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NEW WAY FOR SOLVING A NONLINEAR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATION

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Abstract In this sequel, the Haar wavelet is applied to acquire approximate solutions of nonlinear multi-order fractional differential equations (M-FDEs). The fractional derivative is described in the Caputo sense. Numerical example is stated to find out the efficiency and accuracy of the proposed technique. The results reveal that the method is accurate and easy to use.

1 Introduction

Many phenomena in engineering physics, chemistry, and other sciences can be described very successfully by models that utilize mathematical tools of fractional calculus, i.e. the theory of derivatives and integrals of non-integer order. For example, they have been successfully used in modeling frequency dependent damping behavior of many viscoelastic materials. There are numerous research which demonstrate the applications of fractional derivatives in the areas of electrochemical processes [3]. The organization of this study is classified as follows: In Section 2, Basic Preliminaries is stated. Properties of the Haar basis is introduced in 3. An numerical example is solved in Section 4. Finally, Section 5 will give a conclusion briefly.

2 Basic Preliminaries

Definition 2.1. Let $x : [a,b] \to \mathcal{R}$ be a function, $\alpha > 0$ a real number, and $n = \alpha$, where α denotes the smallest integer greater than or equal to α (see [2]). The left (left RLFI) and right (right RLFI) Riemann-Liouville fractional integrals are defined by

$$
{}_aI_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\,d\tau, \quad \text{(left RLFI)},
$$
$$
{}_tI_b^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)\,d\tau, \quad \text{(right RLFI)},
$$

The left (left RLFD) and right (right RLFD) Riemann-Liouville fractional derivatives are given according to

$$
{}_aD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)\,d\tau, \quad \text{(left RLFD)},
$$
$$
{}_tD_b^{\alpha}x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1}x(\tau)\,d\tau, \quad \text{(right RLFD)}, \tag{2.1}
$$

Moreover, the left (left CFD) and right (right CFD) Caputo fractional derivatives are defined by means of

$$
\begin{aligned}
{}_a^C D_t^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \quad \text{(left CFD)}, \\
{}_t^C D_b^\alpha x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \quad \text{(right CFD)},
\end{aligned} \tag{2.2}
$$

The relation between the right RLFD and the right CFD is as follows:

$$
{}_t^C D_b^\alpha x(t) = {}_t D_b^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \tag{2.3}
$$

Further, it holds

$$
{}_0^C D_t^\alpha c = 0, \tag{2.4}
$$

where c is a constant, and

$$
{}_0^C D_t^\alpha t^n = \begin{cases} 0, \text{ for } n \in \mathbb{N}_0, \text{ and } n < \lceil \alpha \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, \text{ for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil \end{cases} \tag{2.5}
$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We recall that for $\alpha \in N$ the Caputo differential operator coincides with the usual differential operator of integer order.
In this paper, the following fractional differential equation was studied

$$
\begin{aligned}
& D^\alpha x(t) = a(t)x(t) + \sum_{r=1}^{l} b_r(t) D^{\alpha_r} x(q_r t), \; m-1 < \alpha \leq m, \; t \in [0, b], \\
& x^{(i)}(t) = \mu_i, i = 0, 1, \ldots, m-1.
\end{aligned}
$$

Here, $0 < q_r < 1$, $0 \leq \alpha_r < \alpha \leq m$, $r = 1, 2, \ldots, l$; x is an unknown function; $a(t)$ and $b_r(t)$, $r = 1, 2, \ldots, l$, are the known functions defined in $[0, b]$.

3 Properties of the Haar basis

The RH functions $RH(r,t)$, $r = 1, 2, \ldots$, are composed of three values $+1$, -1, 0 and can be defined on the interval $[0, 1)$ by [1] as

$$
RH(r,t) = \begin{cases} 1, & J_1 \leq t \leq J_{\frac{1}{2}} \\ -1, & J_{\frac{1}{2}} \leq t \leq J_0 \\ 0, & otherwise \end{cases}
$$

where

$$
\begin{aligned}
& J_u = \frac{j-u}{2^i}, \; u = 0, \frac{1}{2}, 1, \\
& r = 2^i + j - 1, \; i = 0, 1, 2, 3, \ldots, \; j = 1, 2, 3, \ldots, 2^i.
\end{aligned} \tag{3.1}
$$

$RH(0,t)$ is defined for $i = j = 0$ and is given by

$$
RH(0,t) = 1, \; 0 \leq t \leq 1. \tag{3.2}
$$

A set of the there RH functions is exhibited in Figs. 1,2, and 3, where $r = 3, 4, 5$. A set of there RH functions is shown in Figs. 1,2, and 3, where, $r = 3, 4, 5$. The following orthogonality property is given by

$$
\int_0^1 RH(r,t) RH(v,t)\, dt = \begin{cases} 2^{-i}, & r = v, \\ 0, & r \neq v, \end{cases}
$$

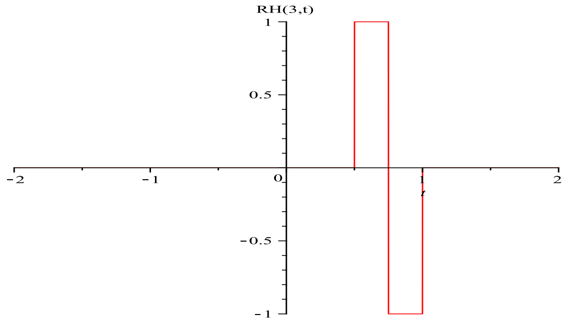

Figure 1. The graph of RH function

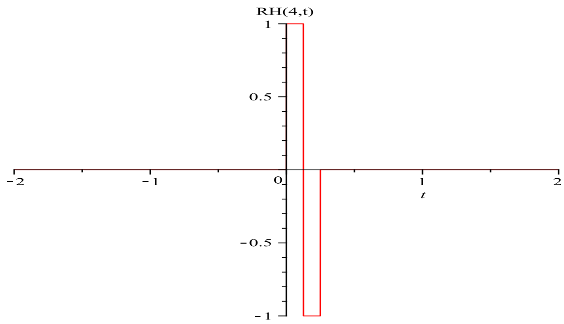

Figure 2. The graph of RH function

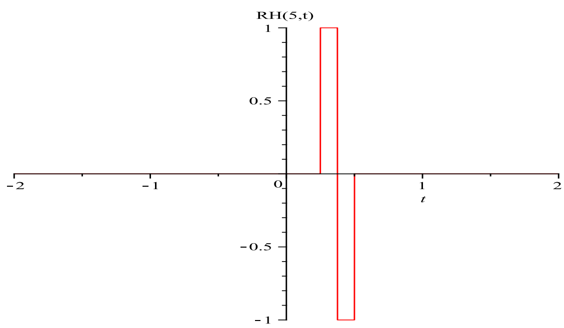

Figure 3. The graph of RH function

4 Numerical example

In this section, a numerical example is stated to solve the fractional neutral pantograph differential equation.

Example 4.1. Consider the fractional neutral pantograph differential equation (see [2])

$$D^{\gamma}x(t) = \frac{3}{4}x(t) + x(\frac{1}{2}t) + D^{\gamma_1}x(\frac{1}{2}t) + \frac{1}{2}D^{\gamma}x(\frac{1}{2}t) - t^2 - t + 1,\ 0 < \gamma_1 < \gamma \leq 2,$$
$$x(0) = x'(0) = 0$$

the graphs of approximated and exact solution $x(t)$ are plotted in Fig. 4.

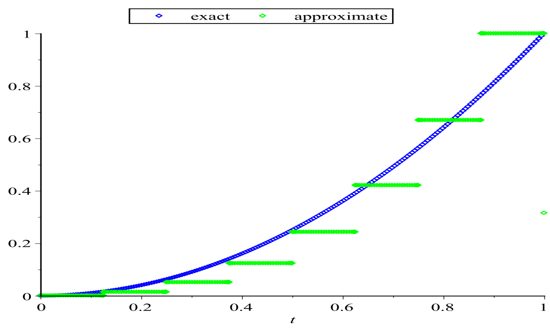

Figure 4. The graphs of approximated and exact solution $x(t)$ for Example 4.1

5 Conclusions

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason one may need a reliable and efficient technique for the solution of fractional differential equations. This paper deals with the approximate solution of a class of multi-order fractional differential equations. The fractional derivatives are described in the Caputo sense. Our main aim is to evaluation of fractional derivative utilizing Haar wavelet collocation method and implementing it to solve the nonlinear multi-order fractional differential equations. Illustrative example is included to demonstrate the validity and applicability of the technique.

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