Palestine Journal of Mathematics

Vol. 9(1)(2020) , 26–30

UPPER BOUNDS OF IRREGULARITY INDIES OF CATEGORICAL PRODUCT OF TWO CONNECTED GRAPHS

Ali Ahmad

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C05, 05C12; Secondary 05C15, 05C31, 05C69

Keywords and phrases: First Zagreb index, Second Zagreb index, irregularity measures.

Abstract Distinctive topological indices are found to be helpful in isomer segregation, structure-property relationship, structure-action relationship, pharmaceutical medication plan etc. in chemistry, biochemistry and nanotechnology.

In this paper, for a connected graph G with n vertices and m edges two novel graph irregularity measures defined as $IRM_1(G) = M_1(G) - \frac{4m^2}{n}$ and $IRM_2(G) = M_2(G) - \frac{4m^3}{n^2}$, where $M_1(G)$ and $M_2(G)$ are the first and second Zagreb indices of graph G, respectively are studied for categorical product of two connected graphs.

1 Introduction

A topological index is a numeric esteem connected with an atomic structure and are utilized to relates the physico-synthetic properties of the molecular graphs. In molecular graph, vertices relate to the molecules and edges compares to the bonds between them. Up to now, a few topological indices have been characterized and different scientific properties and substance applications are additionally researched. Topological indices are utilized for concentrate quantitative structure-activity relationships (QSAR) and quantitative structure property relationships (QSPR) for anticipating diverse properties of synthetic mixtures for demonstrating physicochemical, pharmacologic, toxicologic, organic and different properties of it and thus discovered distinctive applications in science, organic chemistry and nanotechnology. There are numerous strategies to evaluate the molecular structures, of which the topological index is the most well known since it can be acquired directly from molecular structures and quickly processed for huge number of particles [20, 21, 22, 23].

Suppose G is a simple connected graph with $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The notation $|V(G)|$ denotes the number of vertices in G, and $|E(G)|$ denotes the number of edges in G. The edge of the graph G connecting the vertices u and v is denoted by uv. Two vertices are adjacent if there is an edge between them. The set of all adjacent vertices to a vertex u is the neighborhood of u, denoted as $N_G(u)$. The cardinality of the set $N_G(u)$ known as the degree of u, denoted as $d_G(u)$. A vertex u in a connected graph G, the eccentricity $e_G(u)$ of u is the distance between u and a vertex farthest from u in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, which are denoted by $rad(G)$ and $diam(G)$, respectively. The notations used in this paper are mainly taken from book [6].

The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph G are defined in [11, 12, 15, 18] as:

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

$$M_2(G) = \sum_{uv \in E(G)} (d_G(u) \times d_G(v))$$

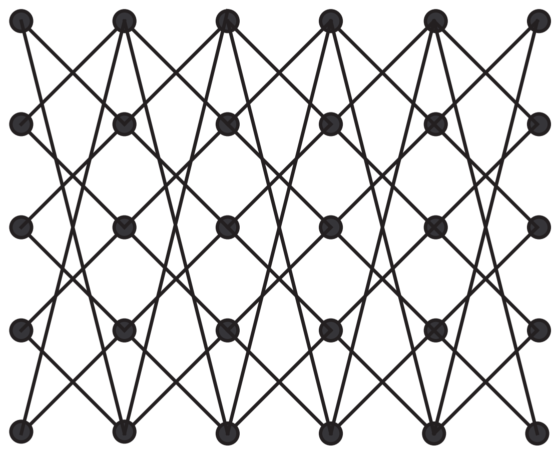

Figure 1. The categorical product of $G = C_5$ with $H = P_6$

For the basic pictorial of an associated graph G with $|V(G)|$ vertices and $|E(G)|$ edges two novel graph irregularity indices are presented, they are characterized as

$$IRM_1(G) = M_1(G) - \frac{4|E(G)|^2}{|V(G)|} \tag{1.1}$$

$$IRM_2(G) = M_2(G) - \frac{4|E(G)|^3}{|V(G)|^2} \tag{1.2}$$

The categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if g_1g_2 are adjacent in G and h_1h_2 are adjacent in H. For illustration, the categorical product of $G = C_5$ with $H = P_6$ is shown in Figure 1.

Graph operations assume a critical part in scientific science, since some artificially fascinating graphs can be gotten from some less difficult graphs by various graph operations. In [14], Khalifeh *et al.* determined some correct expressions for figuring first and second Zagreb indices of a few graph operations. Ashrafi *et al.* [3] determined unequivocal expressions for Zagreb coindices of various graph operations. Das *et al.* [7] inferred some upper limits for multiplicative Zagreb indices of diverse graph operations. Azari and Iranmanesh [5], introduced express formulae for registering the eccentric-distance sum of various graph operations. In [8] and [9], the authors acquired a few limits and correct formulae for the connective eccentric index and for F-index of various graph operations. There are a few different papers concerning topological indices of various graph operations. For more outcomes on topological indices of various graphs operations, intrigued perusers are alluded to the papers [1, 2, 4, 10, 14, 16, 17, 19].

2 Main Results and Discussions

Presently we express the particular properties of categorical product of graphs in type of the accompanying Proposition.

Proposition 2.1. *Let G and H be graphs of order $|V(G)|$ and $|V(H)|$ and size $|E(G)|$ and $|E(H)|$, respectively. Then we have:*

(i) $|V(G \times H)| = |V(G)| \times |V(H)|$

(ii) $|E(G \times H)| = 2(|E(G)| \times |E(H)|)$

(iii) $d_{G\times H}(u, v) = d_G(u) \times d_H(v)$

(iv) The categorical product is commutative and associative.

Theorem 2.2. *Let G and H be graphs of order $|V(G)|$ and $|V(H)|$ and size $|E(G)|$ and $|E(H)|$, respectively. Then Zagreb irregularity index IRM_1 for $G \times H$ is*

$$IRM_1 \leq 4|E(G)|.|E(H)| \left((|V(G)| - rad(G))(|V(H)| - rad(H)) - \frac{4|E(G)|.|E(H)|}{|V(G)|.|V(H)|} \right).$$

Proof. Let G and H be the graphs with vertex sets $\{g_1, g_2, \ldots, g_{|V(G)|}\}$ and $\{h_1, h_2, \ldots, h_{|V(H)|}\}$ respectively. The first Zagreb index of $G \times H$ is defined as:

$$\begin{aligned} M_1(G \times H) &= \sum_{(g,h) \in V(G \times H)} (d_G(g,h))^2 \\ &= \sum_{(g_i,h_j)(g_k,h_l) \in E(G \times H), i \neq k, j \neq l} (d_{G \times H}(g_i, h_j) + d_{G \times H}(g_k, h_l)) \end{aligned}$$

From Proposition 2.1, we get

$$M_1(G \times H) = \sum_{(g_i,h_j)(g_k,h_l) \in E(G \times H), i \neq k, j \neq l} (d_G(g_i) \times d_H(h_j) + d_G(g_k) \times d_H(h_l))$$

We know that $d_G(g) \leq |V(G)| - e_G(g)$ and $rad(G) \leq e_G(g)$.

This implies that $d_G(g) \leq |V(G)| - rad(G)$, similarly $d_H(h) \leq |V(H)| - rad(H)$. Putting these values in the above equation and Proposition 2.1, we get

$$\begin{aligned} M_1(G \times H) \leq\, & 2|E(G)|.|E(H)| \times \Big[(|V(G)| - rad(G)) (|V(H)| - rad(H)) \\ & + (|V(G)| - rad(G)) (|V(H)| - rad(H)) \Big] \\ =& 4|E(G)|.|E(H)| \times \Big[(|V(G)| - rad(G)) (|V(H)| - rad(H)) \Big] \end{aligned}$$

Now, the Zagreb irregularity index IRM_1 for $G \times H$ is defined as:

$$\begin{aligned} IRM_1(G \times H) =& M_1(G \times H) - \frac{4|E(G \times H)|^2}{|V(G \times H)|} \\ \leq& 4|E(G)|.|E(H)| \times \Big[(|V(G)| - rad(G)) (|V(H)| - rad(H)) \Big] \\ &- \frac{4 (2|E(G)|.|E(H)|)^2}{|V(G)||V(H)|} \\ =& 4|E(G)|.|E(H)| \Big[(|V(G)| - rad(G)) (|V(H)| - rad(H)) \\ &- \frac{4 (|E(G)|.|E(H)|)}{|V(G)||V(H)|} \Big]. \end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let G and H be graphs of order $|V(G)|$ and $|V(H)|$ and size $|E(G)|$ and $|E(H)|$, respectively. Then Zagreb irregularity index IRM_2 for $G \times H$ is*

$$IRM_2 \leq 2|E(G)|.|E(H)| \Big[(|V(G)| - rad(G))(|V(H)| - rad(H)) - \left(\frac{4|E(G)|.|E(H)|}{|V(G)|.|V(H)|} \right)^2 \Big].$$

Proof. Let G and H be the graphs with vertex sets $\{g_1, g_2, \ldots, g_{|V(G)|}\}$ and $\{h_1, h_2, \ldots, h_{|V(H)|}\}$ respectively. The second Zagreb index of $G \times H$ is defined as:

$$\begin{aligned} M_2(G \times H) &= \sum_{(g_i,h_j)(g_k,h_l) \in E(G \times H), i \neq k, j \neq l} (d_{G \times H}(g_i, h_j).d_{G \times H}(g_k, h_l)) \\ &= \sum_{(g_i,h_j)(g_k,h_l) \in E(G \times H), i \neq k, j \neq l} (d_G(g_i).d_H(h_j).d_G(g_k).d_H(h_l)) \\ &= \sum_{(g_i,g_k) \in E(G)} \sum_{(h_j,h_l) \in E(H)} (d_G(g_i).d_H(h_j).d_G(g_k).d_H(h_l)) \end{aligned}$$

We know that $d_G(g) \leq |V(G)| - e_G(g)$ and $rad(G) \leq e_G(g)$.

This implies that $d_G(g) \leq |V(G)| - rad(G)$, similarly $d_H(h) \leq |V(H)| - rad(H)$. Putting these values in the above equation and using Proposition 2.1, we get

$$M_2(G \times H) \leq 2|E(G)|.|E(H)|.\Big[(|V(G)| - rad(G)) . (|V(H)| - rad(H)) \\ . (|V(G)| - rad(G)) . (|V(H)| - rad(H)) \Big] \\ =2|E(G)|.|E(H)|. (|V(G)| - rad(G))^2 . (|V(H)| - rad(H))^2$$

Now, the Zagreb irregularity index IRM_2 for $G \times H$ is defined as:

$$\begin{aligned} IRM_2(G \times H) =& M_2(G \times H) - \frac{4|E(G \times H)|^3}{|V(G \times H)|^2} \\ \leq& 2|E(G)|.|E(H)|. (|V(G)| - rad(G)) . (|V(H)| - rad(H)) \\ &- \frac{4\left(2|E(G)|.|E(H)|\right)^3}{|V(G)|^2|V(H)|^2} \\ =& 2|E(G)|.|E(H)|\Big[(|V(G)| - rad(G)) . (|V(H)| - rad(H)) \\ &- \left(\frac{4|E(G)|.|E(H)|}{|V(G)||V(H)|}\right)^2 . \end{aligned}$$

This completes the proof. □

3 Closing Remarks

In this paper, we directed the investigation of irregularity measures of categorical product of graphs. We exhibited the upper limits of irregularity indices of categorical product of connected graphs in the form of its graph elements. Some different items and topological indices can be considered for future review.

Acknowledgement

This work was supported by Deanship of Research, Jazan University, KSA.

References

[1] A. Ahmad, On the degree based topological indices of Benzene ring embedded in the P-type-surface in 2D network, *Hacettepe Journal of Mathematics and Statistics*, **47 (1)**, 9–18 (2018).

[2] A. Ahmad, Computation of certain topological properties of honeycomb networks and Graphene, *Discrete Mathematics, Algorithms and Applications (DMAA)*, **9(5)**, 1750064 (16 pages) (2017).

[3] A. R. Ashrafi, T. Doslic, A. Hamzeh, The Zagreb coindices of graph operations, *Discret. Appl. Math.*, **158**, 1571–1578 (2010).

[4] M. Azari, Sharp lower bounds on the Narumi-Katayama index of graph operations, *Appl. Math. Comput.*, **239**, 409–421 (2014).

[5] M. Azari, A. Iranmanesh, Computing the eccentric-distance sum for graph operations, *Discret. Appl. Math.*, **161**, 2827–2840 (2013).

[6] G. Chartrand, P. Zhang, *A First Course in Graph Theory*, Dover Publications Inc.: Mineola, NY, (2012).

[7] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, I. N. Cangul, The multiplicative Zagreb indices of graph operations, *J. Ineql. Appl.*, doi:10.1186/1029-242X-2013-90 (2013).

[8] N. De, A. Pal, S. M. A. Nayeem, On some bounds and exact formulae for connective eccentric indices of graphs under some graph operations, *Int. J. Comb.*, doi:10.1155/2014/579257 (2014).

[9] N. De, S. M. A. Nayeem, A. Pal, F-index of some graph operations. Personal communication, (2015).

[10] N. De, S. M. A. Nayeem, A. Pal, Total eccentricity index of the generalized hierarchical product of graphs, *Int. J. Appl. Comput. Math.*, doi:10.1007/s40819-014-0016-4 (2014).

[11] I. Gutman, Degree-based topological indices, *Croat. Chem. Acta*, **86**, 351–361 (2013).

[12] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.*, **50**, 83–92 (2004).

[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.*, **56**, 1402–1407 (2008).

[14] M. H. Khalifeha, H. Yousefi-Azaria, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discret. Appl. Math.*, **157**, 804–811 (2009).

[15] S. Nikolić, G. Kovačević, A. Miličevi´c, N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta*, **76**, 113–124 (2003).

[16] M. K. Siddiqui, M. Imran, A. Ahmad, On Zagreb indices, Zagreb polynomials of some nanostar dendrimers, *Applied Mathematics and Computation*, **280**, 132-139 (2016).

[17] M. Tavakoli, F. Rahbarnia, A. R. Ashrafi, Some new results on irregularity of graphs, *J. Appl. Math. Inf.,* **32**, 675–685 (2014).

[18] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, (1992).

[19] M. Veylaki, M. J. Nikmehr, H. A. Tavallaee, The third and hyper-Zagreb coindices of some graph operations, *J. Appl. Math. Comput.,* doi:10.1007/s12190-015-0872-z (2015).

[20] Agrawal, V. K. A novel PI index and its applications to QSPR/QSAR studies, *J. Chem. Inf. Comput. Sci.*, **41**, 934–949 (2001).

[21] S. C. Basak, B. D. Gute and G. D. Grunwald, A hierarchical approach to the development of QSAR models using topological, geometrical and quantum chemical parameters. In Topological Indices and Related Descriptors in QSAR and QSPR (Eds: J. Devillers and A.T. Balaban), *Gordon and Breach Science Publishers*, **15**, 675–696 (1992).

[22] M. S. Miguel, P. G. Facundo, N. M. Ricardo, M. S. Salabert, J. G. M. Francisco, A. Rosa, P. Cercos, and M. G. Teresa, QSAR analysis of hypoglycemic agents using the topological indices, *J. Chem. Inf. Comput. Sci.*, **41**, 1345–1354 (2001).

[23] H. Yuan, and A. L. Parrill, QSAR development to describe HIV-1 integrase inhibition, *Journal of Molecular Structure (THEOCHEM)*, **529**, 273–282 (2000).

Author information

Ali Ahmad, College of Computer Science & Information Systems, Jazan University, Jazan, K. S. A..
E-mail: ahmadsms@gmail.com

Received: May 2, 2017.
Accepted: October 12, 2017