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SOME PROPERTIES OF MULTIGROUPS

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Abstract The concept of multigroups is an algebraic structure of multisets which generalize group theory. This paper establishes some results on multigroups, submultigroups and proposes the concept of multigroupoid. Various types of submultigroup are introduced and some related results are obtained. Some properties of commutative multigroups are presented and the notions of semimultigroups and multimonoids are introduced. The concepts of center and centralizer in multigroups setting are proposed and some homomorphic properties of commutative multigroups are explored.

1 Introduction

The concept of multigroups was proposed by Dresher and Ore [7] as algebraic systems that satisfied all the axioms of group except that the multiplication operation (which is the only operation) is multivalued. This notion of multigroup is neither in conformity with the idea of multisets nor in alignment with other non-classical groups studied in [4, 18, 20, 21]. Other attempts to generalize groups can be found in [3, 16, 19] but, none of these portrait multigroup with multiset in mind.

The invention of the notion of multisets (see [5, 6, 15, 22, 23, 24] for details) as a mathematical framework that allows repeated elements in a collection is a boost to the concept of multigroups which generalizes group theory. Nazmul et al. [17] proposed the concept of multigroups drawn from multisets (and parallel to other non-classical groups), obtained some results and defined the notion of abelian multigroups. For further studies on the concept of multigroups drawn from multisets, see [1, 2, 8, 9, 10, 11, 12, 13, 14] for details.

In this paper, we propose the notion of multigroupoid, present some results on multigroups, and introduce various types of submultigroup. The concept of semimultigroups is proposed and commutative multigroup is studied. The ideas of center and centralizer of multigroups are introduced in multigroup context. Finally, some homomorphic properties of commutative multigroups are considered.

2 Preliminaries

In this section, we present some basic definitions and existing results to be used in the sequel.

Definition 2.1. [22] Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A, that is, a counting function of A (where $C_A(x) = 0$, implies $x \notin Dom(A)$).

Suppose that $X = \{a, b, c\}$ is a set, then the multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a^2, b^2, c^3]$. The set X is called the ground or generic set of the class of all multisets containing objects from X.

A multiset A is said to be regular if $C_A(x) = C_A(y)\ \forall x, y \in X$. Various forms of multiset representations is found in [22]. The set of all multisets over X is denoted by $MS(X)$.

Definition 2.2. [23] Let $A, B \in MS(X)$. Then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3. [24] Let $A, B \in MS(X)$. Then the intersection, union and sum of A and B, denoted by $A \cap B$, $A \cup B$ and $A + B$ respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

(iii) $C_{A+B}(x) = C_A(x) + C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. [24] Let $A, B \in MS(X)$. A and B are comparable to each other if $A \subseteq B$ or $B \subseteq A$, and $A = B$ if $C_A(x) = C_B(x) \forall x \in X$.

Definition 2.5. [17] Let X be a group. A multiset G is called a multigroup of X if the count function of G, that is, $C_G : X \to \mathbb{N}$ satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,

(ii) $C_G(x^{-1}) \geq C_G(x) \forall x \in X$.

It follows immediately that,

$$C_G(x^{-1}) = C_G(x), \forall x \in X$$

since

$$C_G(x) = C_G((x^{-1})^{-1}) \geq C_G(x^{-1}).$$

The set of all multigroups of X is denoted by $MG(X)$.

Remark 2.6. [17] Let X be a group and G be a multiset over X. If

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y),$$

for all $x, y \in X$, then G is called a multigroup of X.

Remark 2.7. [8] Every multigroup is a multiset but the converse is not necessarily true.

Definition 2.8. [17] Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\},$$

where e is the identity element of X.

Proposition 2.9. [17] *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.10. [17] Let $A \in MG(X)$. Then A^{-1} is defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \forall x \in X.$$

Thus, we notice that $A \in MG(X) \Leftrightarrow A^{-1} \in MG(X)$.

Definition 2.11. Let $A, B \in MG(X)$. Then the product $A \circ B$ of A and B is defined to be a multiset over X as follows:

$$C_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz}(C_A(y) \wedge C_B(z)), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

This definition is adapted from [17].

Remark 2.12. [17] Let $A, B \in MG(X)$. Then $A \circ B$ is a multigroup of X if and only if $A \circ B = B \circ A$. Also, $A \circ A = A$.

Proposition 2.13. [17] *Let $A \in MG(X)$. Then the following statements hold.*

(i) $C_A(e) \geq C_A(x) \forall x \in X$*, where e is the identity of X.*

(ii) $C_A(x^n) \geq C_A(x) \forall x \in X, n \in \mathbb{N}$.

Definition 2.14. [10] Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$,

$$C_A(xyx^{-1}) \geq C_A(y).$$

Proposition 2.15. [10] *Let $A, B \in MG(X)$. Then the following statements are equivalent.*

(i) A is a normal submultigroup of B.

(ii) $C_A(xyx^{-1}) = C_A(y) \forall x, y \in X$.

(iii) $C_A(xy) = C_A(yx) \forall x, y \in X$.

Definition 2.16. [17] Let $A \in MG(X)$. Then A is said to be commutative if for all $x, y \in X$,

$$C_A(xy) = C_A(yx).$$

Definition 2.17. [9] Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies

(i) $C_{f(A)}(y_1 y_2) \geq C_{f(A)}(y_1) \wedge C_{f(A)}(y_2) \; \forall y_1, y_2 \in Y$,

(ii) $C_B(f(x_1 x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2)) \; \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \; \forall x \in X.$$

Theorem 2.18. [9] *Let $f : X \to Y$ be an isomorphism. Then $A \in MG(X) \Leftrightarrow f(A) \in MG(Y)$ and $B \in MG(Y) \Leftrightarrow f^{-1}(B) \in MG(X)$.*

3 Multigroupoids and multigroups

Definition 3.1. Let X be a group. A multiset G over X is called a multigroupoid of X if for all $x, y \in X$,

$$C_G(xy) \geq C_G(x) \wedge C_G(y),$$

where C_G denotes count function of G from X into a natural number $\mathbb{N}$.

Definition 3.2. Let X be a group. A multigroupoid G of X is called a multigroup of X if

$$C_G(x^{-1}) = C_G(x) \; \forall x \in X.$$

Definition 3.3. Let G be a multigroup of a group X. The count of an element in G is the number of occurrence of the element in G, and denoted by C_G. The order of G is the sum of the count of each of the elements in G, and is given by

$$|G| = \sum_{i=1}^{n} C_G(x_i)\ \forall x_i \in X.$$

Example 3.4. The following are examples of multigroups with the exception of (iv).

(i) Let $Z_3 = \{0, 1, 2\}$ be a group with respect to addition. Then

$$G = [0^4, 1^3, 2^3]$$

is a multigroup of Z_3. However, it follows that

$$G = [0^4, 1^3, 2^4]$$

is a multigroupoid of Z_3.

(ii) The zeros of $f(x) = x^4 - 2x^3 + 2x - 1$ form a multigroup of a group $X = \{1, -1\}$.

(iii) The zeros of $f(x) = x^8 - 2x^4 + 1$ form a multigroup of a group

$$X = \{1, -1, i, -i\}.$$

(iv) Let $X = \{1, a, a^2, a^3\}$ be a cyclic group by $< a >$ such that $a^4 = 1$. Then

$$A = [(1)^4, (a)^3, (a^2)^2, (a^3)^3]$$

is not a multigroup of X.

(v) Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set

$$S = \{1, 2, 3\}$$

such that

$$\rho_0 = (1),\ \rho_1 = (123),\ \rho_2 = (132),\ \rho_3 = (23),\ \rho_4 = (13),\ \rho_5 = (12).$$

Then $A = [\rho_0^7, \rho_1^4, \rho_2^4, \rho_3^3, \rho_4^3, \rho_5^3]$ is a multigroup of X.

Remark 3.5. From Example 3.4, it implies that,

(i) a group is a special case of multigroup with a unit count.

(ii) every multigroup is a multiset but the converse is not necessarily true.

Lemma 3.6. *Let A be a multigroup of a finite group X. Then $C_A(x^{-1}) = C_A(x^{n-1})\ \forall x \in X$, $n \in \mathbb{N}$.*

Proof. Let $x \in X$, $x \neq e$. Since X is finite, x has finite order, say $n > 1$. Thus $x^n = e$ and so $x^{-1} = x^{n-1}$. Consequently, A is finite since $A \in MG(X)$, then we have

$$\begin{aligned} C_A(x^{-1}) &= C_A(x^{-1}e) = C_A(x^{n-1}x^n) \\ &\geq C_A(x^{n-1}) \wedge C_A(x^n) \\ &= C_A(x^{n-1}) \end{aligned}$$

$\Rightarrow C_A(x^{-1}) \geq C_A(x^{n-1})$,
and

$$\begin{aligned} C_A(x^{n-1}) &= C_A(x^{n-1}x^n) = C_A((x^{n-2}x)x^n) \\ &\geq C_A(x^{n-2}x) \wedge C_A(x^n) \\ &\geq C_A(x^{n-2}) \wedge C_A(x) \\ &\geq C_A(x) \wedge ... \wedge C_A(x) \\ &= C_A(x) = C_A(x^{-1}). \end{aligned}$$

$\Rightarrow C_A(x^{n-1}) \geq C_A(x^{-1})$. Hence, $C_A(x^{-1}) = C_A(x^{n-1})\ \forall x \in X$. □

Theorem 3.7. *A multigroupoid A of a finite group X is a multigroup if $C_A(x^{-1}) = C_A(x^{n-1})$ $\forall x \in X$ and $n \in \mathbb{N}$.*

Proof. Since A is a multigroupoid of X, then $C_A(xy) \geq C_A(x) \wedge C_A(y)$ for all $x, y \in X$. Suppose $C_A(x^{-1}) = C_A(x^{n-1})$ $\forall x \in X$ and $n \in \mathbb{N}$. Using the notion of multigroupoid repeatedly, we get

$$\begin{aligned} C_A(x^{-1}) = C_A(x^{n-2}x) &\geq C_A(x^{n-2}) \wedge C_A(x) \\ &\geq C_A(x) \wedge C_A(x) \wedge ... \wedge C_A(x) \\ &= C_A(x), \end{aligned}$$

that is,

$$C_A(x^{-1}) \geq C_A(x)$$

and by Definition 2.5,

$$C_A(x) = C_A((x^{-1})^{-1}) \geq C_A(x^{-1}),$$

implies

$$C_A(x) \geq C_A(x^{-1}).$$

Hence, $C_A(x^{-1}) = C_A(x)$. Therefore, A is a multigroup of X by Definition 3.2. □

Definition 3.8. Let $\{A_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of multigroups of X. Then

$$C_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} C_{A_i}(x) \; \forall x \in X$$

and

$$C_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} C_{A_i}(x) \; \forall x \in X.$$

The family of multigroups $\{A_i\}_{i\in I}$ of X is said to have inf or sup assuming chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

In [17], it was proved that, if $A, B \in MG(X)$ and $\{A_i\}_{i\in I}$ is a family of multigroups of X, then $A \cap B \in MG(X)$, $\bigcap_{i\in I} A_i \in MG(X)$ and $A \cup B \notin MG(X)$ in general. Now, we show that $\bigcup_{i\in I} A_i \in MG(X)$ if $\{A_i\}_{i\in I}$ have either sup/inf assuming chain.

Theorem 3.9. *Let $\{A_i\}_{i\in I}$ be a family of multigroups of X. If $\{A_i\}_{i\in I}$ have sup/inf assuming chain, then $\bigcup_{i\in I} A_i \in MG(X)$.*

Proof. Let $A = \bigcup_{i\in I} A_i$, then $C_A(x) = \bigvee_{i\in I} C_{A_i}(x)$. We show that

$$C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y) \; \forall x, y \in X.$$

If either $C_A(x) = 0$ or $C_A(y) = 0$, then the inequality holds. Let $C_A(x) > 0$ and $C_A(y) > 0$, then we have

$$\bigvee_{i\in I} C_{A_i}(x) > 0, \bigvee_{i\in I} C_{A_i}(y) > 0.$$

By hypothesis, suppose $\exists\, i_0 \in I$ such that $C_{A_{i_0}}(x) = \bigvee_{i\in I} C_{A_i}(x)$, and also $\exists\, j_o \in I$ such that $C_{A_{j_0}}(x) = \bigvee_{i\in I} C_{A_i}(x)$. Since $\{A_i\}_{i\in I}$ have sup/inf assuming chain, it follows that either (i) $A_{i_0} \subseteq A_{j_0}$ or (ii) $A_{j_0} \subseteq A_{i_0}$.

(i) Suppose $A_{i_0} \subseteq A_{j_0}$, that is, $C_{A_{i_0}}(x) \leq C_{A_{j_0}}(x)$. Then

$$\begin{aligned} C_A(xy^{-1}) &= C_{A_{j_0}}(xy^{-1}) \\ &\geq C_{A_{j_0}}(x) \wedge C_{A_{j_0}}(y) \\ &\geq C_{A_{i_0}}(x) \wedge C_{A_{i_0}}(y) \\ &= \bigvee_{i\in I} C_{A_i}(x) \wedge \bigvee_{i\in I} C_{A_i}(y) \\ &= C_A(x) \wedge C_A(y). \end{aligned}$$

(ii) Suppose $A_{j_0} \subseteq A_{i_0}$, that is, $C_{A_{j_0}}(x) \leq C_{A_{i_0}}(x)$. Then

$$\begin{aligned} C_A(xy^{-1}) &= C_{A_{i_0}}(xy^{-1}) \\ &\geq C_{A_{i_0}}(x) \wedge C_{A_{i_0}}(y) \\ &\geq C_{A_{j_0}}(x) \wedge C_{A_{j_0}}(y) \\ &= \bigvee_{i \in I} C_{A_i}(x) \wedge \bigvee_{i \in I} C_{A_i}(y) \\ &= C_A(x) \wedge C_A(y). \end{aligned}$$

Hence, $A = \bigcup_{i \in I} A_i \in MG(X)$. □

Theorem 3.10. *If $A, B \in MG(X)$, then the sum of A and B is a multigroup of X.*

Proof. Let $x, y \in X$. By Definition 2.3 and Remark 2.6, we have

$$\begin{aligned} C_{A+B}(xy^{-1}) &= C_A(xy^{-1}) + C_B(xy^{-1}) \\ &\geq (C_A(x) \wedge C_A(y)) + (C_B(x) \wedge C_B(y)) \\ &= (C_A(x) + C_B(x)) \wedge (C_A(y) + C_B(y)) \\ &= C_{A+B}(x) \wedge C_{A+B}(y), \end{aligned}$$

$\Rightarrow C_{A+B}(xy^{-1}) \geq C_{A+B}(x) \wedge C_{A+B}(y)$. Hence, $A + B \in MG(X)$. □

Remark 3.11. Let $\{A_i\}_{i \in I} \in MG(X)$. Then $\sum_{i \in I} A_i \in MG(X)$.

Theorem 3.12. *Let $A \in MG(X)$ and if $x, y \in X$ with $C_A(x) \neq C_A(y)$, then*

$$C_A(xy) = C_A(yx) = C_A(x) \wedge C_A(y).$$

Proof. Let $x, y \in X$. Since $C_A(x) \neq C_A(y)$, it implies that $C_A(x) > C_A(y)$ or $C_A(y) > C_A(x)$. Suppose $C_A(x) > C_A(y)$. Then $C_A(xy) \geq C_A(y)$ and

$$\begin{aligned} C_A(y) = C_A(x^{-1}xy) &\geq C_A(x^{-1}) \wedge C_A(xy) \\ &= C_A(x) \wedge C_A(xy) \\ &= C_A(xy). \end{aligned}$$

It follows that

$$\begin{aligned} C_A(y) \geq C_A(xy) &\geq C_A(x) \wedge C_A(y) \\ &= C_A(y). \end{aligned}$$

From here, we see that

$$C_A(xy) \geq C_A(x) \wedge C_A(y)$$

and

$$C_A(x) \wedge C_A(y) \geq C_A(xy).$$

Thus, $C_A(xy) = C_A(x) \wedge C_A(y)$.

Similarly, suppose $C_A(y) > C_A(x)$. We have $C_A(yx) \geq C_A(x)$ and

$$\begin{aligned} C_A(x) = C_A(y^{-1}yx) &\geq C_A(y^{-1}) \wedge C_A(yx) \\ &= C_A(y) \wedge C_A(yx) \\ &= C_A(yx). \end{aligned}$$

Thus, we get

$$\begin{aligned} C_A(x) \geq C_A(yx) &\geq C_A(y) \wedge C_A(x) \\ &= C_A(x). \end{aligned}$$

Clearly, $C_A(yx) = C_A(y) \wedge C_A(x)$.
Therefore, $C_A(xy) = C_A(yx) = C_A(x) \wedge C_A(y)\ \forall x, y \in X$. □

Theorem 3.13. *Let A be a regular multiset defined over a group X. Then A is a multigroup of X if and only if A_* is a subgroup of X.*

Proof. Let X be a group and $x, y \in X$. Suppose A_* is a subgroup of X. Then $xy^{-1} \in A_*$ by Proposition 2.9. Since A is regular and A_* is the root set of A, it follows that

$$C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y) \ \forall x, y \in X.$$

Thus, A is a multigroup of X by Remark 2.6.

Conversely, suppose $A \in MG(X)$. Then by Proposition 2.9, A_* is a subgroup of X. □

Theorem 3.14. *Let A and B be multigroups of a group X. Then*

(i) $A \subseteq A \circ B$ if $C_A(e) \leq C_B(e)$.

(ii) $A \subseteq A \circ B$ and $B \subseteq A \circ B$ if $C_A(e) = C_B(e)$.

Proof. (i) Let $x \in X$. Suppose $C_A(e) \leq C_B(e)$. Then by Definition 2.11, we get

$$C_{A\circ B}(x) = \bigvee_{x=yz} (C_A(y) \wedge C_B(z)) \ \forall y, z \in X.$$

Also, it follows that

$$C_{A\circ B}(x) \geq C_A(x) \wedge C_B(e).$$

Now,

$$\begin{aligned} C_{A\circ B}(x) &= \bigvee_{x=yz} (C_A(y) \wedge C_B(z)) \ \forall y, z \in X \\ &\geq C_A(x) \wedge C_B(e) \\ &\geq C_A(x) \wedge C_A(e) \\ &= C_A(x). \end{aligned}$$

$\Rightarrow C_{A\circ B}(x) \geq C_A(x)$ that is, $A \subseteq A \circ B$.

(ii) Let $x \in X$. Assume that $C_A(e) = C_B(e)$. Then, it follows from (i) that $A \subseteq A \circ B$. Also, the proof of the second part follows; that is

$$\begin{aligned} C_{A\circ B}(x) &= \bigvee_{x=yz} (C_A(y) \wedge C_B(z)) \ \forall y, z \in X \\ &\geq C_A(e) \wedge C_B(x) \\ &= C_B(e) \wedge C_B(x) \\ &= C_B(x). \end{aligned}$$

$\Rightarrow C_{A\circ B}(x) \geq C_B(x)$ that is, $B \subseteq A \circ B$. □

Theorem 3.15. *Let $A, B \in MG(X)$ such that $C_A(e) = C_B(e)$. If $A \circ B$ is a multigroup of X, then $A \circ B$ is generated by A and B.*

Proof. Suppose that $A \circ B \in MG(X)$. Then, we show that $A \circ B$ is the smallest multigroup of X containing A and B. By Theorem 3.14, we see that $A \subseteq A \circ B$ and $B \subseteq A \circ B$.
Let C be any multigroup of X containing both A and B. Let $x \in X$, then we get

$$\begin{aligned} C_{A\circ B}(x) &= \bigvee_{x=yz} (C_A(y) \wedge C_B(z)) \forall y, z \in X \\ &\leq \bigvee_{x=yz} (C_C(y) \wedge C_C(z)) \forall y, z \in X \\ &= C_{C\circ C}(x), \end{aligned}$$

since $C_A(y) \leq C_C(y)$ and $C_B(z) \leq C_C(z)$. Because $C \in MG(X)$ and $C \circ C = C$ by Remark 2.12, we have $A \circ B \subseteq C$. Consequently, $A \circ B$ is a multigroup generated by A and B. □

4 Submultigroups of a multigroup

Definition 4.1. Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Example 4.2. Let $X = \{e, a, b, c\}$ be a Klein 4-group and $A = [e^6, a^4, b^5, c^4]$ be a multigroup generated from X. Then

$$A = [e^6, a^4, b^5, c^4], B = [e^5, a^3, b^4, c^3],$$

$$C = [e^4, a^2, b^3, c^2], D = [e^3, a, b^2, c] \text{ and } E = [e^2, b]$$

are submultigroups of A.
But

$$B = [e^5, a^3, b^4, c^3], C = [e^4, a^2, b^3, c^2],$$

$$D = [e^3, a, b^2, c] \text{ and } E = [e^2, b]$$

are proper submultigroups of A.

Definition 4.3. Let $A \in MG(X)$. Then we define the following types of submultigroup.

(i) A submultigroup B of A is said to be complete if $B_* = A_*$.

(ii) A submultigroup B of A is said to be incomplete if $B_* \neq A_*$.

(iii) A submultigroup B of A is said to be regular complete if B is complete and $C_B(x) = C_B(y) \, \forall x, y \in X$.

(iv) A submultigroup B of A is said to be regular incomplete if B is incomplete and $C_B(x) = C_B(y) \, \forall x, y \in X$.

Remark 4.4. If $A \in MG(X)$ and $B \sqsubseteq A$, then $B \in MG(X)$. Again, suppose $A, B \in MG(X)$, $C \in MS(X)$, $B \sqsubseteq A$ and $C \subseteq B$, respectively. Then $C \sqsubseteq A$ if and only if $C \sqsubseteq B$.

Remark 4.5. Let $A, B \in MG(X)$, then the following statements hold.

(i) $A \sqsubseteq B \Leftrightarrow A^{-1} \sqsubseteq B^{-1}$.

(ii) $A \sqsubseteq A^{-1} \Leftrightarrow A^{-1} \sqsubseteq A$.

Proposition 4.6. *Let $A, B \in MG(X)$ such that $C_A(x) \leq C_B(x) \, \forall x \in X$. Then*

(i) A_* is a subgroup of B_*,

(ii) A^* is a subgroup of B^*.

Proof. (i) Let X be a group and $x \in X$ because $X \neq \emptyset$. Since $A, B \in MG(X)$, then A_* is a subgroup of X, and consequently, B_* is a subgroup of X by Proposition 2.9. Since A is a submultigroup of B, the result follows.

(ii) Follows from (i). □

Proposition 4.7. *If $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$, then*

(i) $A \cap B$ is submultigroup of C,

(ii) $A \cup B$ is submultigroup of C.

Proof. (i) Suppose $A, B, C \in MG(X)$, then $C_{A \cap B}(x) \leq C_C(x) \, \forall x \in X$ since $A \subseteq B \subseteq C$. Thus, $A \cap B$ is submultigroup of C.

(ii) Follows from (i). □

Theorem 4.8. *Let $A \in MG(X)$ and B be a submultiset of A. Then B is a complete submultigroup of A if and only if (i) $B \neq \emptyset$ and (ii) for every $x, y \in X, C_B(xy^{-1}) \geq C_B(x) \wedge C_B(y)$.*

Proof. Suppose that B is a complete submultigroup of A, then $B \neq \emptyset$, that is, B has at least e such that $C_B(e) \geq C_B(x)\ \forall x \in X$. For any $x, y \in X$, we get $C_B(y^{-1}) = C_B(y)$ and so, $C_B(xy^{-1}) \geq C_B(x) \wedge C_B(y)\ \forall x, y \in X$.

Conversely, let $B \subseteq A$ and suppose that, given any $x, y \in X$, we get

$$C_B(xy^{-1}) \geq C_B(x) \wedge C_B(y).$$

Since $B \neq \emptyset$, for any element $x_\circ \in X$, $C_B(x_\circ) = C_B(x_\circ^{-1})$. Then, by the properties of B we have

$$C_B(e) = C_B(x_\circ x_\circ^{-1}) \geq C_B(x_\circ).$$

Now let $x \in X$, then $C_B(x^{-1}) = C_B(ex^{-1})$. Moreover, given $y \in X$, we have $C_B(y^{-1}) = C_B(y)$ and hence

$$C_B(xy) = C_B(x(y^{-1})^{-1}) \geq C_B(x) \wedge C_B(y)\ \forall x, y \in X.$$

Therefore, B is a complete submultigroup of A. □

Proposition 4.9. *Let $A \in MG(X)$ and B be a nonempty submultiset of A. Then the following statements are equivalent.*

(i) B is a submultigroup of A.

(ii) $C_B(xy) \geq C_B(x) \wedge C_B(y)$ and $C_B(x^{-1}) = C_B(x)\ \forall x, y \in X$.

(iii) $C_B(xy^{-1}) \geq C_B(x) \wedge C_B(y)\ \forall x, y \in X$.

Proof. (i)⇒(ii). Suppose $B \sqsubseteq A$. Then from Remark 4.4, it follows that $B \in MG(X)$. Thus, $C_B(xy) \geq C_B(x) \wedge C_B(y)$ and $C_B(x^{-1}) = C_B(x)\ \forall x, y \in X$.

(ii)⇒(iii). We have seen that $B \in MG(X)$. Then it follows that,

$$C_B(xy^{-1}) \geq C_B(x) \wedge C_B(y)\ \forall x, y \in X$$

by Remark 2.6.

(iii)⇒(i). Since $B \subseteq A$ and $B \in MG(X)$, it implies that $B \sqsubseteq A$. □

Theorem 4.10. *Let $A_1, A_2, ..., A_k$ be all the regular incomplete submultigroups of $B \in MG(X)$ such that only $C_{A_1 \cap A_2 \cap ... \cap A_k}(e)$ exists and*

$$C_{A_1+A_2+...+A_k}(e) \leq C_B(e),$$

where e is the identity element of X. Then $A_1 + A_2 + ... + A_k$ is a submultigroup of B.

Proof. Suppose $C_{A_1+A_2+...+A_k}(e) \leq C_B(e)$ for $e \in X$. Since only $C_{A_1 \cap A_2 \cap ... \cap A_k}(e)$ exists, we notice that, the count of each elements of $A_1, A_2, ..., A_k$ is distinct with the exception of e. By Definition 2.3, it follows that

$$C_{A_1+A_2+...+A_k}(x) \leq C_B(x)\ \forall x \in X.$$

Hence, $A_1 + A_2 + ... + A_k$ is a submultigroup of B. □

5 Commutative multigroups

Recall that, a multigroup A of X is said to be commutative or Abelian if for all $x, y \in X$,

$$C_A(xy) = C_A(yx).$$

To validate this, we consider the following examples of multigroup.

Example 5.1. The set of matrices $X = \{e, a, b, c\}$ such that

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a group under matrix multiplication. Then $A = [e^4, a^3, b^4, c^3]$ is a commutative multigroup of X.

Example 5.2. Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ be a group under matrix multiplication such that

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g_6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, g_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then $A = [g_1^{10}, g_2^5, g_3^7, g_4^5, g_5^5, g_6^5, g_7^7, g_8^8]$ is a multigroup of X but not commutative.

Remark 5.3. Let A be a multigroup of X.

(i) If X is an abelian group, then A is commutative.

(ii) If $C_A(x) = C_A(y)\ \forall x, y \in X$, then A is commutative whether X is not abelian.

Proposition 5.4. *Let B be a commutative multigroup of a group X. Then every complete submultigroup of B is a normal submultigroup.*

Proof. Suppose A is a complete submultigroup of $B \in MG(X)$ and B is commutative. Then for all $x, y \in X, C_A(xy) = C_A(yx)$. Consequently,

$$C_A(xyx^{-1}) = C_A(yxx^{-1}) \geq C_A(y).$$

Thus, A is a normal submultigroup of B. □

Theorem 5.5. *Let $A, B \in MG(X)$ such that $C_A(e) = C_B(e)$, where e is the identity element of X. Then B is commutative if and only if A is a commutative multigroup of X.*

Proof. Let X be a group such that $x, y \in X$. Suppose B is commutative. Then, it follows that

$$\begin{aligned} C_B((xy)(xy)^{-1}) = C_B(e) &= C_B((xy)(yx)^{-1}) \\ &= C_A((xy)(yx)^{-1}) = C_A(e), \end{aligned}$$

since $C_A(e) = C_B(e)$. Thus, $C_A(xy) = C_A(yx)\ \forall x, y \in X$.

Conversely, assume A is a commutative multigroup of X. Then, we have $C_B(xy) = C_B(yx)$ $\forall x, y \in X$ using the same logic in the necessity part. □

Definition 5.6. A multiset G over a group X is a semimultigroup if

$$C_G(xyz) = C_G(yxz) \forall\, x, y, z \in X,$$

and a multimonoid of X if in addition to being a semimultigroup of X,

$$C_G(e) \geq C_G(x)\ \forall x \in X,$$

where e is the identity element of X.

Let X be a group and $x, y \in X$. Recall that a commutator of x and y in X is defined by $[x, y] = x^{-1}y^{-1}xy$.

Theorem 5.7. *Let B be a commutative multigroup of a group X. Then*

(i) $C_B([x, y]) = C_B(e)$,

(ii) $C_B([x,y]) \geq C_B(x)$*, where* e *is the identity element of* X*.*

Proof. (i) Let $x, y \in X$ such that x and y commute with each other. Now,

$$\begin{aligned} C_B([x,y]) = C_B(x^{-1}y^{-1}xy) &= C_B(x^{-1}xy^{-1}y) \\ &\geq C_B(x^{-1}x) \wedge C_B(y^{-1}y) \\ &= C_B(e) \wedge C_B(e) \\ &= C_B(e) \end{aligned}$$

$\Rightarrow C_B([x,y]) \geq C_B(e)$, and

$$\begin{aligned} C_B(e) = C_B(xyx^{-1}y^{-1}) &= C_B((xyx^{-1}y^{-1})e) \\ &= C_B((xyx^{-1}y^{-1})(xyx^{-1}y^{-1})) \\ &\geq C_B(xyx^{-1}y^{-1}) \wedge C_B(xyx^{-1}y^{-1}) \\ &= C_B(x^{-1}y^{-1}xy) \\ &= C_B([x,y]) \end{aligned}$$

$\Rightarrow C_B(e) \geq C_B([x,y])$. Hence, $C_B([x,y]) = C_B(e)$.

(ii) Also,

$$\begin{aligned} C_B([x,y]) = C_B(x^{-1}y^{-1}xy) &\geq C_B(x^{-1}) \wedge C_B(y^{-1}xy) \\ &\geq C_B(x) \wedge C_B(x) \\ &= C_B(x). \end{aligned}$$

Thus, $C_B([x,y]) \geq C_B(x)$. □

Corollary 5.8. *Let* A *be a semimultigroup of a group* X*. Then* A *is a multimonoid if*

$$C_A([x,y]) \geq C_A(x) \; \forall x, y \in X.$$

Proof. Let $x, y \in X$ such that $x, y \neq e$, where e is the identity element of X. Since A is a semimultigroup of X, the result follows if we show that $C_A(e) \geq C_A(x) \; \forall x \in X$. Suppose $C_A([x,y]) \geq C_A(x) \; \forall x, y \in X$. By Theorem 5.7, $C_A(e) = C_A([x,y]) \; \forall x, y \in X$. Hence, $C_A(e) \geq C_A(x) \; \forall x \in X$. Therefore, A is a multimonoid of X by Definition 5.6. □

Theorem 5.9. *Let* $A \in MG(X)$ *be commutative and* $n \in \mathbb{N}$*. Then* $C_A((xy)^n) = C_A(x^n y^n)$ *for all* $x, y \in X$*.*

Proof. Let $x, y \in X$, we have

$$\begin{aligned} C_A((xy)^n) &= C_A(xy...xyxyxy) = C_A(xy...xyxy^2x[x,y]) \\ &\geq C_A(xy...xyxy^2x) \wedge C_A([x,y]) = C_A(x^2y...xyxy^2) \\ &= C_A(x^2y...xy^3x) = C_A(x^2y...xy^3x[x,y]) \\ &\geq C_A(x^3y...xy^3) \geq ... \geq C_A(x^{n-1}yxy^{n-1}) \\ &= C_A(x^{n-1}xy^n[x,y^{n-1}]) \geq C_A(x^{n-1}y^nx) \\ &= C_A(x^ny^n) \end{aligned}$$

$\Rightarrow C_A((xy)^n) \geq C_A(x^ny^n)$.
Also,

$$\begin{aligned} C_A(x^ny^n) &= C_A(x^{n-1}y^nx) = C_A(x^{n-1}yxy^{n-1}[y^{n-1},x]) \\ &\geq C_A(x^{n-1}yxy^{n-1}) \geq ... \geq C_A(xy...xyxy^2x) \\ &= C_A(xy...xyxyxy[x,y]) \geq C_A(xy...xyxyxy) \\ &= C_A((xy)^n) \end{aligned}$$

$\Rightarrow C_A(x^ny^n) \geq C_A((xy)^n)$. Hence, $C_A((xy)^n) = C_A(x^ny^n)$. □

Definition 5.10. Let $B \in MG(X)$ and A be a submultiset of B. Then the centralizer of a submultiset A of B is the set

$$Z(A) = \{x \in X \mid C_A(xy) = C_A(yx) \text{ and } C_A(xyz) = C_A(yxz)\ \forall y, z \in X\}.$$

Lemma 5.11. *$B \in MG(X)$ and A be a submultiset of B. Then $x \in Z(A)$ if*

$$C_A(xy_1...y_n) = C_A(y_1xy_2...y_n) = ... = C_A(y_1y_2...y_nx)\ \forall y_1, y_2, ..., y_n \in X.$$

Proof. We prove by induction on n. For $n = 1$, we have

$$C_A(xy_1y_2) = C_A(y_1xy_2)\ \forall y_1, y_2 \in X.$$

Thus, $x \in Z(A)$.

Now, we prove for $n = k + 1$. It follows that,

$$\begin{aligned} C_A(xy_1...(y_ky_{k+1})) &= C_A(y_1xy_2...(y_ky_{k+1})) \\ &= ... \\ &= C_A(y_1y_2...x(y_ky_{k+1})) \\ &= C_A(y_1y_2...(y_ky_{k+1})x) \end{aligned}$$

and

$$\begin{aligned} C_A(x(y_1y_2)...y_ky_{k+1}) &= C_A((y_1y_2)x...y_ky_{k+1}) \\ &= ... \\ &= C_A((y_1y_2)...y_kxy_{k+1}) \\ &= C_A((y_1y_2)...y_ky_{k+1}x)\ \forall y, y_2, ..., y_k, y_{k+1} \in X. \end{aligned}$$

The result follows. □

Lemma 5.12. *$B \in MG(X)$ and A be a submultiset of B, and*

$$T = \{x \in X \mid C_A(xyx^{-1}y^{-1}) = C_A(e)\ \forall y \in X\}.$$

Then $T = Z(A)$.

Proof. Let $x \in T$. Then for all $y, z \in X$, we get

$$\begin{aligned} C_A((xyz(yxz)^{-1}) &= C_A(xyzz^{-1}x^{-1}y^{-1}) \\ &= C_A(xyx^{-1}y^{-1}) \\ &= C_A(e) \end{aligned}$$

$\Rightarrow C_A(xyz) = C_A(yxz)\ \forall y, z \in X$ and so, $x \in Z(A)$. Thus, $T \subseteq Z(A)$.

Again, if $x \in Z(A)$ then $C_A(xy) = C_A(yx) \Rightarrow C_A(xyx^{-1}y^{-1}) = C_A(e)$ $\forall x, y \in X$. So $x \in T$. Thus, $Z(A) \subseteq T$. Hence, $T = Z(A)$. □

Corollary 5.13. *Let $A \in MG(X)$. Then $C_A(xyx^{-1}y^{-1}) = C_A(e)\ \forall x, y \in X$ if and only if A is a commutative multigroup of X.*

Proof. Suppose $C_A(xyx^{-1}y^{-1}) = C_A(e)\ \forall x, y \in X$. Then we have

$$C_A(xy(yx)^{-1}) = C_A(e) \Rightarrow C_A(xy) = C_A(yx)\ \forall x, y \in X.$$

So, A is commutative.

Conversely, let A be a commutative multigroup of X. It follows that

$$C_A(xy) = C_A(yx) \Rightarrow C_A(xyx^{-1}y^{-1}) = C_A(e)\ \forall x, y \in X.$$

□

Theorem 5.14. *Let B be a multiset over a semigroup X and A be a submultiset of B. If $Z(A)$ is nonempty, then $Z(A)$ is a subsemigroup of X. Moreover, if X is a group, then $Z(A)$ is a normal subgroup of X.*

Proof. Let $x_1, x_2 \in Z(A)$. Then for all $y, z \in X$, we have

$$C_A((x_1x_2)yz) = C_A(y(x_1x_2)z)$$

by Lemma 5.11, and clearly $C_A((x_1x_2)y) = C_A(y(x_1x_2))$. Hence, we have $x_1x_2 \in Z(A)$. Thus, $Z(A)$ is a subsemigroup of X. Suppose X is a group. Then $Z(A)$ is nonempty since $e \in Z(A)$. If $x \in Z(A)$, then

$$\begin{aligned} C_A(x^{-1}yz) &= C_A(x^{-1}yx^{-1}xz) \\ &= C_A(xx^{-1}yx^{-1}z) \\ &= C_A(yx^{-1}z)\ \forall y, z \in X \end{aligned}$$

and so, $x^{-1} \in Z(A)$. Hence, $Z(A)$ is a subgroup of X by the first part of the proof. Next, let $x \in Z(A)$ and $g \in X$. Then by Lemma 5.11,

$$\begin{aligned} C_A((g^{-1}xg)yz) &= C_A(xg^{-1}gyz) = C_A(xyg^{-1}gz) \\ &= C_A(yg^{-1}xgz) = C_A(y(g^{-1}xg)z)\ \forall y, z \in X, \end{aligned}$$

and so, $g^{-1}xg \in Z(A)$. Thus, $Z(A)$ is a normal subgroup of X. □

Theorem 5.15. *Let C be a semimultigroup of a group X, and both A and B be submultisets of C. Then $Z(A) \cap Z(B) \subseteq Z(A \cap B)$.*

Proof. Let $x \in Z(A)$ and $x \in Z(B) \Rightarrow x \in Z(A) \cap Z(B)$. For any $y, z \in X$, we get

$$\begin{aligned} C_{A\cap B}(xyz) &= C_A(xyz) \wedge C_B(xyz) \\ &= C_A(gxyzg^{-1}) \wedge C_B(gxyzg^{-1})\ \forall g \in X \\ &= C_A(y(gx)zg^{-1}) \wedge C_B(y(gx)zg^{-1}) \\ &= C_A(y(xg)zg^{-1}) \wedge C_B(y(xg)zg^{-1}) \\ &= C_A(y(xg)g^{-1}z) \wedge C_B(y(xg)g^{-1}z) \\ &= C_A(yxz) \wedge C_B(yxz) \\ &= C_{A\cap B}(yxz). \end{aligned}$$

Also, $C_{A\cap B}(xy) = C_{A\cap B}(yx)$. Hence, $x \in Z(A \cap B)$ and consequently,
$Z(A) \cap Z(B) \subseteq Z(A \cap B)$. □

Corollary 5.16. *Let C be a multigroup of a group X, and both A and B be submultisets of C such that $C_A(e) = C_B(e)$. Then $Z(A) \cap Z(B) = Z(A \cap B)$.*

Proof. By Lemma 5.12, $x \in Z(A \cap B)$
$\Leftrightarrow C_{A\cap B}(e) = C_{A\cap B}(xyx^{-1}y^{-1})\ \forall y \in X$
$\Leftrightarrow C_A(e) = C_B(e) = C_{A\cap B}(e) = C_A(xyx^{-1}y^{-1}) \wedge C_B(xyx^{-1}y^{-1})\ \forall y \in X$
$\Leftrightarrow C_A(xyx^{-1}y^{-1}) = C_A(e)$ and $C_B(xyx^{-1}y^{-1}) = C_B(e)\ \forall y \in X$
$\Leftrightarrow x \in Z(A)$ and $x \in Z(B)$
$\Leftrightarrow x \in Z(A) \cap Z(B)$.
Thus, $Z(A) \cap Z(B) = Z(A \cap B)$. □

Proposition 5.17. *Let C be a multigroup of a group X, and both A and B be submultisets of C. Then $Z(A) \circ Z(B) \subseteq Z(A \circ B)$.*

Proof. Let $x_1 \in Z(A)$ and $x_2 \in Z(B)$. Then for all $y, z \in X$,

$$\begin{aligned}
C_{A\circ B}((x_1x_2)yz) &= \bigvee_{x_1x_2yz=ab} (C_A(a) \wedge C_B(b)) \; \forall a, b \in X \\
&= \bigvee_{x_1x_2yz=ab} (C_A(x_1x_2yzb^{-1}) \wedge C_B(b)) \; \forall b \in X \\
&= \bigvee_{x_1x_2yz=ab} (C_A(x_2yx_1zb^{-1}) \wedge C_B(b)) \; \forall b \in X \\
&= \bigvee_{x_2yx_1z=ab} (C_A(a) \wedge C_B(b)) \; \forall a, b \in X \\
&= \bigvee_{x_2yx_1z=ab} (C_A(a) \wedge C_B(a^{-1}x_2yx_1z)) \; \forall a \in X \\
&= \bigvee_{x_2yx_1z=ab} (C_A(a) \wedge C_B(a^{-1}yx_1x_2z)) \; \forall a \in X \\
&= \bigvee_{yx_1x_2z=ab} (C_A(a) \wedge C_B(b)) \; \forall a, b \in X \\
&= C_{A\circ B}(y(x_1x_2)z).
\end{aligned}$$

Similarly, $C_{A\circ B}((x_1x_2)y) = C_{A\circ B}(y(x_1x_2))$. Hence, $x_1x_2 \in Z(A \circ B)$. Thus, $Z(A) \circ Z(B) \subseteq Z(A \circ B)$. □

Remark 5.18. Let C be a multigroup of a group X, and both A and B be submultisets of C. Suppose $A \subseteq B$, then $Z(A) \subseteq Z(B)$.

Definition 5.19. Let A be a multigroup of a group X. Then the center of A is defined as

$$C(A) = \{x \in X \mid C_A([x, y]) = C_A(e) \; \forall y \in X\}.$$

Theorem 5.20. *If A is a multigroup of a group X, then $C(A)$ is a subgroup of X.*

Proof. $C(A) \neq \emptyset$ since $e \in C(A)$. Let $x, y \in C(A)$. Then
$C_A([x, z]) = C_A(e)$ and $C_A([y, z]) = C_A(e) \; \forall z \in X$. Consequently,

$$\begin{aligned}
C_A([xy, z]) &= C_A([x, z]^y[y, z]) \; (for \; [x, z]^y = yx^{-1}z^{-1}xzy^{-1}) \\
&\geq C_A([x, z]^y) \wedge C_A([y, z]) \\
&\geq C_A([x, z]^y) \; (since \, C_A([y, z]) = C_A(e)) \\
&= C_A(y[x, z]y^{-1}) = C_A([x, z]) = C_A(e).
\end{aligned}$$

Thus, $xy \in C(A)$.

Again, let $x \in C(A)$. Then $C_A([x, z]) = C_A(e) \; \forall z \in X$. Hence,

$$\begin{aligned}
C_A([x^{-1}, z]) &= C_A(xz^{-1}x^{-1}z) = C_A(xz^{-1}x^{-1}zxx^{-1}) \\
&= C_A(z^{-1}x^{-1}zxx^{-1}x) = C_A([z, x]) \\
&= C_A([x, z]^{-1}) = C_A([x, z]) = C_A(e).
\end{aligned}$$

Thus, $x^{-1} \in C(A)$. Therefore, $C(A)$ is a subgroup of X. □

Remark 5.21. Let A be a multigroup of X. We notice that, $C(A) = A_*$ whenever A is either commutative or regular. Otherwise, $C(A) \subseteq A_*$.

Now, some homomorphic properties of commutative multigroups are explored.

Theorem 5.22. *Let f be an isomorphism of an abelian group X onto an abelian group Y. Let A and B be multigroups of X and Y, respectively. If A and B are commutative, then* *(i)* $f(A)$ *is commutative.*

(ii) $f^{-1}(B)$ *is commutative.*

Proof. By Theorem 2.18, $f(A) \in MG(Y)$ and $f^{-1}(B) \in MG(X)$.
(i) Let $x, y \in Y$. Since f is an isomorphism, then for some $a \in X$ we have $f(a) = x$. Thus,

$$\begin{aligned} C_{f(A)}(xyx^{-1}) &= C_A(f^{-1}(xyx^{-1}) = C_A(f^{-1}(y) \\ &= C_{f(A)}(y). \end{aligned}$$

From Proposition 2.15, $f(A)$ is commutative.
(ii) Let $a, b \in X$, then we have

$$\begin{aligned} C_{f^{-1}(B)}(aba^{-1}) &= C_B(f(aba^{-1})) = C_B(f(b)) \\ &= C_{f^{-1}(B)}(b) \end{aligned}$$

$\Rightarrow C_{f^{-1}(B)}(aba^{-1}) = C_{f^{-1}(B)}(b)$. The result follows from Proposition 2.15. □

Theorem 5.23. *Let f be a homomorphism of a group X onto a group Y. Let C and D be multigroups of X and Y, respectively. Suppose A is a submultiset of C, then $f(Z(A)) \subseteq Z(f(A))$.*

Proof. Let $x \in f(Z(A))$. Then $\exists u \in Z(A)$ such that $f(u) = x$. For all $y, z \in Y$,

$$\begin{aligned} C_{f(A)}(xyz) &= C_A(f^{-1}(xyz)) = C_A(f^{-1}(x)f^{-1}(y)f^{-1}(z)) \\ &= C_A(f^{-1}(f(u))f^{-1}(f(v))f^{-1}(f(w))) = C_A(uvw) \\ &= C_A(vuw) = C_A(f^{-1}(y)f^{-1}(x)f^{-1}(z)) \\ &= C_A(f^{-1}(yxz)) = C_{f(A)}(yxz), \end{aligned}$$

where $v, w \in X$ such that $f(v) = y$ and $f(w) = z$. Thus, $x \in Z(f(A))$. Hence,

$$f(Z(A)) \subseteq Z(f(A)).$$

□

Theorem 5.24. *Let $f : X \to Y$ be an isomorphism of groups. Let C and D be multigroups of X and Y, respectively. Suppose B is a submultiset of D, then $f^{-1}(Z(B)) = Z(f^{-1}(B))$.*

Proof. Let $x \in f^{-1}(Z(B))$. Then for all $y, z \in X$,

$$\begin{aligned} C_{f^{-1}(B)}(xyz) &= C_B(f(xyz)) = C_B(f(x)f(y)f(z)) \\ &= C_B(f(y)f(x)f(z)) = C_B(f(yxz)) \\ &= C_{f^{-1}(B)}(yxz). \end{aligned}$$

Similarly, $C_{f^{-1}(B)}(xy) = C_{f^{-1}(B)}(yx)$. Thus, $x \in Z(f^{-1}(B))$. Hence, $f^{-1}(Z(B)) \subseteq Z(f^{-1}(B))$.

Again, let $x \in Z(f^{-1}(B))$ and $f(x) = u$. Then for all $v, w \in Y$,

$$\begin{aligned} C_B(uvw) &= C_B(f(x)f(y)f(z)) = C_B(f(xyz)) \\ &= C_{f^{-1}(B)}(xyz) = C_{f^{-1}(B)}(yxz) \\ &= C_B(f(yxz)) = C_B(f(y)f(x)f(z)) \\ &= C_B(vuw), \end{aligned}$$

where $y, z \in X$ such that $f(y) = v$ and $f(z) = w$. Similarly, we have $C_B(uv) = C_B(vu)$. Thus, $u \in Z(B)$, that is, $x \in f^{-1}(Z(B))$. Hence, it implies that $Z(f^{-1}(B)) \subseteq f^{-1}(Z(B))$. Therefore, $f^{-1}(Z(B)) = Z(f^{-1}(B))$. □

6 Conclusion

The concepts of multigroups and submultigroups have been studied and some results were established. The notion of multigroupoid was proposed and various types of submultigroup based on their formations were introduced. We also explored commutative multigroups, proposed semi-multigroups and multimonoids with some related results. The concepts of center and centralizer of a multigroup were introduced. Finally, we considered homomorphic image and homomorphic preimage of commutative multigroups. Nonetheless, other group theoretic notions could be exploited in multigroup setting.

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