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FEW RESULTS ON q-SAKAGUCHI TYPE FUNCTIONS

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Abstract In this paper we study classes of functions defined by using the concept of q-derivative and Sakaguchi functions. In particular we derive coefficient inequalities, distortion inequalities, coefficient estimates etc.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \;\; and \;\; |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function that are univalent in $\mathcal{U}$.
In[3], Jackson introduced and studied the concept of the q-derivative operator $\partial_q f(z)$ as follows:

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{1.2}$$

Equivalently (1.2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad z \neq 0,$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Note that as $q \to 1, \; [n]_q \to n$.
By combining the concept of q-derivative with Sakaguchi type function we will define the following:

Definition 1.1. For arbitrary fixed numbers q, α and t, $0 \leq q < 1, \; 0 \leq \alpha < 1, \; -1 \leq t < 1$, let $\mathcal{S}_q(\alpha, t)$ denote the family of functions $f \in \mathcal{A}$ which satisfies

$$\Re\left\{\frac{(1-t)z\partial_q f(z)}{f(z) - f(tz)}\right\} > \alpha, \qquad \text{for all } \; z \in \mathcal{U}. \tag{1.3}$$

For special cases for the parameters q, α and t the class $\mathcal{S}_q(\alpha, t)$ yield several known subclasses of $\mathcal{A}$, namely $\mathcal{S}_1(\alpha, t) = \mathcal{S}(\alpha, t)$ the class introduced and studied by Owa et al.[4], $\mathcal{S}_1(0, -1)$ = $\mathcal{S}(0, -1)$ the class introduced and studied by Sakaguchi [6] and $\mathcal{S}_1(0, t) = \mathcal{S}(t)$ the class introduced Rnning [5].
We denote by $\mathcal{T}_q(\alpha, t)$ the subclass of $\mathcal{A}$ consisting of all functions f such that:

$$z\partial_q f(z) \in \mathcal{S}_q(\alpha, t). \tag{1.4}$$

We need the following lemma to prove our main results.

Lemma 1.2. *[2] Let* $P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (z \in \mathcal{U})$, *with the condition* $\Re\{p(z)\} > 0$, *then*

$$|p_n| \leq 2, \quad (n \geq 1).$$

2 Main results

Theorem 2.1. *If the function $f \in \mathcal{A}$ and satisfies*

$$\sum_{n=2}^{\infty} \{|[n]_q - u_n| + (1-\alpha)|u_n|\}\, |a_n| \leq 1-\alpha, \quad u_n = 1+t+t^2+...+t^{n-1}, \tag{2.1}$$

then $f \in \mathcal{S}_q(\alpha, t)$.

Proof. Equivalently we show that

$$\left| \frac{(1-t)z\partial_q f(z)}{f(z)-f(tz)} - 1 \right| < 1-\alpha.$$

Consider

$$\frac{(1-t)z\partial_q f(z)}{f(z)-f(tz)} - 1 = \frac{\sum_{n=2}^{\infty}([n]_q - u_n)a_n z^n}{z+\sum_{n=2}^{\infty} a_n u_n z^n} = \frac{\sum_{n=2}^{\infty}([n]_q - u_n)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n u_n z^{n-1}},$$

which implies that

$$\left| \frac{(1-t)z\partial_q f(z)}{f(z)-f(tz)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty}|[n]_q - u_n||a_n|}{1-\sum_{n=2}^{\infty} |a_n||u_n|}.$$

Therefore if f satisfies (2.1), then we have

$$\left| \frac{(1-t)z\partial_q f(z)}{f(z)-f(tz)} - 1 \right| < 1-\alpha.$$

This completes the proof of Theorem 2.1. □

As $q \to 1$ we get the following result introduced by Owa S. et al. [4].

Corollary 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{|n - u_n| + (1-\alpha)|u_n|\}\, |a_n| \leq 1-\alpha, \quad u_n = 1+t+t^2+...+t^{n-1}, \tag{2.2}$$

for $0 \leq \alpha < 1$, then $f \in \mathcal{S}(\alpha, t)$.

Theorem 2.3. *If the function $f \in \mathcal{A}$ and defined by the form (1.1) and satisfies*

$$\sum_{n=2}^{\infty} [n]_q \{|[n]_q - u_n| + (1-\alpha)|u_n|\}\, |a_n| \leq 1-\alpha, \quad u_n = 1+t+t^2+...+t^{n-1}, \tag{2.3}$$

for $0 \leq \alpha < 1$, then $f \in \mathcal{T}_q(\alpha, t)$.

Proof. Since $f \in \mathcal{T}_q(\alpha, t)$ if and only if $z\partial_q f(z) \in \mathcal{S}_q(\alpha, t)$, the result follows. □

As $q \to 1$ we get the following result proved by Owa S. et al. [4].

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n\, \{|n - u_n| + (1-\alpha)|u_n|\}\, |a_n| \leq 1-\alpha, \quad u_n = 1+t+t^2+...+t^{n-1}, \tag{2.4}$$

then $f \in \mathcal{T}(\alpha, t)$.

Now we discuss the coefficient inequalities for function f in $\mathcal{S}_q(\alpha, t)$ and $\mathcal{T}_q(\alpha, t)$.

Theorem 2.5. *If* $f \in \mathcal{S}_q(\alpha, t)$, *then*

$$|a_n| \leq \prod_{j=1}^{n-1} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}, \tag{2.5}$$

Proof. We define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{(1-t)z\partial_q f(z)}{f(z)-f(tz)} - \alpha\right) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

where $p(z)$ is Carathéodory function and $f(z) \in \mathcal{S}_q(\alpha, t)$.

Since

$$(1-t)z\partial_q f(z) = (f(z) - f(tz))(\alpha + (1-\alpha)p(z)),$$

we have

$$\sum_{n=2}^{\infty}([n]_q - u_n)a_n z^n = \left(z + \sum_{n=2}^{\infty} a_n u_n z^n\right)\left(1 + (1-\alpha)\sum_{n=1}^{\infty} p_n z^n\right)$$

where $u_n = 1 + t + t^2 + ... + t^{n-1}$.

Equating coefficints of z^n on both sides we have

$$a_n = \frac{(1-\alpha)}{([n]_q - u_n)} \sum_{j=1}^{n-1} u_{n-j} a_{n-j} p_j, \quad a_1 = 1.$$

By Lemma1.2, we get

$$|a_n| \leq \frac{2(1-\alpha)}{|[n]_q - u_n|} \sum_{j=1}^{n-1} |u_j| a_j|, \quad a_1 = 1. \tag{2.6}$$

Now we prove that

$$\frac{(1-\alpha)}{|[n]_q - u_n|} \sum_{j=1}^{n-1} |u_j| a_j| \leq \prod_{j=1}^{n-1} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}. \tag{2.7}$$

We proof (2.7) by the induction method.
For n=2, from (2.6), we have

$$|a_2| \leq \frac{2(1-\alpha)}{|[2]_q - u_2|}.$$

(2.5) yields

$$|a_2| \leq \frac{2(1-\alpha)|u_1| + |[1]_q - u_1|}{|[2]_q - u_2|} \leq \frac{2(1-\alpha)}{|[2]_q - u_2|}.$$

For n=3, from (2.6), we get

$$|a_3| \leq \frac{2(1-\alpha)}{|[3]_q - u_3|}(1 + |u_2||a_2|)$$

$$\leq \frac{2(1-\alpha)}{|[3]_q - u_3|}\left(1 + |u_2|\frac{2(1-\alpha)}{|[2]_q - u_2|}\right).$$

Also from (2.5), we derive

$$|a_3| \leq \left(\frac{2(1-\alpha)}{|[2]_q - u_2|}\right)\left(\frac{2(1-\alpha)|u_2| + |[2]_q - u_2|}{|[3]_q - u_3|}\right)$$

$$\leq \left(\frac{2(1-\alpha)}{|[3]_q - u_3|}\right)\left(\frac{2(1-\alpha)|u_2|}{|[2]_q - u_2|} + 1\right).$$

Let the hypothesis be true for $n = m$. From (2.6), we have

$$|a_m| \leq \frac{2(1-\alpha)}{|[m]_q - u_m|} \sum_{j=1}^{m-1} |u_j| a_j|, \quad a_1 = 1.$$

From (2.5), we have

$$|a_m| \leq \prod_{j=1}^{m-1} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}.$$

By the induction hypothesis , we have

$$\frac{2(1-\alpha)}{|[m]_q - u_m|} \sum_{j=1}^{m-1} |u_j| a_j| \leq \prod_{j=1}^{m-1} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}.$$

Multiplying both sides by

$$\frac{2(1-\alpha)|u_m| + |[m]_q - u_m|}{|[m+1]_q - u_{m+1}|},$$

we have

$$\prod_{j=1}^{m} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|} \geq \frac{2(1-\alpha)}{|[m]_q - u_m|} \frac{2(1-\alpha)|u_m| + |[m]_q - u_m|}{|[m+1]_q - u_{m+1}|} \sum_{j=1}^{m-1} |u_j| a_j|$$

$$= \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}|} \left\{ \frac{2(1-\alpha)|u_m|}{|[m]_q - u_m|} \sum_{j=1}^{m-1} |u_j| a_j| + \sum_{j=1}^{m-1} |u_j| a_j| \right\}$$

$$\geq \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}|} \left\{ |u_m||a_m| + \sum_{j=1}^{m-1} |u_j| a_j| \right\}$$

$$\geq \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}|} \sum_{j=1}^{m} |u_j| a_j|.$$

Hence

$$\frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}|} \sum_{j=1}^{m} |u_j| a_j| \leq \prod_{j=1}^{m} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}.$$

Which shows that the inequality (2.7) is true for $n = m + 1$, and the result is true. □

Theorem 2.6. *If $f \in \mathcal{T}_q(\alpha, t)$, then*

$$|a_n| \leq \frac{1}{[n]_q} \prod_{j=1}^{n-1} \frac{2(1-\alpha)|u_j| + |[j]_q - u_j|}{|[j+1]_q - u_{j+1}|}, \quad \textit{for } n \geq 2. \tag{2.8}$$

The proof follows by using Theorem 2.5 and (1.4).

Now we define $\mathcal{S}_{0,q}(\alpha, t)$ and $\mathcal{T}_{0,q}(\alpha, t)$ as follows: $\mathcal{S}_{0,q}(\alpha, t) = \{f(z) \in \mathcal{A} : f(z)$ satisfies (2.1)$\}$ and $\mathcal{T}_{0,q}(\alpha, t) = \{f(z) \in \mathcal{A} : f(z)$ satisfies (2.3)$\}$. For functions f in the classes $\mathcal{S}_{0,q}(\alpha, t)$ and $\mathcal{T}_{0,q}(\alpha, t)$ we derive the following results.

Theorem 2.7. *If $f \in \mathcal{S}_{0,q}(\alpha, t)$, then*

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - A_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j|z|^{j+1}, \tag{2.9}$$

where
$$A_j = \frac{1-\alpha-\sum_{n=2}^{\infty}\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n|}{j+1-\alpha|u_{j+1}|} \quad (j\geq 2). \tag{2.10}$$

Proof. From the inequality (2.9), we know that

$$\sum_{n=j+1}^{\infty}\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n| \leq 1-\alpha-\sum_{n=2}^{j}\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n|.$$

On the other hand
$$|[n]_q-u_n|+(1-\alpha)|u_n| \geq [n]_q-\alpha|u_n|,$$
and hence $[n]_q-\alpha|u_n|$ is monotonically increasing with respect to n. Thus we deduce

$$(j+1-\alpha|u_{j+1}|)\sum_{n=j+1}^{\infty}|a_n| \leq 1-\alpha-\sum_{n=2}^{j}\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n|.$$

which implies that
$$\sum_{n=j+1}^{\infty}|a_n| \leq A_j.$$

Therefore we have that
$$|f(z)| \leq |z|+\sum_{n=2}^{j}|a_n||z|^n+A_j|z|^{j+1}$$

and
$$|f(z)| \geq |z|-\sum_{n=2}^{j}|a_n||z|^n-A_j|z|^{j+1}.$$

This completes the proof of the theorem. □

Analogously we prove

Theorem 2.8. *If $f\in\mathcal{T}_{0,q}(\alpha,t)$ then*

$$|z|-\sum_{n=2}^{j}|a_n||z|^n-B_j|z|^{j+1} \leq |f(z)| \leq |z|+\sum_{n=2}^{j}|a_n||z|^n+B_j|z|^{j+1},$$

and

$$1-\sum_{n=2}^{j}[n]_q|a_n||z|^{n-1}-C_j|z|^{j-1} \leq |f'(z)| \leq 1+\sum_{n=2}^{j}[n]_q|a_n||z|^{n-1}+C_j|z|^{j-1}.$$

Where
$$B_j = \frac{1-\alpha-\sum_{n=2}^{j}[n]_q\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n|}{(j+1)\{j+1-\alpha|u_{j+1}|\}}, \quad (j\geq 2).$$

and
$$C_j = \frac{1-\alpha-\sum_{n=2}^{j}[n]_q\{|[n]_q-u_n|+(1-\alpha)|u_n|\}|a_n|}{j+1-\alpha|u_{j+1}|}, \quad (j\geq 2).$$

Remark. By the definitions of the classes $\mathcal{S}_{0,q}(\alpha,t)$, and $\mathcal{T}_{0,q}(\alpha,t)$, evidently we have
$\mathcal{S}_{0,q}(\alpha,t)\subset\mathcal{S}_{0,q}(\beta,t) \qquad (0\leq\beta\leq\alpha<1)$,
and
$\mathcal{T}_{0,q}(\alpha,t)\subset\mathcal{T}_{0,q}(\beta,t) \qquad (0\leq\beta\leq\alpha<1)$.
Now we derive a relation between $\mathcal{S}_{0,q}(\beta,t)$ and $\mathcal{T}_{0,q}(\alpha,t)$.

Theorem 2.9. *If* $f \in \mathcal{T}_{0,q}(\alpha, t)$, *then* $\in \mathcal{S}_{0,q}(\frac{1+\alpha}{2}, t)$.

Proof. Let $f(z) \in \mathcal{T}_{0,q}(\alpha, t)$.
Then if β satisfies

$$\frac{|[n]_q - u_n| + (1-\beta)|u_n|}{1-\beta} \leq [n]_q \frac{|[n]_q - u_n| + (1-\alpha)|u_n|}{1-\alpha} \tag{2.11}$$

for all $n \geq 2$, then we have that $f(z) \in \mathcal{S}_{0,q}(\beta, t)$. From 2.9, we have

$$\beta \leq 1 - \frac{(1-\alpha)|[n]_q - u_n|}{[n]_q|[n]_q - u_n| + (1-\alpha)([n]_q - 1)|u_n|}.$$

Furthermore, since for all $n \geq 2$

$$\frac{|[n]_q - u_n|}{[n]_q|[n]_q - u_n| + (1-\alpha)([n]_q - 1)|u_n|} \leq \frac{1}{n} \leq \frac{1}{2},$$

we obtain

$$f(z) \in \mathcal{S}_{0,q}(\frac{1+\alpha}{2}, t).$$

□

As $q \to 1$ in the above theorem we get the following result proved by Owa S. et al. [4].

Corollary 2.10. *If* $f \in \mathcal{T}_0(\alpha, t)$, *then* $\in \mathcal{S}_0(\frac{1+\alpha}{2}, t)$.

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