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Numerical solution for nonlinear volterra integro-differential equations

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Abstract. In this paper, a method for solving nonlinear volterra integro-differential equations (NVIDEs) is stated. The main idea behind this work is the use of the Bezier curve method (BCM). To show the efficiency of the developed method, numerical results are presented.

1 Introduction

Nonlinear volterra integral-differential equations (IDEs) are based on many problems of theoretical physics, and many disciplines. therefore application of numerical techniques for solving them are attractive. There are many techniques to solve system of NVIDEs, such as chebyshev wavelets, block-pulse functions, differential transforms, Tau technique [6], successive approximation method, Adomian decomposition method (ADM), Chebyshev and Taylor collocation methods, Haar Wavelet method (HWM), Wavelet Galerkin method (WGM), monotone iterative technique, and Walsh series method. Also, the methods based on Legendre polynomials (LP) may be used for solving linear and nonlinear differential and Fredholm-Volterra integral and integro-differential-difference equations [3].
BCM is used for solving dynamical systems, (see [4]). Also BCM is used for solving delay differential equations and switched systems (see [4]). Authors in [5] proposed the utilization of BCM on some linear optimal control systems with pantograph delays. Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, BCM is utilized (see [5]). In this study, BCM is extended for solving NVIDEs as follows:

$$\sum_{j=1}^{M} F_{ij}(t, y_j, \ldots, y_j^{(\gamma_{ij})}) + \sum_{j=1}^{M} \int_0^t K_{ij}(t,x)\Phi_{ij}(t, y_j(x), \ldots, y_j^{(\lambda_{ij})}(x))\, dx = f_i(t),$$
$$y_j^{(k)}(t_0) = y_{j0}^k,\ k = 0, 1, \ldots, s-1, i, j = 1, 2, \ldots, M, \tag{1.1}$$

The approach used in this article reduces the CPU time and the computer memory comparing with other existing methods (see examples).
The outline of this sequel is as follows: In Section 2, function approximation is stated. Section 3 is devoted to numerical examples for the precision of the proposed technique. Finally, the conclusion is presented in Section 4.

2 Function approximation

Utilizing Bezier curves, this technique is to approximate the solutions $x(t)$ where $x(t)$ is given in Eq. (2.1). Define the Bezier polynomials of degree n that approximate over the interval

$t \in [t_0, t_f]$ as follows:

$$x \approx P^n x = \sum_{i=0}^{n} c_i B_{i,n}\left(\frac{t - t_0}{h}\right) = C^T B(t), \tag{2.1}$$

where $h = t_f - t_0$, $t_0 = 0$, $t_f = 1$, $C^T = [c_0, c_1, \ldots, c_n]^T$, and

$$B^T(t) = [B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)]^T, \tag{2.2}$$
$$B_{i,n}(\frac{t - t_0}{h}) = \binom{n}{i} \frac{1}{h^n}(t_f - t)^{n-i}(t - t_0)^i,$$

is the Bernstein polynomial with degree n for $t \in [t_0, t_f]$, and c_r is the control point.

3 Numerical application

In this section, some numerical examples are presented to illustrate the proposed method.

Example 3.1. The following NVIDEs is considered (see [6]):

$$u'''(t) + u'(t) + \int_0^t u''^2(x) + v''^2(x)\, dx = t,$$
$$v'''(t) - \int_0^t u''(x)v(x)dx = sint + \frac{1}{2}sin^2(t),$$
$$u(0) = 0, u'(0) = 1, u''(0) = 0, v(0) = 1, v'(0) = 0,$$
$$v''(0) = -1,$$
$$u_{exact} = sin(t), v_{exact} = cos(t),$$

Using the described technique with $n = 3$, one may have the following $u_{approx}(t)$, $v_{approx}(t)$, and Figs 1, 2.

$$u_{approx}(t) = 1.093276186t - 0.2858952349t^2 + 0.03409003344t^3$$
$$v_{approx}(t) = 1. + 0.01472832853t - 0.5638267963t^2 + 0.08940077360t^3$$

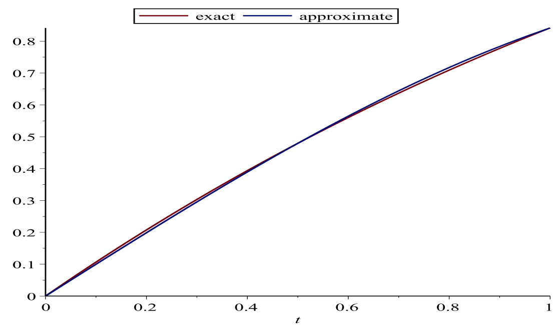

Figure 1. The graphs of approximated and exact solution u for Example 3.1

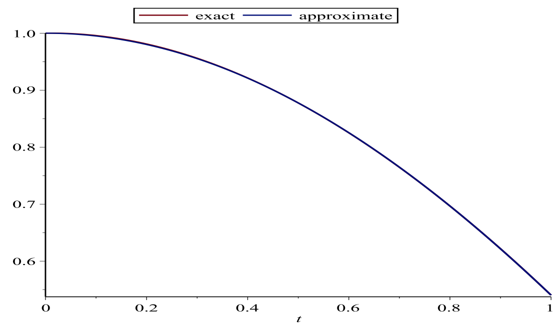

Figure 2. The graphs of approximated and exact solution v for Example 3.1

Example 3.2. The following NVIDEs is considered (see [6]):

$$
\begin{aligned}
& u'(t) + \frac{1}{2}v'^2(t) - \int_0^t (t-x)v(x) + v(x)u(x)dx = 1, \\
& v'(t) - \int_0^t (t-x)u(x) - v^2(x) + u^2(x)dx = 2t, \\
& u(0) = 0, v(0) = 1, \\
& u_{exact}(t) = sinh(t), v_{exact} = cosh(t),
\end{aligned}
$$

Using the described technique with $n = 3$, one may have the following $u_{approx}(t)$, $v_{approx}(t)$, and Figs 3, 4 and Table 1.

$$
\begin{aligned}
u_{approx}(t) &= 2 \times 10^{-9} t(5 \times 10^8 - 3.219375 \times 10^6 t + 9.0819972 \times 10^7 t^2) \\
v_{approx}(t) &= 1 + 0.476231022t^2 + 0.066849613t^3.
\end{aligned}
$$

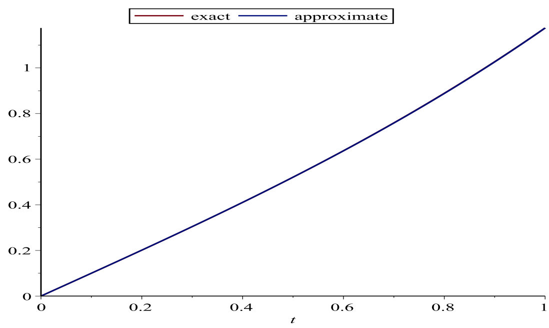

Figure 3. The graphs of approximated and exact solution u for Example 3.2

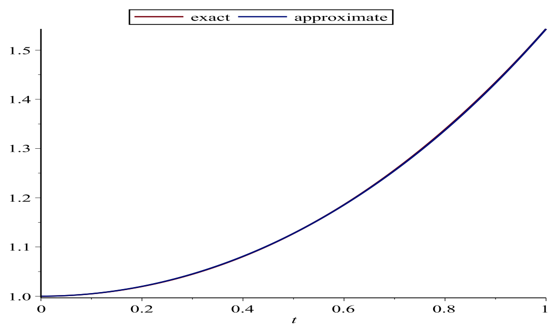

Figure 4. The graphs of approximated and exact solution v for Example 3.2

Table 1. The error of approximation solution of the this method for Example 3.2

t	error for approximate $u(t)$	error for approximate $v(t)$
0.1	$0.4949755600 \times 10^{-4}$	0.0001750083650
0.2	$0.1404329480 \times 10^{-3}$	0.0004827179200
0.3	0.0001955023600	0.0006727828600
0.4	0.0001575693800	0.0005970332600
0.5	0.0	0.0002120078000
0.6	0.0002626959000	0.0004174664000
0.7	0.0005638115000	0.001113612000
0.8	0.0007728691000	0.001579909900
0.9	0.0006834057000	0.001394111000
1.0	0.0	0.0

Table 2. The error errors between CAS wavelet method [2] and Legendre polynomial method [3] for Example 3.3

t	error in [2]	error in [3]
0.1	3.37×10^{-3}	4.8×10^{-4}
0.3446	4.72×10^{-3}	5.7×10^{-5}
0.7075	5.87×10^{-3}	3.4×10^{-4}
0.9178	3.42×10^{-2}	2.13×10^{-6}
1.0	6.20×10^{-2}	5.8×10^{-5}

Example 3.3. The following NVIDEs is considered (see [3]):

$$\begin{aligned}
&3(t-1)u(t) + t^2u'(t) = f(t) + \int_0^t (t-x)u(x)dx,\\
&f(t) = 3(t-1)(t-t^2) + t^2(1-2t) - \frac{1}{4}t^4 + \frac{1}{3}(t+1)t^3 - \frac{1}{2}t^3,\\
&u_{exact}(t) = t - t^2,
\end{aligned}$$

Using the described technique with $n = 3$, one may have $u_{approx}(t) = t - t^2$ with error zero. Comparison of absolute errors between CAS wavelet method [2] and Legendre polynomial method [3] is shown in Table 2.

Example 3.4. The following NVIDEs is considered (see [3]):

$$\begin{aligned}
&(t-1)u'(t) = 3(t-1)t^2 - \frac{1}{3}t + \frac{1}{3}t\cos(t^3) + \int_0^t tx^2 \sin(u(x))dx,\\
&f(t) = 3(t-1)(t-t^2) + t^2(1-2t) - \frac{1}{4}t^4 + \frac{1}{3}(t+1)t^3 - \frac{1}{4}t^3,\\
&u_{exact}(t) = t^3,
\end{aligned}$$

Using the described technique with $n = 3$, one may have $u_{approx}(t) = t^3$ with error zero. Comparison of absolute errors between Block-Pulse functions method (BPFM) [1] and Legendre polynomial method (LPM) [3] is shown in Table 3.

4 Conclusions

In this study, BCM is used to solve a class of NVIDEs. The achieved results by the BCM are in good agreement with the given exact solutions. The study shows that the method is effective and is a simple technique to solve NVIDEs.

Table 3. The error errors between BPFM [1] and LPM [3] for Example 3.4

t	error in [1]	error in [3]
0.1	4.654×10^{-4}	3.585×10^{-6}
0.3537	8.098×10^{-5}	1.726×10^{-7}
0.6101	6.675×10^{-5}	6.052×10^{-7}
0.9500	3.581×10^{-5}	2.138×10^{-7}

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