Palestine Journal of Mathematics

Vol. 9(1)(2020) , 327–336

VALUE DISTRIBUTION AND UNIQUENESS OF DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS

Biswajit Saha

Communicated by S. P. Goyal

MSC 2010 Classifications: Primary 30D35; Secondary 39A10.

Keywords and phrases: Uniqueness, Entire functions, Difference polynomial.

Abstract. In this paper we deal with distribution of zeros of certain types of difference polynomial and in addition to this we investigate the uniqueness results when two difference products of entire functions of finite order share one value counting and ignoring multiplicities by considering that the functions share the value zero, counting multiplicities and obtain some results which improve and generalize some recent results of H. Wang and H.Y. Xu [Revista De Mathematica, 22(2) 2015, 223-254] and P Sahoo and S. Seikh [Mathematical Sciences and Applications E-Notes, 4(2) 2016, 29-36].

1 Introduction, Definitions and Results

In this paper, a meromorphic function $f(z)$ means meromorphic in the complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [7], [8] and [17]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f-a$ and $g-a$ coincide in locations and multiplicity, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f-a$ and $g-a$ coincide only in their locations, then we say that f and g share the value a IM (ignoring multiplicities). For a positive integer p, we denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. A meromorphic function $\alpha(\not\equiv 0,\infty)$ is called a small function with respect to f, if $T(r,\alpha) = S(r,f)$.

Recently, the topic of difference equation and difference product in the complex plane $\mathbb{C}$ has attracted many mathematicians, a large number of papers have focused on value distribution of differences and differences operator analogues of Nevanlinna theory (including [4], [5], [6], and [9]), and many people paid their attention to the uniqueness of differences and difference polynomials of meromorphic function and obtained many interesting results.
In 2010, X.G. Qi, L.Z. Yang and K. Liu [12] studied the problem on the difference polynomials of entire functions of finite order and obtained the following result.

Theorem A. Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share the value 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.

In 2012, M.R. Chen and Z.X. Chen [3] further studied a certain type of difference polynomials and obtained the following theorem.

Theorem B. Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, c be nonzero finite complex numbers. If $n \geq m+8\sigma$, n, m, s, $\mu_j (j = 1, 2, ..., s)$ and $\sigma = \sum_{j=1}^{s} \mu_j$ are integers, and $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = t^{m+\sigma} = 1$.

Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex constants. We denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2, \Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. Throughout the paper we denote $d = \gcd(\lambda_0, \lambda_1, ..., \lambda_n)$, where $\lambda_i = n + 1$ if $a_i = 0$, $\lambda_i = i + 1$ if $a_i \neq 0$. In 2011 L. Xundan and W.C. Lin [15] considered the zeros of certain type of difference polynomial and obtained the following result.

Theorem C. Let f be a transcendental entire function of finite order and c be a fixed nonzero complex constants. Also suppose that $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex constants, and m is the number of distinct zeros of $P(z)$. Then for $n > m$, $P(f(z))f(z+c) - \alpha(z) = 0$ has infinitely many solutions, where $\alpha(z)(\not\equiv 0)$ is a small function with respect to f.

In the same paper the author investigated the uniqueness of complex difference polynomials of entire functions sharing one value and obtained the following result corresponding to theorem C.

Theorem D. Let f and g be two transcendental entire functions of finite order, c be a nonzero complex constant, and $n > 2\Gamma_2 + 1$ be an integer. If $P(f(z))f(z+c)$ and $P(g(z))g(z+c)$ share the value 1 CM, then one of the following cases hold:
(i) $f = tg$ where $t^d = 1$;
(ii) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z + c) - P(w_2)w_2(z + c)$;
(iii) $f = e^{\alpha}$, $g = e^{\beta}$, where α and β are two polynomials and $\alpha + \beta = b$, b is a constant satisfying $a_n^2e^{(n+1)b} = 1$.

We recall the example which shows that the second case of theorem D may occur.

Example 1.1. Let $P(z) = (z - 1)^6(z + 1)^6z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $c = 2\pi$. It is easy to see that $n > 2\Gamma_1 + 1$ and $P(f(z))f(z + c) \equiv P(g(z))g(z + c)$ so $P(f(z))f(z + c)$ and $P(g(z))g(z + c)$ share 1 CM. It is also clear that we get $f = tg$ for a constant t such that $t^m = 1$, where $m \in Z^+$, but f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = P(w_1)w_1(z + c) - P(w_2)w_2(z + c)$.

Regarding above example it is natural to ask the following question:

Question 1.1. What condition on f and g can guarantee that the case (ii) of theorem D may not occur ?

Keeping the above question in mind, H. Wang and H.Y. Xu [14] obtained the following theorem in 2015.

Theorem E. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM. Suppose that c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers and $n(\geq 1)$, n, s, $\mu_j(j = 1, 2, ..., s)$ are integers. If $P(f)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}$ and $P(g)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$ share 1 CM and $n > 2\Gamma_1 + \sigma$, then one of the following cases holds:
(i) $f \equiv tg$ for a constant t such that $t^l = 1$ where $l = \gcd(\lambda_0 + \sigma, \lambda_1 + \sigma, ..., \lambda_n + \sigma)$ and $\lambda_i = n$ if $a_i = 0$, $\lambda_i = i$ if $a_i \neq 0$, $i = 1, 2, ..., n$.
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial, η is a complex constant satisfying $a_n^2\eta^{n+\sigma} \equiv 1$.

Remark 1.1. From theorem D it is evident that the condition "f and g share 0 CM" in theorem E is necessary.

In 2016, W.L. Li and X.M. Li [10] also took into consideration the above problem and proved the following results.

Theorem F. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM, let c be a nonzero complex constant, and let $n > 2\Gamma_2 + 1$ be an integer. If

$P(f(z))f(z+c)$ and $P(g(z))g(z+c)$ share the value 1 CM, then one of the following two cases holds:
(i) $f = tg$ where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

Theorem G. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM, let c be a nonzero complex constant, and let $n > 3\Gamma_1 + 2\Gamma_2 + 4$ be an integer. If $P(f(z))f(z+c)$ and $P(g(z))g(z+c)$ share the value 1 IM, then one of the following two cases holds:
(i) $f = tg$ where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

In the same year P. Sahoo and S. Seikh [13] considered the difference polynomial of the form $(P(f(z))f(z+c))^{(k)}$ where $k(\geq 0)$ is an integer and obtained the following results.

Theorem H. Let f be a transcendental entire function of finite order and $\alpha(z)(\neq 0)$ be a small function with respect to f. Suppose that c is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers. Also suppose that $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex constants. Then for $n > \Gamma_1 + km_2$, $(P(f(z))f(z+c))^{(k)} - \alpha(z) = 0$ has infinitely many solutions.

Theorem I. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM. Suppose that c is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n > 2\Gamma_2 + 2km_2 + 1$. If $(P(f(z))f(z+c))^{(k)}$ and $(P(g(z))g(z+c))^{(k)}$ share the value 1 CM, then one of the following two cases holds:
(i) $f = tg$ where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

Theorem J. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM. Suppose that c is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$. If $(P(f(z))f(z+c))^{(k)}$ and $(P(g(z))g(z+c))^{(k)}$ share the value 1 IM, then one of the following two cases holds:
(i) $f = tg$ where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

Now it is natural to ask the following questions which are the motivation of the paper.

Question 1.2. What happen if one consider the difference polynomials of the form $\left(P(f)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)}$, where $f(z)$ is a transcendental entire function of finite order, $c_j(j = 1, 2, ..., s)$, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, s and $\mu_j(j = 1, 2, ..., s)$ are integers, $\sigma = \sum_{j=1}^{s} \mu_j$ in theorem E and in theorem H-J ?

In the paper, our main concern is to find the possible answer of the above question. The following are the main results of the paper.

Theorem 1.1. Let f be a transcendental entire function of finite order and $\alpha(z)(\neq 0)$ be a small function with respect to f. Suppose that c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers and $n(\geq 1)$, s, μ_j $(j = 1, 2, ..., s)$ and $k(\geq 0)$ are nonnegative integers. Also suppose that $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex constants. Then for $n > \Gamma_1 + km_2$, $\left(P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)} - \alpha(z) = 0$ has infinitely many solutions.

Remark 1.2. Theorem 1.1 improves Theorem C.

Remark 1.3. Theorem 1.1 extends and generalizes Theorem H.

Theorem 1.2. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM. Suppose that c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers and $n(\geq 1)$, s, μ_j $(j = 1, 2, ..., s)$ and $k(\geq 0)$ are nonnegative integers satisfying $n > 2\Gamma_2 + 2km_2 + \sigma$. If $\left(P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)}$ and $\left(P(g(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)}$ share the value 1 CM, then one of the following two cases holds:
(i) $f = tg$ where $t^l = 1$ where $l = \gcd(\lambda_0 + \sigma, \lambda_1 + \sigma, ..., \lambda_n + \sigma)$ and $\lambda_i = n$ if $a_i = 0$, $\lambda_i = i$ if $a_i \neq 0$, $i = 1, 2, ..., n$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

Remark 1.4. Theorem 1.2 improves Theorem E.

Remark 1.5. Theorem 1.2 extends and generalizes Theorem I.

Theorem 1.3. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM. Suppose that c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers and $n(\geq 1)$, s, μ_j $(j = 1, 2, ..., s)$ and $k(\geq 0)$ are nonnegative integers satisfying $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4\sigma$. If $\left(P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)}$ and $\left(P(g(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)}$ share the value 1 IM, then one of the following two cases holds:
(i) $f = tg$ where $t^l = 1$ where $l = \gcd(\lambda_0 + \sigma, \lambda_1 + \sigma, ..., \lambda_n + \sigma)$ and $\lambda_i = n$ if $a_i = 0$, $\lambda_i = i$ if $a_i \neq 0$, $i = 1, 2, ..., n$;
(ii) $f = e^{\alpha}$, $g = \eta e^{-\alpha}$, where α is a nonconstant polynomial and η is a constant satisfying $a_n^2 \eta^{n+1} = 1$.

Remark 1.6. Theorem 1.3 improves Theorem E.

Remark 1.7. Theorem 1.3 extends and generalizes Theorem J.

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [2] Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\{r^{\rho-1+\epsilon}\}.$$

Lemma 2.2. [11] Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then

$$\begin{aligned}
N(r, 0; f(z+c)) &\leq N(r, 0; f) + S(r, f),\\
N(r, \infty; f(z+c)) &\leq N(r, \infty; f) + S(r, f),\\
\overline{N}(r, 0; f(z+c)) &\leq \overline{N}(r, 0; f) + S(r, f),\\
\overline{N}(r, \infty; f(z+c)) &\leq \overline{N}(r, \infty; f) + S(r, f),
\end{aligned}$$

outside of possible exceptional set with finite logarithmic measure.

Lemma 2.3. Let f be a transcendental entire function of finite order. Suppose that c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers and $n(\geq 1)$, s, μ_j $(j = 1, 2, ..., s)$ are nonnegative integers and $F_1 = P(f(z)) \prod_{j=1}^{s} f(z+c_j)^{\mu_j}$. Then

$$T(r, F_1) = (n+\sigma)T(r, f) + S(r, f).$$

From the lemma it is clear that $S(r, F_1) = S(r, f)$.

Proof. Since f is an entire function of finite order we deduce from Lemma 2.1 and the standard Valiron Mohon'ko theorem that

$$\begin{aligned}
(n+\sigma)T(r,f) &= T(r, f(z)^{\sigma}P(f(z))) + S(r,f) \\
&= m(r, f(z)^{\sigma}P(f(z))) + S(r,f) \\
&\leq m\left(r, \frac{f(z)^{\sigma}P(f(z))}{P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}}\right) + m(r, F_1) + S(r,f) \\
&\leq \sum_{j=1}^{s} \mu_j m\left(r, \frac{f(z)}{f(z+c_j)}\right) + m(r, F_1) + S(r,f) \\
&\leq T(r, F_1) + S(r,f). \qquad (2.1)
\end{aligned}$$

On the other hand, by Lemma 2.2 and the fact that f is a transcendental entire function of finite order, we obtain

$$\begin{aligned}
T(r,f) &\leq T(r, P(f(z))) + T\left(r, f(z)^{\sigma}\prod_{j=1}^{s}\frac{f(z+c_j)^{\mu_j}}{f(z)^{\mu_j}}\right) + S(r,f) \\
&\leq nT(r,f) + \sigma T(r, f(z)) + \sum_{j=1}^{s}\mu_j T\left(r, \frac{f(z+c_j)}{f(z)}\right) + S(r,f) \\
&\leq (n+\sigma)T(r,f) + S(r,f). \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2) we can prove this lemma easily. □

Lemma 2.4. [18] Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r,f) + N_{p+k}(r, 0; f) + S(r,f). \qquad (2.3)$$

and

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r,f). \qquad (2.4)$$

Lemma 2.5. [16] Let f and g be two nonconstant meromorphic functions sharing the value 1 CM. Then one of the following three cases holds:
(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,
(ii) $f = g$,
(iii) $fg = 1$,
Where $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.6. [1] Let F and G be two nonconstant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then
$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G)$,
and the same inequality holds for $T(r,G)$.

Lemma 2.7. Let f and g be two entire functions and $n(\geq 1)$, $k(\geq 0)$, be integers, and let $F = \left(P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)}$ and $G = \left(P(g(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)}$. If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, c_2; G) = \overline{N}(r, 0; F)$, then $n \leq 2\Gamma_1 + 2km_2 + \sigma$.

Proof. We put $F_1 = P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, $G_1 = P(f(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{2.5}$$

Using (2.3), (2.4), (2.5), Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned} (n+\sigma)T(r,f) &\leq T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (m_1 + m_2 + km_2 + \sigma)(T(r,f) + T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{2.6}$$

Similarly,

$$(n+\sigma)T(r,g) \leq (m_1 + m_2 + km_2 + \sigma)(T(r,f) + T(r,g)) + S(r,f) + S(r,g). \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$(n - 2m_1 - 2m_2 - 2km_2 - \sigma)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which gives

$$n \leq 2\Gamma_1 + 2km_2 + \sigma.$$

This proves the lemma. □

3 Proof of the Theorems

Proof of Theorem 1.1. Let $F_1 = P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$. Then F_1 is a transcendental entire function. If possible, we may assume that $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then we have

$$N(r;\alpha;F_1^{(k)}) = O\{logr\} = S(r,f). \tag{3.1}$$

Using (2.3), (3.1) and Nevanlinna's three small function theorem we obtain

$$\begin{aligned} T(r,F_1^{(k)}) &\leq \overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,\alpha;F_1^{(k)}) + S(r,f) \\ &\leq T(r,F_1^{(k)}) - T(r,F_1) + N_{k+1}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.2}$$

Applying Lemma 2.3 we obtain from (3.2)

$$\begin{aligned} (n+\sigma)T(r,f) &\leq N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq (m_1 + m_2 + km_2 + \sigma)T(r,f) + S(r,f). \end{aligned}$$

This gives

$$(n - m_1 - m_2 - km_2)T(r,f) \leq S(r,f),$$

a contradiction with the assumption that $n \geq \Gamma_1 + km_2$. This proves the theorem. □

Proof of Theorem 1.2. Let $F_1 = P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, $G_1 = P(g(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$, $F = F_1^{(k)}$ and $G = G_1^{(k)}$. Then F and G are transcendental entire functions that share the value 1 CM. Using (2.3) and Lemma 2.3 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq T(r,F_1^{(k)}) - (n+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

From this we get

$$(n+\sigma)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f). \tag{3.3}$$

Again by (2.4) we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.4}$$

Suppose, if possible, that (i) of Lemma 2.5 holds. Then using (3.4) we obtain from (3.3)

$$\begin{aligned} (n+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) \\ &\quad + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (m_1 + 2m_2 + km_2 + \sigma)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Similarly,

$$(n+\sigma)T(r,g) \leq (m_1 + 2m_2 + km_2 + \sigma)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$(n - 2m_1 - 4m_2 - 2km_2 - \sigma)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the fact that

$$n > 2\Gamma_2 + 2km_2 + \sigma.$$

Therefore, by Lemma 2.5 we have either $FG = 1$ or $F = G$. Let $FG = 1$. Then

$$\left(P(f(z))\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)} \left(P(g(z))\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)} = 1 \tag{3.7}$$

Since f and g are entire functions, from (3.7) we deduce that $P(f(z)) \neq 0$ and $P(g(z)) \neq 0$. If possible, we assume that $P(z) = 0$ has two distinct roots, say, z_1 and z_2. Then

$$P(f(z)) = a_n(f - z_1)^{n_1}(f - z_2)^{n_2}$$

where n_1, n_2 are positive integers with $n_1 + n_2 = n$. Therefore $N(r,z_1;f) = O\{\log r\}$ and $N(r,z_2;f) = O\{\log r\}$. By using Nevanlinna second fundamental theorem, we can get a contradiction easily. Next we suppose that $P(z) = 0$ has only one root. Then $P(f(z)) = a_n(f-a)^n$ and $P(g(z)) = a_n(g-a)^n$, where a is a complex constant. Hence from the assumption that f and g are two transcendental entire functions of finite order, we have $f(z) = e^{\alpha(z)} + a$ and $g(z) = e^{\beta(z)} + a$, $\alpha(z)$, $\beta(z)$ being nonconstant polynomials. From (3.7), we also see that $f(z+c_j) \neq 0$ and $g(z+c_j) \neq 0$ for $j = 1, 2, ..., s$. Thus it follows $a = 0$, that is, $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$ and $P(z) = a^n z^n$. Then from (3.7) we obtain

$$\left(a_n \exp\left\{n\alpha(z) + \sum_{j=1}^{s} \mu_j \alpha(z+c_j)\right\}\right)^{(k)} \left(a_n \exp\left\{n\beta(z) + \sum_{j=1}^{s} \mu_j \beta(z+c_j)\right\}\right)^{(k)} = 1.$$

If $k=0$, then since $\alpha(z)$ and $\beta(z)$ are two nonconstant polynomials, we get $\alpha+\beta\equiv\eta$, where η is a constant. Hence we can easily get $f(z)=e^{\alpha(z)}$ and $g(z)=\eta e^{-\alpha(z)}$, where $\alpha(z)$ is a nonconstant polynomial, η is a complex constant satisfying $a_n^2\eta^{n+\sigma}=1$.
If $k\geq 1$ then we deduce

$$\begin{aligned}
&\left(a_n\exp\left\{n\alpha(z)+\sum_{j=1}^{s}\mu_j\alpha(z+c_j)\right\}\right)^{(k)}\\
=&\left(a_n\exp\left\{n\alpha(z)+\sum_{j=1}^{s}\mu_j\alpha(z+c_j)\right\}\right)P\left(\alpha^{'},{\alpha_{c_j}}^{'},...,\alpha^{(k)},{\alpha_{c_j}}^{(k)}\right),
\end{aligned}$$

where $\alpha_{c_j}=\alpha(z+c_j)$, $j=1,2,...,s$. Obviously, $P\left(\alpha^{'},{\alpha_{c_j}}^{'},...,\alpha^{(k)},{\alpha_{c_j}}^{(k)}\right)$ has infinite zeros, so it is impossible. Next we assume that $F=G$. Then

$$\left(P(f)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\right)^{(k)}=\left(P(g)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}\right)^{(k)}.$$

Integrating above we obtain

$$\left(P(f)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\right)^{(k-1)}=\left(P(g)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}\right)^{(k-1)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, using Lemma 2.7 it follows that $n\leq 2\Gamma_1+2(k-1)m_2+\sigma$, a contradiction as $n>2\Gamma_2+2km_2+\sigma$ and $\Gamma_2\geq\Gamma_1$. Hence $c_{k-1}=0$. Repeating the process k-times, we deduce that

$$P(f)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}=P(g)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}.$$

Then arguing similarly as in case 2 in the proof of Theorem 1.7 [14] we obtain $f=tg$ where $t^l=1$ where $l=\gcd(\lambda_0+\sigma,\lambda_1+\sigma,...,\lambda_n+\sigma)$ and $\lambda_i=n$ if $a_i=0$, $\lambda_i=i$ if $a_i\neq 0$, $i=1,2,...,n$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 1.2. Then F and G are transcendental entire functions that share the value 1 IM. We assume, if possible, that $H\not\equiv 0$. Using Lemma 2.6 and (3.4) we obtain from (3.3)

$$\begin{aligned}
(n+\sigma)T(r,f)\ &\leq\ N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
&\quad+N_{k+2}(r,0;F_1)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\
&\leq\ N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\
&\quad+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\
&\leq\ (3m_1+4m_2+3km_2+3\sigma)T(r,f)+(2m_1+3m_2+2km_2+2\sigma)+T(r,g)\\
&\quad+S(r,f)+S(r,g)\\
&\leq\ (5m_1+7m_2+5km_2+5\sigma)T(r)+S(r). \qquad (3.8)
\end{aligned}$$

Similarly,

$$(n+\sigma)T(r,g)\ \leq\ (5m_1+7m_2+5km_2+5\sigma)T(r)+S(r). \qquad (3.9)$$

(3.8) and (3.9) together gives

$$(n-5m_1-7m_2-5km_2-4\sigma)T(r)\leq S(r),$$

contradicting with the fact that

$$n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4\sigma.$$

We now assume that $H = 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.10}$$

where $A(\neq 0)$ and B are constants. From (3.10) it is obvious that F, G share the value 1 CM. Therefore $n > 2\Gamma_2 + 2km_2 + \sigma$. We now discuss the following three cases separately.

Case 1. Suppose that $B \neq 0$ and $A = B$. Then from (3.10) we obtain

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.11}$$

If $B = -1$, then from (3.11) we obtain

$$FG = 1,$$

which is a contradiction as in the proof of Theorem 1.2.
If $B \neq -1$, from (3.11), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Using (2.3), (2.4) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{B+1}; G\right) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) \\ &\quad -(n+\sigma)T(r,g) + S(r,g). \end{aligned}$$

This gives

$$(n+\sigma)T(r,g) \leq (m_1 + m_2 + km_2 + \sigma)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n - 2m_1 - 2m_2 - 2km_2 - \sigma)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n > 2\Gamma_2 + 2km_2 + \sigma$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.10) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.10) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, it follows that $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Now applying Lemma 2.7 it can be shown that $n \leq 2\Gamma_1 + 2km_2 + \sigma$, which is a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 1.2. This completes the proof of Theorem 1.3. □

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Author information

Biswajit Saha, Department of Mathematics, Government General Degree College Muragachha, Nadia, West Bengal, 741154, India.
E-mail: sahaanjan11@gmail.com, sahabiswajitku@gmail.com

Received: August 22, 2017.
Accepted: March 19, 2018