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# On a subclass of spiral-like functions at the boundary

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**Abstract**. In this paper, a boundary version of the Schwarz lemma for classes $\mathcal{M}(\alpha,\theta)$ is investigated. Let $f(z) = z + c_2z^2 + c_3z^3 + ...$ be a holomorphic function in the unit disc $U = \{z : |z| < 1\}$. We estimate a modulus of the angular derivative of $\frac{zf'(z)}{f(z)}$ function at the boundary point to $c$ with $\frac{cf'(c)}{f(c)} = \frac{1-e^{-2i\theta}}{2}$, $|\theta| < \frac{\pi}{2}$. The sharpness of these inequalities is also proved.

## 1 Introduction

Let $U = \{z : |z| < 1\}$ be the unit disc in $\mathbb{C}$. The classical Schwarz lemma in one complex variable is as follows:

Let $f : U \to U$ be a holomorphic mapping with $f(0) = 0$. Then the following statements hold:

$i-)$ $|f(z)| \le |z|$ for any $z \in U$,

$ii-)$ $|f'(0)| \le 1$,

$iii-)$ if there exists $z_1 \in U\setminus\{0\}$ such that $|f(z_1)| = |z_1|$, or $|f'(0)| = 1$, then there exists a complex number $\lambda$ of modulus 1 such that $f(z) = \lambda z$ and $f$ is an automorphism of $U$ [9]. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to ([3], [8])

Let $f(z) = z + c_2z^2 + c_3z^3 + ...$ be a holomorphic function in the unit disc $U$ and $\alpha$ be a real number. Then $f(z)$ is said to be $\alpha-$spiral-convex function if and only if it satisfies the estimate

$$\Re\left\{\left(e^{i\theta} - \alpha\cos\theta\right)\frac{zf'(z)}{f(z)} + \alpha\cos\theta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

in $U$, for some $|\theta| < \frac{\pi}{2}$. Denote this class by $\mathcal{M}(\alpha,\theta)$. In [22], a subclass of spiral-like function, which for different values of $\alpha$ and $\theta$ leads to the class of $\alpha-$convex function and spiral-convex functions introduced. In our study, the behavior of $\alpha-$spiral-like functions at the boundary of the unit disc will be examined. Also, the Schwarz lemma in class $\mathcal{M}(\alpha,\theta)$ will be expressed.

In order to show our main results, we need the following lemma due to Jack's Lemma [10].

**Lemma 1.1** (Jack's Lemma). *Let $f(z)$ be a non-constant and holomorphic function in the unit disc $E$ with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0$, then*

$$\frac{z_0f'(z_0)}{f(z_0)} = k,$$

*where $k \ge 1$ is a real number.*

Let $f(z) \in \mathcal{M}(\alpha,\theta)$ be a holomorphic function in the unit disc $U$ with $|\theta| < \frac{\pi}{2}$. Consider the function

$$\phi(z) = \frac{1 - h(z)}{h(z) + e^{-2i\theta}}, \tag{1.1}$$

where $h(z) = \frac{zf'(z)}{f(z)}$, $\theta$ is a real number and $|\theta| < \frac{\pi}{2}$. Clearly, the function $\phi(z)$ is a holomorphic in $U$ and $\phi(0) = 0$. Now, let us show that the function $|\phi(z)| < 1$ in $U$. We suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|} |\phi(z)| = |\phi(z_0)| = 1.$$

From the Jack's lemma, we have

$$\phi(z_0) = e^{i\gamma} \quad \text{and} \quad z_0\phi'(z_0) = k\phi(z_0).$$

Also, from (1.1), we get

$$e^{i\theta}h(z) = \frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta. \tag{1.2}$$

If we show that

$$\Re\left\{\left(e^{i\theta} - \alpha\cos\theta\right)\frac{zf'(z)}{f(z)} + \alpha\cos\theta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} = 0$$

at the point $z_0$, then it contradicts the fact that $f(z) \in \mathcal{M}(\alpha, \theta)$.

From (1.2), we take

$$\begin{aligned}
&\Re\left\{\left(e^{i\theta} - \alpha\cos\theta\right)\frac{zf'(z)}{f(z)} + \alpha\cos\theta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\
= \; &\Re\left\{e^{i\theta}\frac{zf'(z)}{f(z)} - \alpha\cos\theta\frac{zf'(z)}{f(z)} + \alpha\cos\theta + \alpha\cos\theta\frac{zf''(z)}{f'(z)}\right\} \\
= \; &\Re\left\{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta + \alpha\cos\theta\left(1 + \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right)\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(\frac{zf'(z)}{f(z)}\right)' &= e^{-i\theta}\left(\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta\right)' \\
\frac{f'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) &= e^{-i\theta}\frac{2\phi'(z)}{(1-\phi(z))^2}\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} &= \frac{f(z)}{f'(z)}\left(e^{-i\theta}\frac{2\phi'(z)}{(1-\phi(z))^2}\cos\theta\right) \\
&= \frac{z\left(e^{-i\theta}\frac{2\phi'(z)}{(1-\phi(z))^2}\cos\theta\right)}{e^{-i\theta}\left(\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta\right)} = \frac{\frac{2z\phi'(z)}{(1-\phi(z))^2}\cos\theta}{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\Re\left\{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta + \alpha\cos\theta\left(1 + \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right)\right\} \\
= \; &\Re\left\{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta + \alpha\cos\theta\left(\frac{\frac{2z\phi'(z)}{(1-\phi(z))^2}\cos\theta}{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta}\right)\right\} \\
= \; &\Re\left\{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta + \alpha\cos^2\theta\left(\frac{\frac{2z\phi'(z)}{(1-\phi(z))^2}}{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta}\right)\right\}.
\end{aligned}$$

In the last equality, if we take $m(z) = \frac{1+\phi(z)}{1-\phi(z)}$ and $m'(z) = \frac{2\phi'(z)}{(1-\phi(z))^2}$, so we have

$$\begin{aligned}
&\Re\left\{\frac{1+\phi(z)}{1-\phi(z)}\cos\theta + i\sin\theta + \alpha\cos\theta\left(1 + \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right)\right\} \\
= \; &\Re\left\{m(z)\cos\theta + i\sin\theta + \alpha\cos^2\theta\left(\frac{zm'(z)}{m(z)\cos\theta + i\sin\theta}\right)\right\}.
\end{aligned}$$

Therefore, by Jack's lemma we have

$$m(z_0)=\frac{1+\phi(z_0)}{1-\phi(z_0)}=\frac{1+e^{i\gamma}}{1-e^{i\gamma}}=i\frac{\sin\gamma}{1-\cos\gamma}$$

and

$$\begin{aligned} z_0m'(z_0) &= \frac{2z_0\phi'(z_0)}{(1-\phi(z_0))^2}=\frac{2k\phi(z_0)}{(1-\phi(z_0))^2}=2k\frac{e^{i\gamma}}{(1-e^{i\gamma})^2} \\ &= 2k\frac{e^{i\gamma}}{1-2e^{i\gamma}+e^{2i\gamma}}=2k\frac{1}{e^{-i\gamma}-2+e^{i\gamma}} \\ &= 2k\frac{1}{2\cos\gamma-2}=\frac{k}{\cos\gamma-1}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\Re\left\{m(z_0)\cos\theta+i\sin\theta+\alpha\cos^2\theta\left(\frac{z_0m'(z_0)}{m(z_0)\cos\theta+i\sin\theta}\right)\right\} \\ =\ &\Re\left\{i\frac{\sin\gamma}{1-\cos\gamma}\cos\theta+i\sin\theta+\alpha\cos^2\theta\left(\frac{\frac{k}{\cos\gamma-1}}{i\frac{\sin\gamma}{1-\cos\gamma}\cos\theta+i\sin\theta}\right)\right\} \\ =\ &0. \end{aligned}$$

This contradicts the condition $f(z)\in\mathcal{M}(\alpha,\theta)$. This means that there is no point $z_0\in U$. Therefore, $|\phi(z)|<1$ for $|z|<1$. By the Schwarz lemma, we obtain

$$|c_2|\leq 2\cos\theta. \tag{*}$$

The result is sharp and the extremal function is

$$f(z)=\frac{z}{(1+z)^{1+e^{-2i\theta}}},$$

where $\theta$ is real number with $|\theta|<\frac{\pi}{2}$.

It is an elementary consequence of Schwarz lemma that if $f$ extends continuously to some boundary point $c$ with $|c|=1$, and if $|f(c)|=1$ and $f'(c)$ exists, then $|f'(c)|\geq 1$, which is known as the Schwarz lemma on the boundary. The equality in $|f'(c)|\geq 1$ holds if and only if $f(z)=ze^{i\sigma}$, $\sigma$ real. This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature.

More than the last decade, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [2], [3], [6], [7], [16] [17], [18], and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(c)|=1$ condition of the boundary of the unit circle.

Osserman [16] has given the inequalities which are called the boundary Schwarz lemma. He has first showed that

$$|f'(c)|\geq\frac{2}{1+|f'(0)|} \tag{1.3}$$

and

$$|f'(c)|\geq 1 \tag{1.4}$$

under the assumption $f(0)=0$, where $f$ is a holomorphic function mapping the unit disc into itself and $c$ is a boundary point which $f$ extends continuosly, and $|f(c)|=1$. In addition, the equality in (1.4) holds if and only if $f(z)=ze^{i\sigma}$, where $\sigma$ is a real number. Also, Inequality (1.3) is sharp, with equality possible for each value of $|f'(0)|$.

The following set is called a Stolz angle at $c\in\partial U$

$$\triangle=\{z\in U:|\arg(1-\overline{c}z)|<\varsigma,\ |z-c|<r\},\ \left(0<\varsigma<\frac{\pi}{2},\ r<2\cos\varsigma\right).$$

Let $f$ be a function from $U$ to $\overline{\mathbb{C}}$. It is said that $f$ has an angular limit $d \in \overline{\mathbb{C}}$ at $c \in \partial U$ if

$$f(z) \to d \text{ as } z \to c, z \in \triangle$$

for each stolz angle $\triangle$ at $c$. The number $2\varsigma$ which is length of $\triangle$ can be any number less than $\pi$. It is said that $f$ has the unrestricted limit $d \in \overline{\mathbb{C}}$ at $b$ if

$$f(z) \to d \text{ as } z \to c, z \in U.$$

Clearly, in the last fact, if the function $f$ which is continuous in $U$ is defined at the point $b$ as $f(c) = d$ then $f$ becomes continuous in $U \cup \{c\}$.

Let $f$ be a functionfrom $U$ to $U$ and $\nu$ be its angular limit at the point $c$. If there exists a point $\zeta$ such that

$$\lim_{z \to c,\ z \in \triangle} \frac{f(z) - \nu}{z - c} = \zeta$$

for every Stolz anagle $\triangle$ at the point $c$ then $\zeta$ is called the angular derivative of the function $f$ at $c$ and iy is shown with $f'(c)$ [19].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [19]).

**Lemma 1.2** (Julia-Wolff lemma). *Let $f$ be a holomorphic function in $U$, $f(0) = 0$ and $f(U) \subset U$. If, in addition, the function $f$ has an angular limit $f(c)$ at $c \in \partial U$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

**Corollary 1.3.** *The holomorphic function f has a finite angular derivative $f'(c)$ if and only if $f'$ has the finite angular limit $f'(c)$ at $c \in \partial U$.*

D. M. Burns and S. G. Krantz [4] and D. Chelst [5] studied the uniqueness part of the Schwarz lemma. The similar types of results which are related with the subject of the paper can be found in ([13], [14] and [15]).

The inequality $(1.3)$ is a particular case of a result due to Vladimir N. Dubinin in [6], who strengthened the inequality $|f'(c)| \geq 1$ by involving zeros of the function $f$.

X. Tang, T. Liu and J. Lu [20] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk $E^n$ in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, M. Jeong [12] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [11] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc. Furthermore, X. Tang, T. Liu and W. Zhang [21] established a new type of the classical Schwarz lemma at the boundary for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$, and then give the boundary version of the rigidity theorem. S.L. Wail and W.M. Shah [23] established some results by using a boundary refinement of the classical Schwarz lemma.

## 2 Main Results

In this section, a boundary version of the Schwarz lemma for classes $\mathcal{M}(\alpha, \theta)$ is investigated. We establish a new type of the classical Schwarz lemma at the boundary for holomorphic in class $\mathcal{M}(\alpha, \theta)$. Let $f(z) = z + c_2 z^2 + c_3 z^3 + ...$ be a holomorphic function in the unit disc $U = \{z : |z| < 1\}$ with $|\theta| < \frac{\pi}{2}$. We estimate a modulus of the angular derivative of $\frac{zf'(z)}{f(z)}$ function at the boundary point to $c$ with $\frac{cf'(c)}{f(c)} = \frac{1 - e^{-2i\theta}}{2}$. The sharpness of these inequalities is also proved.

**Theorem 2.1.** *Let $f(z) \in \mathcal{M}(\alpha, \theta)$. Suppose that, for some $c \in \partial U$, $f$ has an angular limit $f(c)$ at $c$, $\frac{cf'(c)}{f(c)} = \frac{1 - e^{-2i\theta}}{2}$. Then we have the inequality*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)'_{z=c} \right| \geq \frac{\cos\theta}{2}. \tag{2.1}$$

*The inequality* $(2.1)$ *is sharp with extremal function*

$$f(z)=\frac{z}{(1+z)^{1+e^{-2i\theta}}},$$

*where* $\theta$ *is real number with* $|\theta|<\frac{\pi}{2}$.

*Proof.* Let us consider the following function

$$\phi(z)=\frac{1-h(z)}{h(z)+e^{-2i\theta}},$$

where $h(z)=\frac{zf'(z)}{f(z)}$, $\theta$ is a real number and $|\theta|<\frac{\pi}{2}$. Then $\phi(z)$ is holomorphic function in the unit disc $U$ and $\phi(0)=0$. By the Jack's lemma and since $f(z)\in\mathcal{M}(\alpha,\theta)$, we take $|\phi(z)|<1$ for $|z|<1$. Also, we have $|\phi(c)|=1$ for $c\in\partial U$. It is clear that

$$\phi'(z)=-\frac{h'(z)\left(1+e^{-2i\theta}\right)}{\left(h(z)+e^{-2i\theta}\right)^2}.$$

Therefore, for $h(c)=\frac{1-e^{-2i\theta}}{2}$, we take from $(1.4)$, we obtain

$$1\leq|\phi'(c)|=\left|\frac{h'(c)\left(1+e^{-2i\theta}\right)}{\left(h(c)+e^{-2i\theta}\right)^2}\right|=\frac{|h'(c)|\left|1+e^{-2i\theta}\right|}{\left|\frac{1-e^{-2i\theta}}{2}+e^{-2i\theta}\right|^2},$$

$$\begin{aligned}
1 &\leq 4\frac{|h'(c)|}{|1+e^{-2i\theta}|}=4\frac{|h'(c)|}{|1+\cos 2\theta-i\sin 2\theta|}\\
&= 4\frac{|h'(c)|}{\sqrt{(1+\cos 2\theta)^2+\sin^2 2\theta}}\\
&= \frac{4|h'(c)|}{\sqrt{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}}\\
&= \frac{4|h'(c)|}{\sqrt{2(1+\cos 2\theta)}}=\frac{4|h'(c)|}{\sqrt{2(1+2\cos^2\theta-1)}}\\
&= \frac{2|h'(c)|}{\cos\theta}
\end{aligned}$$

and

$$|h'(c)|\geq\frac{\cos\theta}{2}.$$

Now, we shall show that the inequality $(2.1)$ is sharp. Let

$$f(z)=\frac{z}{(1+z)^{1+e^{-2i\theta}}}. \tag{2.2}$$

Differentiating $(2.2)$ logarithmically, we obtain

$$\ln f(z)=\ln\frac{z}{(1+z)^{1+e^{-2i\theta}}}=\ln z-\left(1+e^{-2i\theta}\right)\ln(1+z),$$

$$\frac{f'(z)}{f(z)}=\frac{1}{z}-\frac{\left(1+e^{-2i\theta}\right)}{1+z}$$

and

$$h(z)=\frac{zf'(z)}{f(z)}=1-\frac{z\left(1+e^{-2i\theta}\right)}{1+z}=\frac{1-ze^{-2i\theta}}{1+z}.$$

Thus, we take

$$h'(z) = -\frac{1+e^{-2i\theta}}{(1+z)^2}$$

and

$$|h'(1)| = \frac{\left|1+e^{-2i\theta}\right|}{4} = \frac{\cos\theta}{2}.$$

□

**Theorem 2.2.** *Let $f(z) \in \mathcal{M}(\alpha,\theta)$. Suppose that, for some $c \in \partial U$, $f$ has an angular limit $f(c)$ at c, $\frac{cf'(c)}{f(c)} = \frac{1-e^{-2i\theta}}{2}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right| \geq \frac{2\cos^2\theta}{2\cos\theta + |c_2|}. \tag{2.3}$$

*The inequality* (2.3) *is sharp with extremal function*

$$f(z) = \frac{z}{(1+2az+z^2)^{\frac{1+e^{-2i\theta}}{2}}},$$

*where $\theta$ is real number with $|\theta| < \frac{\pi}{2}$ and $a = \frac{|c_2|}{2\cos\theta}$ is an arbitrary number from $[0,1]$ $(see, *)$.*

*Proof.* Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Therefore, we take from (1.3), we obtain

$$\frac{2}{1+|\phi'(0)|} \leq |\phi'(c)| = \frac{2\,|h'(c)|}{\cos\theta}.$$

Since

$$\begin{aligned}\phi(z) &= \frac{1-h(z)}{h(z)+e^{-2i\theta}} = \frac{1-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+e^{-2i\theta}}\\ &= \frac{1-\left(1+c_2z+\left(2c_3-c_2^2\right)z^2+...\right)}{\left(1+c_2z+\left(2c_3-c_2^2\right)z^2+...\right)+e^{-2i\theta}},\end{aligned}$$

$$\phi(z) = -\frac{c_2z+\left(2c_3-c_2^2\right)z^2+...}{1+e^{-2i\theta}+c_2z+\left(2c_3-c_2^2\right)z^2+...}$$

and

$$|\phi'(0)| = \frac{|c_2|}{|1+e^{-2i\theta}|} = \frac{|c_2|}{2\cos\theta},$$

we take

$$\frac{2}{1+\frac{|c_2|}{2\cos\theta}} \leq \frac{2\,|h'(c)|}{\cos\theta}$$

and

$$|h'(c)| \geq \frac{2\cos^2\theta}{2\cos\theta+|c_2|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{z}{(1+2az+z^2)^{\frac{1+e^{-2i\theta}}{2}}}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we obtain

$$\begin{aligned}\ln f(z) &= \ln\frac{z}{(1+2az+z^2)^{\frac{1+e^{-2i\theta}}{2}}}\\ &= \ln z - \ln\left(1+2az+z^2\right)^{\frac{1+e^{-2i\theta}}{2}}\\ &= \ln z - \frac{1+e^{-2i\theta}}{2}\ln\left(1+2az+z^2\right),\end{aligned}$$

$$\frac{f'(z)}{f(z)}=\frac{1}{z}-\frac{1+e^{-2i\theta}}{2}\frac{2a+2z}{1+2az+z^2}$$

and

$$h(z)=\frac{zf'(z)}{f(z)}=1-\frac{1+e^{-2i\theta}}{2}\frac{2az+2z^2}{1+2az+z^2}.$$

Therefore, we get

$$h'(z)=-\frac{1+e^{-2i\theta}}{2}\left[\frac{(2a+4z)\left(1+2az+z^2\right)-(2a+2z)\left(2az+2z^2\right)}{\left(1+2az+z^2\right)^2}\right],$$

$$h'(1)=-\frac{1+e^{-2i\theta}}{2}\left[\frac{(2a+4)\,(1+2a+1)-(2a+2)\,(2a+2)}{(1+2a+1)^2}\right]$$

and

$$|h'(1)|=\frac{\left|1+e^{-2i\theta}\right|}{2}\frac{1}{1+a}=\frac{2\cos\theta}{2\,(1+a)}=\frac{\cos\theta}{1+a}.$$

Thus, since $a=\frac{|c_2|}{2\cos\theta}$, we obtain

$$|h'(1)|=\frac{\cos\theta}{1+a}=\frac{\cos\theta}{1+\frac{|c_2|}{2\cos\theta}}=\frac{2\cos^2\theta}{2\cos\theta+|c_2|}.$$

□

An interesting special case of Theorem 2.2 is when $c_2=0$, in which case inequality (2.3) implies

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\cos\theta.$$

Clearly equality holds for

$$f(z)=\frac{z}{\left(1+z^2\right)^{\frac{1+e^{-2i\theta}}{2}}}.$$

The inequality (2.3) can be strengthened as below by taking into account $c_3$ which is third coefficient in the expansion of the function $f(z)$.

**Theorem 2.3.** *Let $f(z)\in\mathcal{M}(\alpha,\theta)$. Suppose that, for some $c\in\partial U$, $f$ has an angular limit $f(c)$ at $c$, $\frac{cf'(c)}{f(c)}=\frac{1-e^{-2i\theta}}{2}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\frac{\cos\theta}{2}\left(1+\frac{2\,(2\cos\theta-|c_2|)^2}{4\cos^2\theta-|c_2|^2+\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}\right).\qquad(2.5)$$

*The equality in (2.5) occurs for the function*

$$f(z)=\frac{z}{(1+z)^{1+e^{-2i\theta}}},$$

*where $\theta$ is real number with $|\theta|<\frac{\pi}{2}$.*

*Proof.* Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$g(z)=\frac{\phi(z)}{B(z)},$$

where $B(z)=z$. The function $g(z)$ is holomorphic in $U$. According to the maximum princible, we have $|g(z)|<1$ for each $z\in U$. In particular, we have

$$|g(0)|=\frac{|c_2|}{2\cos\theta}\leq 1\qquad(2.6)$$

and

$$|g'(0)| = \frac{\left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|}{4\cos^2\theta}.$$

Furthermore, it can be seen that

$$\frac{c\phi'(c)}{\phi(c)} = |\phi'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)}.$$

Consider the function

$$\Upsilon(z) = \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)}.$$

This function is holomorphic in $U$, $|\Upsilon(z)| \leq 1$ for $|z| < 1$, $\Upsilon(0) = 0$, and $|\Upsilon(c)| = 1$ for $c \in \partial U$. From $(1.3)$, we obtain

$$\begin{aligned} \frac{2}{1 + |\Upsilon'(0)|} &\leq |\Upsilon'(c)| = \frac{1 - |g(0)|^2}{\left|1 - \overline{g(0)}g(c)\right|^2}|g'(c)| \\ &\leq \frac{1 + |g(0)|}{1 - |g(0)|}\{|\phi'(c)| - |B'(c)|\}. \end{aligned}$$

Since

$$\Upsilon'(z) = \frac{1 - |g(0)|^2}{\left(1 - \overline{g(0)}g(z)\right)^2}g'(z)$$

and

$$|\Upsilon'(0)| = \frac{|g'(0)|}{1 - |g(0)|^2} = \frac{\frac{\left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|}{4\cos^2\theta}}{1 - \left(\frac{|c_2|}{2\cos\theta}\right)^2} = \frac{\left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|}{4\cos^2\theta - |c_2|^2},$$

we take

$$\begin{aligned} \frac{2}{1 + \frac{\left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|}{4\cos^2\theta - |c_2|^2}} &\leq \frac{1 + \frac{|c_2|}{2\cos\theta}}{1 - \frac{|c_2|}{2\cos\theta}}\left\{\frac{2|h'(c)|}{\cos\theta} - 1\right\}, \\ \frac{2\left(4\cos^2\theta - |c_2|^2\right)}{4\cos^2\theta - |c_2|^2 + \left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|} &\leq \frac{2\cos\theta + |c_2|}{2\cos\theta - |c_2|}\left\{\frac{2|h'(c)|}{\cos\theta} - 1\right\} \\ \frac{2\left(2\cos\theta - |c_2|\right)^2}{4\cos^2\theta - |c_2|^2 + \left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|} &\leq \frac{2|h'(c)|}{\cos\theta} - 1 \\ 1 + \frac{2\left(2\cos\theta - |c_2|\right)^2}{4\cos^2\theta - |c_2|^2 + \left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|} &\leq \frac{2|h'(c)|}{\cos\theta} \end{aligned}$$

and

$$|h'(c)| \geq \frac{\cos\theta}{2}\left(1 + \frac{2\left(2\cos\theta - |c_2|\right)^2}{4\cos^2\theta - |c_2|^2 + \left|2c_3 + \left(2c_3 - c_2^2\right)e^{-2i\theta}\right|}\right).$$

Now, we shall show that the inequality $(2.5)$ is sharp. Let

$$f(z) = \frac{z}{(1+z)^{1+e^{-2i\theta}}}.$$

Then

$$h(z) = \frac{zf'(z)}{f(z)} = \frac{1 - ze^{-2i\theta}}{1+z}$$

and

$$|h'(1)| = \frac{\left|1 + e^{-2i\theta}\right|}{4} = \frac{\cos\theta}{2}.$$

Since $|c_2| = 2\cos\theta$, $(2.5)$ is satisfied with equality. □

If $f(z)-z$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.5) can be further strengthened. This is given by the following Theorem.

**Theorem 2.4.** *Let $f(z)\in\mathcal{M}(\alpha,\theta)$, $f(z)-z$ has no zeros in $U$ except $z=0$ and $c_2>0$. Suppose that, for some $c\in\partial U$, $f$ has an angular limit $f(c)$ at $c$, $\frac{cf'(c)}{f(c)}=\frac{1-e^{-2i\theta}}{2}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\frac{\cos\theta}{2}\left(1-\frac{4c_2\cos\theta\ln^2\left(\frac{c_2}{2\cos\theta}\right)}{4c_2\cos\theta\ln\left(\frac{c_2}{2\cos\theta}\right)-\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}\right). \tag{2.7}$$

*The equality in (2.7) occurs for the function*

$$f(z)=\frac{z}{(1+z)^{1+e^{-2i\theta}}},$$

*where $\theta$ is real number with $|\theta|<\frac{\pi}{2}$.*

*Proof.* Let $c_2>0$ and let us consider the function $g(z)$ as in Theorem 2.3. Taking account of the equality (2.6), we denote by $\ln g(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln g(0)=\ln\left(\frac{c_2}{2\cos\theta}\right)=\ln\left|\frac{c_2}{2\cos\theta}\right|+i\arg\left(\frac{c_2}{2\cos\theta}\right)<0,\ \ c_2>0$$

and

$$\ln\left(\frac{c_2}{2\cos\theta}\right)<0.$$

Take the following auxiliary function

$$\Omega(z)=\frac{\ln g(z)-\ln g(0)}{\ln g(z)+\ln g(0)}.$$

It is obvious that $\Omega(z)$ is a holomorphic function in $U$, $\Omega(0)=0$, $|\Omega(z)|\leq1$ for $|z|<1$, and also $|\Omega(c)|=1$ for $c\in\partial U$. So, we can apply (1.3) to the function $\Omega(z)$. Since

$$\Omega'(z)=2\ln g(0)\frac{g'(z)}{g(z)\left(\ln g(z)+\ln g(0)\right)^2},$$

and

$$\Omega'(c)=2\ln g(0)\frac{g'(c)}{g(c)\left(\ln g(c)+\ln g(0)\right)^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1+|\Omega'(0)|} &\leq |\Omega'(c)|=\frac{2\left|\ln g(0)\right|}{\left|\ln g(c)+\ln g(0)\right|^2}\left|\frac{g'(c)}{g(c)}\right|,\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(c)}\left|\frac{\phi'(c)}{B(c)}-\frac{\phi(c)B'(c)}{B(c)^2}\right|\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(c)}\left|\frac{\phi(c)}{c^2}\right|\left|\frac{c\phi'(c)}{\phi(c)}-\frac{cB'(c)}{B(c)}\right|\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(c)}\left\{|\phi'(c)|-|B'(c)|\right\}\\
&\leq \frac{-2\ln g(0)}{\ln^2 g(0)}\left\{\frac{2\left|h'(c)\right|}{\cos\theta}-1\right\}\ (\mathit{Replacing}\ \arg^2 g(c)\ \mathit{by\ zero})\\
&= \frac{-2}{\ln\left(\frac{c_2}{2\cos\theta}\right)}\left\{\frac{2\left|h'(c)\right|}{\cos\theta}-1\right\}.
\end{aligned}$$

Since

$$\Omega'(0) = \frac{g'(0)}{2g(0)\ln g(0)}$$

and thus,

$$|\Omega'(0)| = \frac{\frac{\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}{4\cos^2\theta}}{-2\frac{c_2}{2\cos\theta}\ln\left(\frac{c_2}{2\cos\theta}\right)} = \frac{\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}{-4c_2\cos\theta\ln\left(\frac{c_2}{2\cos\theta}\right)},$$

we have

$$\frac{2}{1-\frac{\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}{4c_2\cos\theta\ln\left(\frac{c_2}{2\cos\theta}\right)}} \leq \frac{-2}{\ln\left(\frac{c_2}{2\cos\theta}\right)}\left\{\frac{2\left|h'(c)\right|}{\cos\theta}-1\right\},$$

$$1-\frac{4c_2\cos\theta\ln^2\left(\frac{c_2}{2\cos\theta}\right)}{4c_2\cos\theta\ln\left(\frac{c_2}{2\cos\theta}\right)-\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|} \leq \frac{2\left|h'(c)\right|}{\cos\theta},$$

and

$$\left|h'(c)\right| \geq \frac{\cos\theta}{2}\left(1-\frac{4c_2\cos\theta\ln^2\left(\frac{c_2}{2\cos\theta}\right)}{4c_2\cos\theta\ln\left(\frac{c_2}{2\cos\theta}\right)-\left|2c_3+\left(2c_3-c_2^2\right)e^{-2i\theta}\right|}\right).$$

Since $|c_2| = 2\cos\theta$, $(2.7)$ is satisfied with equality. □

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