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THE WEIGHTED MEAN FOR STEPANOV ALMOST PERIODIC FUNCTIONS

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Abstract In this paper, we present some sufficient conditions which do guarantee the existence of a weighted mean for Stepanov almost periodic functions, which will then coincide with the classical Bohr mean. Moreover, we will show that under those conditions, the corresponding weighted Bohr transform exists.

1 Introduction And Preliminaries

Although the concept of Stepanov almost periodic functions was introduced more than 90 years ago, some of their properties which play an important role in discussing the solutions of differential equations were not established until recently. In particular, Does the mean value; under a weight μ; exist for all almost periodic functions?

Let $\mathbb{U}$ denotes the collection of all functions (weights) $\mu : \mathbb{R} \to (0, \infty)$ which are locally integrable over $\mathbb{R}$ such that $\mu(x) > 0$ for almost each $x \in \mathbb{R}$. From now on, if $\mu \in \mathbb{U}$ and for $r > 0$, we then set $Q_r := [-r, r]$; $Q_{r+a} := [-r + a, r + a]$ and

$$\mu(Q_r) := \int_{Q_r} \mu(t)\, dt.$$

As in the particular case when $\mu(x) = 1$ for each $x \in \mathbb{R}$, we are exclusively interested in those weights, $\mu(x) = 1$, for which $\mu(Q_r) \to \infty \quad \text{as} \quad r \to \infty$ Consequently, we define the set of weights $\mathbb{U}_\infty$ by

$$\mathbb{U}_\infty := \{\mu \in \mathbb{U} : \lim_{r\to+\infty} \mu(Q_r) = \infty\}.$$

Suppose $\mu \in \mathbb{U}_\infty$ and let $\mathbb{X}$ be a Banach space. If $f : \mathbb{R} \to \mathbb{X}$ is a bounded measurable function, we define its weighted mean , if the limit exists, by

$$\mathcal{M}(f; \mu) := \lim_{r\to+\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} f(t)\mu(t)\, dt.$$

Let $L^p(\mathbb{R}, \mathbb{X})$ denote the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $\|f\| \in L^p(\mathbb{R})$. Let $L^p_{loc}(\mathbb{R}, \mathbb{X})$ denote the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that the restriction of every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, \mathbb{X})$.

Definition 1.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is called (Bohr) almost periodic ($AP(\mathbb{X})$) if for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a numbers τ with the property that $\|f(t + \tau) - f(t)\| < \varepsilon$ for each $t \in \mathbb{R}$.

Definition 1.2. The Bochner transform $f^b(t, s), t \in \mathbb{R},\ s \in [0, 1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) := f(t + s)$.

Remark 1.3. A function $\varphi(t, s), t \in \mathbb{R}, s \in [0, 1]$, is the Bochner transform of a certain function $f, \varphi(t, s) = f^b(t, s)$, if and only if
$\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}, s \in [0, 1]$ and $\tau \in [s - 1, s]$.

Definition 1.4. Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{\frac{1}{p}}$$

Definition 1.5. A function $f \in BS^p(\mathbb{X})$ is said to be S^p-almost periodic (or Stepanov almost periodic) if $f^b \in AP\big(L^p((0,1),\mathbb{X})\big)$. That is, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}} \left(\int_0^1 \left\| f^b(t+\tau,s) - f^b(t,s) \right\|^p ds \right)^{1/p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| f(s+\tau) - f(s) \right\|^p ds \right)^{1/p} < \varepsilon.$$

The collection of such functions will be denoted by $S^p_{ap}(\mathbb{X})$.

Theorem 1.6. ([1]) Every S^p-almost periodic function can be represented by its Fourier series, given by

$$f(x) \sim \sum_n a(\lambda_n, f) e^{-i\lambda_n x}$$

Theorem 1.7. ([1]) The mean value of every S^p-almost periodic function exists and

a) $\mathcal{M}(f) = \lim\limits_{r\to\infty} \dfrac{1}{r} \displaystyle\int_0^r f(t)\, dt = \lim\limits_{r\to\infty} \dfrac{1}{r} \displaystyle\int_{-r}^0 f(t)\, dt$

b) $\mathcal{M}(f) = \lim\limits_{r\to\infty} \dfrac{1}{2r} \displaystyle\int_{a-r}^{a+r} f(t)\, dt$ uniformly in $a \in \mathbb{R}$.

Consequently, since for every S^p-almost periodic function f and for every real number λ, the fonction $f(t)e^{-i\lambda t}$ is still a S^p-almost periodic function, the following limit:

$$a(f,\lambda) := \mathcal{M}(f(t)e^{-i\lambda t}) = \lim_{r\to+\infty} \frac{1}{2r} \int_{Q_r} f(t) e^{-i\lambda t}\, dt$$

always exists and is called the Bohr transform of f.

It is well-known ([2], [4], [1]) that for every S^p-almost periodic function f, there exists at most a countable infinite set of the Fourier-Bohr exponents λ, for which $a(f,\lambda) \neq 0$. The set defined by

$$\sigma_b(f) = \{\lambda \in \mathbb{R} : a(f,\lambda) \neq 0\}$$

is called the Bohr spactrum of f.

Consider an $\mathbb{X}$-valued trigonometric polynomial function $P_n : \mathbb{R} \to \mathbb{X}$ of the form

$$P_n(t) = \sum_{k=1}^n a_k e^{i\lambda_k t}$$

where $\lambda_k \in \mathbb{R}$ and $a_k \in \mathbb{X}$ for $k = 1, ..., n$. We have the following approximation theorem

Theorem 1.8. ([1],[4]) Let f be a S^p-almost periodic function. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial

$$P_\varepsilon(t) = \sum_{k=1}^n a_k e^{i\lambda_k t}$$

where $\lambda_k \in \sigma_b(f)$ and $a_k \in \mathbb{X}$, such that

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| f(s) - P_\varepsilon(s) \right\|^p ds \right)^{1/p} < \varepsilon.$$

Remark 1.9. $S^p_{ap}(\mathbb{X})$ can be seen as the space obtained as the closure in $BS^p(\mathbb{X})$ of the space of all trigonometric polynomials.

2 Weighted Mean For Stepanov Almost Periodic Functions

Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the sets of weights $\mathbb{U}_\infty^q$ and $\mathbb{U}_\infty^0$ defined by

$$\mathbb{U}_\infty^q = \left\{\mu \in \mathbb{U}_\infty \cap L_{loc}^q(\mathbb{R}) : \limsup_{r\to\infty} \frac{\mu(Q_r; q)}{\mu(Q_r)} < \infty\right\}$$

where $\mu(Q_r; q) = \left(\int_{Q_r} (\mu(t))^q \, dt\right)^{\frac{1}{q}}$ and

$$\mathbb{U}_\infty^0 = \left\{\mu \in \mathbb{U}_\infty^q : \lim_{r\to\infty} \frac{\mu(Q_{r+\tau})}{\mu(Q_r + \tau)} < \infty \quad \text{for all } \tau \in \mathbb{R}\right\}$$

Next, we establish the following main result:

Theorem 2.1. Fix $\mu \in \mathbb{U}_\infty^q$. If $f : \mathbb{R} \mapsto \mathbb{X}$ is a S^p-almost periodic function such that

$$\lim_{r\to\infty} \left| \frac{1}{\mu(Q_r)} \int_{Q_r} e^{i\lambda t}\mu(t) \, dt\right| = 0 \tag{2.1}$$

for all $0 \neq \lambda \in \sigma_b(f)$. Then the weighted mean of f,

$$\mathcal{M}(f, \mu) = \lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} f(t)\mu(t) \, dt,$$

exists and $\mathcal{M}(f, \mu) = \mathcal{M}(f)$.

Proof. If f is a trigonometric polynomial, say, $f(t) = \sum_{k=0}^{n} a_k e^{i\lambda_k t}$ where $a_k \in \mathbb{X}$ and $\lambda_k \in \mathbb{R}$ for $k = 1, ..., n$, then $\sigma_b(f) = \{\lambda_k : k = 1, ..., n\}$. Moreover,

$$\begin{aligned}\frac{1}{\mu(Q_r)} \int_{Q_r} f(t)\mu(t) \, dt &= a_0 + \frac{1}{\mu(Q_r)} \int_{Q_r} \left[\sum_{k=1}^{n} a_k e^{i\lambda_k t}\right]\mu(t) \, dt \\ &= a_0 + \sum_{k=1}^{n} a_k \left[\frac{1}{\mu(Q_r)} \int_{Q_r} e^{i\lambda_k t}\mu(t) \, dt\right]\end{aligned}$$

and hence

$$\left\|\frac{1}{\mu(Q_r)} \int_{Q_r} f(t)\mu(t) \, dt - a_0\right\| \leq \sum_{k=1}^{n} \|a_k\| \left|\frac{1}{\mu(Q_r)} \int_{Q_r} e^{i\lambda_k t}\mu(t) \, dt\right|$$

which by Eq.(2.1) yields

$$\left\|\frac{1}{\mu(Q_r)} \int_{Q_r} f(t)\mu(t) \, dt - a_0\right\| \to 0 \quad \text{as} \quad r \to \infty$$

and therefore $\mathcal{M}(f, \mu) = a_0 = \mathcal{M}(f)$.

If in the finite sequence of λ_k there exist $\lambda_{n_k} = 0$ for $k = 1, ..., l$ with $a_m \in \mathbb{X} - 0$ for all $m \neq n_k (k = 1, ..., l)$, then it can be easly shown that

$$\mathcal{M}(f, \mu) = \sum_{k=1}^{l} a_{n_k} = \mathcal{M}(f).$$

Since $\mu \in \mathbb{U}_\infty^q$, there exist constants $c, r_1 > 0$ such that for all $r > r_1$

$$\frac{1}{\mu(Q_{r_1})} \left(\int_{Q_{r_1}} (\mu(t))^q \, dt\right)^{\frac{1}{q}} < c$$

Now, if f is an arbitrary S^p-almost periodic function, then for every $\varepsilon > 0$ there exists a trigonometric polynomial

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $\lambda_k \in \sigma_b(f)$ and $a_k \in \mathbb{X}$, such that

$$\|f - P_\varepsilon\|_{S^p} < \frac{\varepsilon}{3c}.$$

By using the well-known convergence criterion of Cauchy, Indeed, proceeding as in ([3]) it follows that there exists r_0 such that for all $r_2, r_3 > r_0$,

$$\left\|\frac{1}{\mu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)\mu(t)\,dt - \frac{1}{\mu(Q_{r_3})}\int_{Q_{r_3}} P_\varepsilon(t)\mu(t)\,dt\right\| = \big\|\mathcal{M}(P_\varepsilon,\mu) - \mathcal{M}(P_\varepsilon,\mu)\big\| = 0 < \frac{\varepsilon}{3}.$$

In view of the above and by using Hölder inequality, it follows that for all $r_2, r_3 > r_0$,

$$\begin{aligned}
\left\|\frac{1}{\mu(Q_{r_2})}\int_{Q_{r_2}} f(t)\mu(t)\,dt - \frac{1}{\mu(Q_{r_3})}\int_{Q_{r_3}} f(t)\mu(t)\,dt\right\| &\le \frac{1}{\mu(Q_{r_2})}\int_{Q_{r_2}} \big\|f(t) - P_\varepsilon(t)\big\|\mu(t)\,dt \\
&+ \left\|\frac{1}{\mu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)\mu(t)\,dt - \frac{1}{\mu(Q_{r_3})}\int_{Q_{r_3}} P_\varepsilon(t)\mu(t)\,dt\right\| \\
&+ \frac{1}{\mu(Q_{r_3})}\int_{Q_{r_3}} \big\|f(t) - P_\varepsilon(t)\big\|\mu(t)\,dt \\
&\le \frac{\mu(Q_{r_2};q)}{\mu(Q_{r_2})}\Big\|f - P_\varepsilon\Big\|_{S^p} \\
&+ \left\|\frac{1}{\mu(Q_{r_2})}\int_{Q_{r_2}} P_\varepsilon(t)\mu(t)\,dt - \frac{1}{\mu(Q_{r_3})}\int_{Q_{r_3}} P_\varepsilon(t)\mu(t)\,dt\right\| \\
&+ \frac{\mu(Q_{r_3};q)}{\mu(Q_{r_3})}\Big\|f - P_\varepsilon\Big\|_{S^p} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Now, for all $r > r_0$, one has

$$\left\|\frac{1}{\mu(Q_r)}\int_{Q_r} f(t)\mu(t)\,dt - \frac{1}{\mu(Q_r)}\int_{Q_r} P_\varepsilon(t)\mu(t)\,dt\right\| < \frac{\varepsilon}{3}$$

and hence $\mathcal{M}(f,\mu) = \mathcal{M}(P_\varepsilon,\mu) = \mathcal{M}(P_\varepsilon) = \mathcal{M}(f)$. This completes the proof. □

Example 2.2. Suppose that $\mu(t) = 1 + |t|$ for all $t \in \mathbb{R}$. It is easy to check that $\mu \in \mathbb{U}_\infty^q$ and that Eq.(2.1) holds for all (nonconstant) S^p-almost periodic functions $\varphi : \mathbb{R} \mapsto \mathbb{X}$. Using Theorem (2.1), it follows that the weighted mean $\mathcal{M}(\varphi,\mu)$ exists and

$$\lim_{r\to\infty} \frac{1}{r^2 + 2r}\int_{Q_r} \varphi(t)(1 + |t|)\,dt = \lim_{r\to\infty} \frac{1}{2r}\int_{Q_r} \varphi(t)\,dt.$$

Corollary 2.3. Fix $\mu \in \mathbb{U}_\infty^0$. If f is a S^p-almost periodic function such that Eq.(2.1) holds, then

$$\mathcal{M}(f_a,\mu_a) = \mathcal{M}(f,\mu) = \mathcal{M}(f)$$

uniformly in $a \in \mathbb{R}$, where

$$\mathcal{M}(f_a,\mu_a) = \lim_{r\to\infty} \frac{1}{\mu_a(Q_r)}\int_{Q_r} f_a(t)\mu_a(t)\,dt = \lim_{r\to\infty} \frac{1}{\mu(Q_r + a)}\int_{Q_r} f(t+a)\mu(t+a)\,dt.$$

Proof. The proof is similar to that one of corollary (2.5) in ([5]). However, for the sake of clarity, we reproduce it here. First of all, let us mention that the existence of $\mathcal{M}(f,\mu)$ is a straightforward consequence of Theorem (2.1). It is sufficient to suppose that $a>0$. Now since the space $S^p_{ap}(\mathbb{X})$ is translation invariant, it follows that $f_a : t \mapsto f(t+a)$ belongs to $S^p_{ap}(\mathbb{X})$, too. On the other hand, it is not difficult to see that the weight μ_a defined by $\mu_a(t) := \mu_a(t+a)$ for all $t\in\mathbb{R}$ belongs to $\mu\in\mathbb{U}^0_\infty$. Now

$$\left|\int_{Q_r} e^{i\lambda t}\mu_a(t)\,dt\right| = \left|\int_{Q_r+a} e^{i\lambda(t-a)}\mu(t)\,dt\right| = \left|\int_{Q_r+a} e^{i\lambda t}\mu(t)\,dt\right| \le \left|\int_{Q_{r+a}} e^{i\lambda t}\mu(t)\,dt\right|$$

and hence

$$\begin{aligned}
\lim_{r\to\infty}\left|\frac{1}{\mu_a(Q_r)}\int_{Q_r} e^{i\lambda t}\mu_a(t)\,dt\right| &= \lim_{r\to\infty}\left|\frac{\mu(Q_r)}{\mu_a(Q_r)}\times\frac{1}{\mu(Q_r)}\int_{Q_r} e^{i\lambda t}\mu_a(t)\,dt\right| \\
&\le \lim_{r\to\infty}\left|\frac{\mu(Q_r)}{\mu_a(Q_r)}\times\frac{1}{\mu(Q_r)}\int_{Q_{r+a}} e^{i\lambda t}\mu(t)\,dt\right| \\
&= \lim_{r\to\infty}\left|\frac{\mu(Q_r)}{\mu_a(Q_r)}\times\frac{\mu(Q_{r+a})}{\mu(Q_r)}\times\frac{1}{\mu(Q_{r+a})}\int_{Q_{r+a}} e^{i\lambda t}\mu(t)\,dt\right| \\
&= \lim_{r\to\infty}\left|\frac{\mu(Q_{r+a})}{\mu(Q_r+a)}\right|\times\lim_{r\to\infty}\left|\frac{1}{\mu(Q_{r+a})}\int_{Q_{r+a}} e^{i\lambda t}\mu(t)\,dt\right| \\
&= 0
\end{aligned}$$

for all $0\neq\lambda\in\sigma_b(f)$.

Now using Theorem (2.1) it follows that the weighted mean of f defined by

$$\mathcal{M}(f,\mu_a) = \lim_{r\to\infty}\frac{1}{\mu_a(Q_r)}\int_{Q_r} f(t)\mu_a(t)\,dt$$

exists for all $a\in\mathbb{R}$. Furthermore, $\mathcal{M}(f,\mu_a)=\mathcal{M}(f)$ for all $a\in\mathbb{R}$.

Similarly, using the fact $\sigma_b(f)=\sigma_b(f_a)$ for all $a\in\mathbb{R}$ (see Bohr [3]) and Theorem (2.1) it follows that the weighted mean of f_a relatively to μ_a exists, too. Moreover, $\mathcal{M}(f_a,\mu_a)=\mathcal{M}(f_a)$ uniformly in $a\in\mathbb{R}$. Again from Bohr [3], $\mathcal{M}(f_a)=\mathcal{M}(f)$ uniformly in $a\in\mathbb{R}$, which completes the proof. □

Let $f:\mathbb{R}\mapsto\mathbb{X}$ be a S^p-almost periodic function and $t\mapsto f_\omega(t) := f(t)e^{-i\omega t}$. Clearly, $f_\omega\in S^p_{ap}(\mathbb{X})$, and

$$\sigma_b(f_\omega)=\sigma_b(f)-\omega=\{\lambda-\omega:\lambda\in\sigma_b(f)\}$$

Definition 2.4. Fix $\mu\in\mathbb{U}^q_\infty$. If $f:\mathbb{R}\mapsto\mathbb{X}$ is a S^p-almost periodic function such that

$$\lim_{r\to\infty}\left|\frac{1}{\mu(Q_r)}\int_{Q_r} e^{i(\lambda-\omega)t}\mu(t)\,dt\right| = 0 \tag{2.2}$$

for all $\lambda\in\sigma_b(f)$ with $\omega\neq\lambda$, we then define its weighted Bohr transform by

$$\widehat{a}_\mu(f)(\omega) := \lim_{r\to\infty}\frac{1}{\mu(Q_r)}\int_{Q_r} f(t)e^{-i\omega t}\mu(t)\,dt.$$

By Theorem (2.1), one can see that

$$\widehat{a}_\mu(f)(\omega)=\mathcal{M}(f(\cdot)e^{-i\omega\cdot})=a(f,\omega).$$

that is, under Eq.(2.2),

$$\widehat{a}_\mu(f)(\omega)=\lim_{r\to\infty}\frac{1}{\mu(Q_r)}\int_{Q_r} f(t)e^{-i\omega t}\mu(t)\,dt = \lim_{r\to\infty}\frac{1}{2r}\int_{Q_r} f(t)e^{-i\omega t}\mu(t)\,dt = a(f,\omega).$$

Clearly, $\widehat{a}_\mu(f)(\omega)=a(f,\omega)$ is nonzero at most at countably many points and therefore,

$$\sigma^\mu_b(f)=\{\omega\in\mathbb{R}:\widehat{a}_\mu(f)(\omega)\neq 0\}$$

coincides with the Bohr spectrum $\sigma_b(f)$, that is

$$\sigma^\mu_b(f)=\sigma_b(f).$$

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