**Palestine Journal of Mathematics**

**Vol. 9(1)(2020)** , 245–253 

# Ultrametric hyperstability of a Cauchy-Jensen type functional equation

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Communicated by H. M. Srivastava

MSC 2010 Classifications: Primary 39B82; Secondary 39B52, 47H10.

Keywords and phrases: stability , hyperstability, ultrametric space, Cauchy-Jensen functional equation.

**Abstract** In this paper, we establish some hyperstability results concerning the following Cauchy - Jensen functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z)$$

in ultrametric Banach spaces.

## 1 Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [32] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

*Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$ for all $x \in G_1$.*

The first partial answer, in the case of Cauchys equation in Banach spaces, to Ulams question was given by Hyers [22]. Later, the result of Hyers was significantly generalized by Rassias [30] and Găvruţa [19]. Since then, the stability problems of several functional equations have been extensively investigated.
We say a functional equation is *hyperstable* if any function $f$ satisfying the equation approximately (in some sense) must be actually solutions to it. It seems that the first hyperstability result was published in [11] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [26]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [1]-[5], [8], [15]-[18], [21], [26], [29], [31].

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $N_{m_0}$ the set of integers $\geq m_0$, $\mathbb{R}_+ := [0, \infty)$ and we use the notation $X_0$ for the set $X \setminus \{0\}$.

Let us recall (see, for instance, [25]) some basic definitions and facts concerning non-Archimedean normed spaces.

**Definition 1.1.** By a *non-Archimedean* field we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(i) $|r| = 0$ if and only if $r = 0$,

(ii) $|rs| = |r||s|$,

(iii) $|r + s| \leq \max\{|r|, |s|\}$.

The pair $(\mathbb{K}, |.|)$ is called a *valued field*.

In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}_0$. In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are $p$-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, $p$-adic strings and superstrings.

**Definition 1.2.** Let $X$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_* : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $\|x\|_* = 0$ if and only if $x = 0$,

(ii) $\|rx\|_* = |r| \; \|x\|_* \;\; (r \in \mathbb{K}, x \in X)$,

(iii) The strong triangle inequality (ultrametric); namely

$$\|x + y\|_* \leq \max\left\{\|x\|_*, \|y\|_*\right\} \;\; x, y \in X.$$

Then $(X, \|\cdot\|_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space*.

**Definition 1.3.** Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$.

(i) A sequence$\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;

(ii) The sequence $\{x_n\}$ is said to be *convergent* if, there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer $N$ such that $\|x_n - x\|_* \leq \varepsilon$, for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $lim_{n\to\infty} x_n = x$;

(iii) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space* or an *ultrametric Banach space*.

Let $X$, $Y$ be normed spaces. A function $f : X \to Y$ is Cauchy-Jensen provided it satisfies the functional equation

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) = f(x) + 2f(z) \quad \text{for all } \; x, y, z \in X \tag{1.1}$$

and we can say that $f : X \to Y$ is Cauchy-Jensen on $X_0$ if it satisfies (1.1) for all $x, y, z \in X_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$.

Recently, interesting results concerning Cauchy-Jensen functional equation (1.1) have been obtained in [6] and [27].

In 2013, A. Bahyrycz and al. [7] used the fixed point theorem from [12, Theorem 1] to prove the stability results for a generalization of $p$-Wright affine equation in ultrametric spaces. Recently, corresponding results for more general functional equations (in classical spaces) have been proved in [9], [10], [33] and [34].

In this paper, using the fixed point method derived from [8], [15] and [14], we present some hyperstability results for the equation (1.1) in ultrametric Banach spaces. Before proceeding to the main results, we state Theorem 1.4 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(**H1**) $X$ is a nonempty set, $Y$ is an ultrametric Banach space over a non-Archimedean field, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\right\|_* \leq \max_{1\leq i\leq k}\left\{L_i(x)\left\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\right\|_*\right\}, \qquad \xi, \mu \in Y^X, \quad x \in X.$$

**(H3)** $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1\leq i\leq k}\left\{L_i(x)\delta\Big(f_i(x)\Big)\right\}, \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

Thanks to a result due to J. Brzdęk and K. Ciepliński [13, Remark 2], we state an analogue of the fixed point theorem [12, Theorem 1] in ultrametric spaces. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

**Theorem 1.4.** *Let hypotheses (H1)-(H3) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\|_* \leq \varepsilon(x), \qquad x \in X,$$

$$\lim_{n\to\infty} \Lambda^n\varepsilon(x) = 0, \qquad x \in X.$$

*Then there exists a unique fixed point $\psi \in Y^X$ of $\mathcal{T}$ with*

$$\|\varphi(x) - \psi(x)\|_* \leq \sup_{n\in\mathbb{N}_0} \Lambda^n\varepsilon(x), \qquad x \in X.$$

*Moreover*

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

## 2 Main results

In this section, using Theorem 1.4 as a basic tool to prove the hyperstability results of the Cauchy-Jensen functional equation (1.1) in ultrametric Banach spaces.

**Theorem 2.1.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space respectively, $c \geq 0$, $p, q, r \in \mathbb{R}$, $p + q + r < 0$ and let $f : X \to Y$ satisfies*

$$\left\|f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - 2f(z) - f(x)\right\|_* \leq c\,\|x\|^p\,\|y\|^q\,\|z\|^r, \tag{2.1}$$

*for all $x, y, z \in X_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$. Then $f$ is Cauchy-Jensen on $X_0$.*

**Proof.** Take $m \in \mathbb{N}$ such that

$$\alpha_m := m^{p+q+r} < 1 \text{ and } m \geq m_0.$$

Since $p + q + r < 0$, it is sufficient to consider only the case $p + q < 0$. Replace $x$ by $mx$, $y$ by $mx$ and $z$ by $x$ in (2.1). Thus

$$\Big\|f\big((m+1)x\big) - f(mx) - f(x)\Big\|_* \leq c\, m^{p+q}\|x\|^{p+q+r}, \quad x \in X_0. \tag{2.2}$$

Define operators $\mathcal{T}_m : Y^{X_0} \to Y^{X_0}$ and $\Lambda_m : \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ by

$$\mathcal{T}_m\xi(x) := \xi\Big((m+1)x\Big) - \xi(mx), \quad \xi \in Y^{X_0},\ x \in X_0, \tag{2.3}$$

$$\Lambda_m\delta(x) := \max\left\{\ \delta\Big((m+1)x\Big)\ ,\ \delta(mx)\right\}, \quad \delta \in \mathbb{R}_+^{X_0},\ x \in X_0 \tag{2.4}$$

and write

$$\varepsilon_m(x) := c\, m^{p+q}\ \|x\|^{p+q+r}, \quad x \in X_0. \tag{2.5}$$

It is easily seen that $\Lambda_m$ has the form described in (**H3**) with $k = 2$, $f_1(x) = (m + 1)x$, $f_2(x) = mx$ and $L_1(x) = 1$, $L_2(x) = 1$. Further, (2.2) can be written in the following way

$$\|\mathcal{T}_m f(x) - f(x)\|_* \leq \varepsilon_m(x), \quad x \in X_0.$$

Moreover, for every $\xi, \mu \in Y^{X_0}$, $x \in X_0$

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\Big\|_* &= \Big\|\xi\Big((m+1)x\Big) - \xi(mx) - \mu\Big((m+1)x\Big) + \mu(mx)\Big\|_* \\ &\le \max\left\{ \Big\|\xi\Big((m+1)x\Big) - \mu\Big((m+1)x\Big)\Big\|_* , \Big\|\xi(mx) - \mu(mx)\Big\|_* \right\}.\end{aligned}$$

So, (**H2**) is valid.

By using mathematical induction, we will show that for each $x \in X_0$ we have

$$\Lambda_m^n \varepsilon_m(x) = c\, m^{p+q} \|x\|^{p+q+r}\ \alpha_m^n \tag{2.6}$$

where $\alpha_m = m^{p+q+r}$. From (2.5), we obtain that (2.6) holds for $n = 0$. Next, we will assume that (2.6) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) = \Lambda_m\left(\Lambda_m^k \varepsilon_m(x)\right) &= \max\left\{ \Lambda_m^k\varepsilon_m\Big((m+1)x\Big) , \Lambda_m^k\varepsilon_m(mx)\right\} \\ &= c\, m^{p+q}\|x\|^{p+q+r}\ \alpha_m^k \max\left\{ \Big(m+1\Big)^{p+q+r} , m^{p+q+r}\right\} \\ &= c\, m^{p+q}\|x\|^{p+q+r}\ \alpha_m^{k+1}, \quad x \in X_0.\end{aligned}$$

This shows that (2.6) holds for $n = k+1$. Now we can conclude that the inequality (2.6) holds for all $n \in \mathbb{N}_0$. From (2.6), we obtain

$$\lim_{n\to\infty} \Lambda^n \varepsilon_m(x) = 0,$$

for all $x \in X_0$. Hence, according to Theorem 1.4, there exists a unique solution $J_m : X_0 \to Y$ of the equation

$$J_m(x) = J_m\Big((m+1)x\Big) - J_m(mx), \quad x \in X_0 \tag{2.7}$$

such that

$$\|f(x) - J_m(x)\|_* \le \sup_{n\in\mathbb{N}_0} \left\{ c\, m^{p+q}\|x\|^{p+q+r}\ \alpha_m^n \right\}, \quad x \in X_0. \tag{2.8}$$

Moreover,

$$J_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x)$$

for all $x \in X_0$. Now we show that

$$\left\|\mathcal{T}_m^n f\left(\frac{x+y}{2} + z\right) + \mathcal{T}_m^n f\left(\frac{x-y}{2} + z\right) - 2\mathcal{T}_m^n f(z) - \mathcal{T}_m^n f(x)\right\|_* \le c\,\alpha_m^n \|x\|^p\ \|y\|^q\ \|z\|^r, \tag{2.9}$$

for every $x, y, z \in X_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$. Since the case $n = 0$ is just (2.1), take $k \in \mathbb{N}$ and assume that (2.9) holds for $n = k$ and every $x, y, z \in X_0$ such that $\frac{x+y}{2} + z \neq 0$

and $\frac{x-y}{2}+z \neq 0$. Then

$$\left\|\mathcal{T}_m^{k+1} f\left(\frac{x+y}{2}+z\right)+\mathcal{T}_m^{k+1} f\left(\frac{x-y}{2}+z\right)-2\mathcal{T}_m^{k+1} f(z)-\mathcal{T}_m^{k+1} f(x)\right\|_* =$$

$$\left\|\mathcal{T}_m^{k} f\left((m+1)\left(\frac{x+y}{2}+z\right)\right)-\mathcal{T}_m^{k} f\left(m\left(\frac{x+y}{2}+z\right)\right)+\mathcal{T}_m^{k} f\left((m+1)\left(\frac{x-y}{2}+z\right)\right)\right.$$

$$\left.-\mathcal{T}_m^{k} f\left(m\left(\frac{x-y}{2}+z\right)\right)-2\mathcal{T}_m^{k} f((m+1)z)+2\mathcal{T}_m^{k} f(mz)-\mathcal{T}_m^{k} f((m+1)x)+\mathcal{T}_m^{k} f(mx)\right\|_* =$$

$$\leq \max\left\{\left\|\mathcal{T}_m^{k} f\left((m+1)\left(\frac{x+y}{2}+z\right)\right)+\mathcal{T}_m^{k} f\left((m+1)\left(\frac{x-y}{2}+z\right)\right)-2\mathcal{T}_m^{k} f((m+1)z)\right.\right.$$

$$\left.-\mathcal{T}_m^{k} f((m+1)x)\right\|_*,$$

$$\left.\left\|\mathcal{T}_m^{k} f\left(m\left(\frac{x+y}{2}+z\right)\right)+\mathcal{T}_m^{k} f\left(m\left(\frac{x-y}{2}+z\right)\right)-2\mathcal{T}_m^{k} f(mz)-\mathcal{T}_m^{k} f((mx)\right\|_*\right\}$$

$$\leq \max\left\{\, c\,\alpha_m^k\|x\|^p\;\|y\|^q\;\|z\|^r\;(m+1)^{p+q+r}\;,\; c\,\alpha_m^k\|x\|^p\;\|y\|^q\;\|z\|^r\;m^{p+q+r}\right\}$$

$$= c\,\alpha_m^k\|x\|^p\;\|y\|^q\;\|z\|^r \max\left\{\,(m+1)^{p+q+r}\;,\;m^{p+q+r}\right\}$$

$$\leq c\,\alpha_m^{k+1}\|x\|^p\;\|y\|^q\;\|z\|^r$$

for all $x, y, z \in X_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$. Thus, by induction we have shown that (2.9) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (2.9), we obtain that

$$J_m\left(\frac{x+y}{2}+z\right)+J_m\left(\frac{x-y}{2}+z\right)=2J_m(z)+J_m(x),$$

for all $x, y, z \in X_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$. In this way we obtain a sequence $\{J_m\}_{m\geq m_0}$ of Cauchy-Jensen functions on $X_0$ such that

$$\|f(x)-J_m(x)\|_* \leq \sup_{n\in\mathbb{N}_0}\left\{\,c\,m^{p+q}\|x\|^{p+q+r}\;\alpha_m^n\right\},\quad x\in X_0,$$

this implies that

$$\|f(x)-J_m(x)\|_* \leq c\,m^{p+q}\|x\|^{p+q+r},\quad x\in X_0,$$

It follows, with $m \to \infty$, that $f$ is Cauchy-Jensen on $X_0$. □

In a similar way we can prove the following theorem.

**Theorem 2.2.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space respectively, $c \geq 0$, $p, q, r \in \mathbb{R}$, $p+q+r > 0$ and let $f : X \to Y$ satisfy*

$$\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-2f(z)-f(x)\right\|_* \leq c\,\|x\|^p\;\|y\|^q\;\|z\|^r, \tag{2.10}$$

*for all $x, y, z \in X_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$. Then $f$ is Cauchy-Jensen on $X_0$.*

**Proof.** Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left(\frac{m+3}{2m}\right)^{p+q+r} < 1 \;\text{ and }\; m \geq m_0.$$

Since $p+q+r > 0$, one of $p$, $q+r$ must be positive; let $q+r > 0$ and replace $y$ and $z$ by $\frac{1}{m}x$ in (2.10). Thus

$$\left\|f\left(\left(\frac{m+3}{2m}\right)x\right)+f\left(\left(\frac{m+1}{2m}\right)x\right)-2f\left(\frac{x}{m}\right)-f(x)\right\|_* \leq c\,m^{-q-r}\|x\|^{p+q+r},\quad x\in X_0. \tag{2.11}$$

Write

$$\mathcal{T}_m\xi(x) := \xi\left(\left(\frac{m+3}{2m}\right)x\right) + \xi\left(\left(\frac{m+1}{2m}\right)x\right) - 2\xi\left(\frac{x}{m}\right) \quad \xi \in Y^{X_0},\ x \in X_0, \tag{2.12}$$

and

$$\varepsilon_m(x) := c\, m^{-q-r}\|x\|^{p+q+r}, \quad x \in X_0, \tag{2.13}$$

then (2.11) takes form

$$\|\mathcal{T}_m f(x) - f(x)\|_* \le \varepsilon_m(x), \quad x \in X_0.$$

Define

$$\Lambda_m\delta(x) := \max\left\{\delta\left(\left(\frac{m+1}{2m}\right)x\right),\ \delta\left(\left(\frac{m-1}{2m}\right)x\right),\ 2\delta\left(\frac{x}{m}\right)\right\}, \quad \delta \in \mathbb{R}_+^{X_0},\ x \in X_0. \tag{2.14}$$

Then it is easily seen that $\Lambda_m$ has the form described in (**H3**) with $k = 3$, $f_1(x) = \left(\frac{m+3}{2m}\right)x$, $f_2(x) = \left(\frac{m+1}{2m}\right)x$, $f_3(x) = \frac{x}{m}$, $L_1(x) = L_2(x) = 1$ and $L_3(x) = 2$.

Moreover, for every $\xi, \mu \in Y^{X_0}$, $x \in X_0$

$$\begin{aligned}
\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\Big\|_* &= \Bigg\|\xi\left(\left(\frac{m+3}{2m}\right)x\right) + \xi\left(\left(\frac{m+1}{2m}\right)x\right) - 2\xi\left(\frac{x}{m}\right) \\
&\quad -\mu\left(\left(\frac{m+3}{2m}\right)x\right) - \mu\left(\left(\frac{m+1}{2m}\right)x\right) + 2\mu\left(\frac{x}{m}\right)\Bigg\|_* \\
\le \max\Bigg\{ \Bigg\|\xi\left(\left(\frac{m+3}{2m}\right)x\right) - \mu\left(\left(\frac{m+3}{2m}\right)x\right)\Bigg\|_*&,\ \Bigg\|\xi\left(\left(\frac{m+1}{2m}\right)x\right) - \mu\left(\left(\frac{m+1}{2m}\right)x\right)\Bigg\|_*, \\
&\quad 2\left\|\xi\left(\frac{x}{m}\right) - \mu\left(\frac{x}{m}\right)\right\|_*\Bigg\}.
\end{aligned}$$

So, (**H2**) is valid.

By using mathematical induction, we will show that for each $x \in X_0$ we have

$$\Lambda_m^n\varepsilon_m(x) = c\, m^{-q-r}\|x\|^{p+q+r}\ \alpha_m^n \tag{2.15}$$

where $\alpha_m = \left(\frac{m+3}{2m}\right)^{p+q+r}$. From (2.13), we obtain that (2.15) holds for $n = 0$. Next, we will assume that (2.15) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x) &= \max\left\{\Lambda_m^k\varepsilon_m\left(\left(\frac{m+3}{2m}\right)x\right),\ \Lambda_m^k\varepsilon_m\left(\left(\frac{m+1}{2m}\right)x\right),\ 2\Lambda_m^k\varepsilon_m\left(\frac{x}{m}\right)\right\} \\
&= c\, m^{-q-r}\|x\|^{p+q+r}\ \alpha_m^k \max\left\{\left(\frac{m+3}{2m}\right)^{p+q+r},\ \left(\frac{m+1}{2m}\right)^{p+q+r},\ 2\left(\frac{1}{m}\right)^{p+q+r}\right\} \\
&= c\, m^{-q-r}\|x\|^{p+q+r}\ \alpha_m^{k+1}, \quad x \in X_0.
\end{aligned}$$

This shows that (2.15) holds for $n = k + 1$. Now we can conclude that the inequality (2.15) holds for all $n \in \mathbb{N}_0$. From (2.15), we obtain

$$\lim_{n\to\infty} \Lambda^n\varepsilon_m(x) = 0,$$

for all $x \in X_0$. Hence, according to Theorem 1.4, there exists a unique solution $J_m : X_0 \to Y$ of the equation

$$J_m(x) = J_m\left(\left(\frac{m+3}{2m}\right)x\right) + J_m\left(\left(\frac{m+1}{2m}\right)x\right) + 2J_m\left(\frac{x}{m}\right), \quad x \in X_0 \tag{2.16}$$

such that

$$\|f(x) - J_m(x)\|_* \le \sup_{n\in\mathbb{N}_0}\left\{c\, m^{-q-r}\|x\|^{p+q+r}\ \alpha_m^n\right\}, \quad x \in X_0. \tag{2.17}$$

Moreover,

$$J_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x)$$

for all $x \in X_0$. The rest of the proof is similar to the proof of Theorem 2.1. □

In the following theorem, we prove the hyperstability of the Cauchy-Jensen equation (1.1) on the set containing 0.

**Theorem 2.3.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space respectively, $c \geq 0$, $p, q, r > 0$, and let $f : X \to Y$ satisfy*

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x) \right\|_* \leq c\, \|x\|^p\, \|y\|^q\, \|z\|^r, \tag{2.18}$$

*for all $x, y, z \in X$. Then $f$ is Cauchy-Jensen on $X$.*

**Proof.** Replacing $y$ and $z$ by $x$ in (2.18), we get that

$$f(x) = \frac{1}{2} f(2x), \quad x \in X. \tag{2.19}$$

The function $f$ satisfies (2.18) and by using (2.18) and (2.19) we can prove by induction that for every $n \in \mathbb{N}_0$

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x) \right\|_* \leq c \left(\frac{1}{2^{p+q+r}}\right)^n \|x\|^p\, \|y\|^q\, \|z\|^r \tag{2.20}$$

for all $x, y, z \in X$.
Indeed, for $n = 0$ (2.20) is simply (2.18). So, take $k \in \mathbb{N}$ and assume that (2.20) holds for $n = k$. Then using (2.19) to (2.20) we have

$$\left\| \frac{1}{2} f(x+y+2z) + \frac{1}{2} f(x-y+2z) - f(2z) - \frac{1}{2} f(2x) \right\|_* \leq c \left(\frac{1}{2^{p+q+r}}\right)^k \|x\|^p \|y\|^q \|z\|^r,$$

for all $x, y, z \in X$ and

$$\frac{1}{|2|} \|f(x+y+2z) + f(x-y+2z) - 2f(2z) - f(2x)\|_* \leq c \left(\frac{1}{2^{p+q+r}}\right)^k \|x\|^p \|y\|^q \|z\|^r,$$

for all $x, y, z \in X$. Replacing $x$ by $\frac{x}{2}$, $y$ by $\frac{y}{2}$ and $z$ by $\frac{z}{2}$ in the last inequality, we obtain

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x) \right\|_* \leq c \left(\frac{1}{2^{p+q+r}}\right)^{k+1} \|x\|^p\, \|y\|^q\, \|z\|^r$$

for all $x, y, z \in X$, so (2.20) holds for every $n \in \mathbb{N}_0$. With $n \to \infty$ in the inequality (2.20), we obtain that $f$ is Cauchy-Jensen on $X$. □

The above theorems imply in particular the following corollary, which shows their simple application.

**Corollary 2.4.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space respectively, $G : X^3 \to Y$ and $G(x, y, z) \neq 0$ for some $x, y, z \in X$ and*

$$\left\| G(x, y, z) \right\|_* \leq c\, \|x\|^p\, \|y\|^q\, \|z\|^r, \quad x, y, z \in X \tag{2.21}$$

*where $c \geq 0$, $p, q, r \in \mathbb{R}$. Assume that the numbers $p, q, r$ satisfy one of the following conditions:*

*(i) $p + q + r < 0$, and (2.1) holds for all $x, y, z \in X_0$,*

*(ii) $p + q + r > 0$, and (2.10) holds for all $x, y, z \in X_0$,*

*(iii)* $p, q, r > 0$ *and (2.18) holds for all* $x, y, z \in X$.

*Then the functional equation*

$$g\left(\frac{x+y}{2}+z\right)+g\left(\frac{x-y}{2}+z\right)=2g(z)+g(x)+G(x,y,z), \quad x,y,z \in X \tag{2.22}$$

*has no solution in the class of functions* $g : X \to Y$.

**Proof.** Suppose that $g : X \to Y$ is a solution to (2.22). Then (1.1) holds, and consequently, according to the above theorems, $g$ is Cauchy-Jensen on $X_0$, which means that $G(x,y,z) = 0$ for some $x, y, z \in X$. This is a contradiction. □

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Received: December 28, 2017.
Accepted: July 21, 2018.