. Phaneendra, Y.N. Reddy, A seventh order numerical method for singular perturbation problems. *Appl. Math. Comp*, **186**, 860–871 (2007), DOI: 10.1016/j.amc.2006.08.022.

[2] M.G. Gasparo, M. Maconi, Initial value methods for second order singularly perturbed boundary value problems. *Journal of Optimization Theory and Applications*, **66**, 197–210 (1990).

[3] M. Ǵulsu, Y. Öztürk, M. Sezer, Approximate solution of the singular-perturbation problem on Chebyshev-Gauss grid, *Journal of Advanced Research in Differential Equations*, **3(4)**, 1–13 (2011).

[4] A. Isah, C. Phang, New operational matrix of derivative for solving non-linear fractional differential equations via Genocchi polynomials, *Journal of King Saud University-Science*, (2017) DOI: 10.1016/j.jksus.2017.02.001.

[5] A. Isah, C. Phang, Operational matrix based on Genocchi polynomials for solution of delay differential equations, *Ain Shams Engineering Journal*, (2017) DOI: 10.1016/j.asej.2016.09.015.

[6] M.M. Kadalbajoo, K.C. Patidar. ϵ-Uniformly convergent fitted mesh finite difference methods for general singular perturbation problems, *Appl. Math. Comp*, **179**, 248–266 (2006), DOI: 10.1016/j.amc.2005.11.096.

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