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Zeckendorf Arithmetic For Lucas Numbers

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Abstract In this article we present some new algorithms of addition, subtraction, multiplication and division of two positive integers using Zeckendorf form, moreover, we calculated their complexity. Such results find application in coding theory.

1 Introduction

There are few previous work in this area. Graham, Knuth and Patashnik (see[6]) discussing the addition of 1 in the Zeckendorf representation, but have not talked about the actual arithmetic. Fliponi (see[4]) did for addition and multiplication, and Freitag philips(see[2]) for the subtraction and division (see[3]). Thus, no previous work has discussed arithmetic as a coherent whole, covering all major operations, including multiplication and division. All these algorithms have been implemented and tested on a computer. Most algorithms are developed by analogy with conventional arithmetic methods. For example, multiplication is carried out by adding appropriate multiples of the multiplicand, depending on the selected bit pattern of the multiplier. The division will use a sequence of test subtraction, as in the normal long division. Here we use arithmetic complexity models, where cost is measured by the number of machine instructions performed on a single processor with addition and subtractions of m-bit integers that costs $O(m)$ (see[7])

2 Zeckendorf theorems for Lucas numbers

Lucas numbers are defined by the recursion formula:

$$\begin{cases} L_n = L_{n-1} + L_{n-2}, n \geq 2 \\ L_0 = 2, L_1 = 1. \end{cases}$$

and for all $n \geq 0$, we have the well-known
$L_n = F_{n+1} + F_{n-1}$ where F_n is the nth Fibonacci number.

Theorem 2.1. *Let n be an integer satisfying $0 < n \leq L_k$ for $k \geq 1$. Then $n = \sum_{i=0}^{k-1} \alpha_i L_i$ where $\alpha_i \in \{0, 1\}$ such that*

$$\begin{cases} \alpha_i \alpha_{i+1} = 0, \ for\ any\ i > 0; \\ \alpha_0 \alpha_2 = 0. \end{cases}$$

this representation is unique.

Proof: (see[1])

3 Zeckendorf decomposition method

To decompose an integer m of the Zeckendorf form $m = \sum_{n=0}^{\infty} \alpha_n L_n$, proceeding as follows :

(i) Find the greater Lucas number $L_n \leq m$.

(ii) Do subtraction $M = m - L_n$, assign a 1 to e_n and keep this coefficient.

(iii) Assign M to m and repeat steps 1 et 2 until M have a zero.

(iv) Assign of 0 to e_i where $0 \leq i \leq n$ and $e_i \neq 1$.

The result of this decomposition is a vector of n elements that contains the coefficients e_i decomposition. Example decomposition of 50, this table shows the performance:

Lucas sequence	2	1	3	4	7	11	18	29	47	76
Vector e_r	0	0	1	0	0	0	0	0	1	0

So $50_L = 001000001$

Proposition 3.1. *Let $m > 1$, if $L_n \leq m$ then $n \leq \frac{ln(m-1)}{ln(\varphi)}$, where φ is the golden ratio.*

Proof: let $L_n = \varphi^n + \frac{1}{\varphi^n} \leq \varphi^n + 1$.

While $L_n \leq m$ then $\varphi^n + 1 \leq m$

$$\Rightarrow \varphi^n \leq m - 1$$
$$\Rightarrow ln(\varphi^n) \leq ln(m-1)$$
$$\Rightarrow n \leq \frac{ln(m-1)}{ln(\varphi)}.$$

We conclude that the bit Zeckendorf number for representation n is at most equal to $\lfloor \dfrac{ln(m-1)}{ln(\varphi)} \rfloor$.

4 Addition

We take two positive integers a and b written in the form of Zeckendorf, obtainable form of a + b Zeckendorf repeating adding, at the same time, numbers of Lucas occupant in one of two numbers , say b , to another number a. This gives an initial amount for which figures are $d_i \in \{0, 1\}$, where each d_i is L_i its number of Lucas.
For $d_i = 2$ does not exist because $n = 1 \to 2L_1 = L_0$, we replace 020 by 001 and $n \geq 2 \to 2L_n = L_{n+1} + L_{n-2}$, we replace 00200 by 01001. In way is equivalent model x 2 y z figures transforms to $(1 + x)0y(1 + z)$.
This rule does not apply to terms with a weight of 1, which is covered by the special case below. If the combination 011 exists in the vector e_r, we will substitute it by 001. This step must be performed by scanning left to right through the performance. Here is a table that summarizes all possible cases of the addition in the representation of Zeckendorf:

Addition	Lucas weight	L_{i+1}	L_i	L_{i-1}	L_{i-2}
Consecutive 1		x	y	1	1
	becomes	x	$y+1$	0	0
Eliminate a 2	here $x \geq 2$	w	x	y	z
	becomes	$w+1$	$x-2$	y	$z+1$
Add,right bits		L_2	L_1	L_0	
$d_2 \geq 2$	here $x \geq 2$		x	0	
	becomes		0	1	
$d_2 \geq 2$			0	x	
	becomes	1	1	0	

Table 1. Adjustments and corrections in addition

Theorem 4.1. *The complexity of the addition algorithm is $O(ln(a))$.*

Proof: Let a and b two integers such that $b \leq a$. We take n and n' the bit Zeckendorf number for a and b respectively. Then the addition $(a+b)$ cost $O(Max(n, n^{'}))$. The total number $T(a)$ of operation is given by:

$$T(a) = O(\frac{ln(a-1)}{ln(\varphi)}) = O(ln(a)).$$

This table shows the two additions examples $33+19$ and $12+19$ in Zeckendorf representation:

a		1	0	0	0	1	0	0	0	= 33
b			1	0	0	0	0	1	0	= 19
initial sum		1	1	0	0	1	0	1	0	= 52
consecutive 1	1	0	0	0	0	1	0	1	0	= 52
becomes	1	0	0	0	0	1	0	1	0	= 52
check $33+19=52$										

Table 2. Example of addition $(33+19)$

a			1	0	0	0	1	0	= 12
b		1	0	0	0	0	1	0	= 19
initial sum		1	1	0	0	0	2	0	= 31
		1	1	0	0	0	0	1	= 31
consecutive 1		1	1	0	0	0	0	1	= 31
	1	0	0	0	0	0	0	1	= 31
becomes	1	0	0	0	0	0	0	1	= 31
check $12+19=31$									

Table 3. Example of addition $(12+19)$

5 Subtraction

For subtraction, $a-b=z$, where $b<a$ and Z the difference. We start by subtracting all figures $a_i - b_i = z_i$, where
$z_i \in \{0, 1. -1\}$ Values 0 and 1 have no problem, as they are valid representation in Zeckendorf. Where $z_i = -1$ is the most difficult. If in this case, go to the next bit 1 and is written in the Fibonacci rule $100 \rightarrow 011$ and write bit 1 rightmost pairs 1 is repeated until the bit 1 bit coincides with the -1 in the same position and eliminates replacing the box by 0 then 1 consecutive passes.

Subtraction	Lucas weights	L_{i+2}	L_{i+1}	L_i	L_{i-1}	L_{i-2}
eliminate -1		1	0	0	0	-1
		0	1	1	0	-1
	becomes	0	1	0	1	0

Table 4. Adjustments and corrections in subtraction

Theorem 5.1. *The complexity of the subtraction algorithm is $O(ln(a))$.*

Proof: Let a and b two integers such that $b \leq a$. We take n and n' the bit Zeckendorf number for a and b respectively. Then the addition $(a-b)$ cost $O(Max(n, n^{'}))$. The total number $T(a)$ of operation is given by:

$$T(a) = O(\frac{ln(a-1)}{ln(\varphi)}) = O(ln(a)).$$

This table shows the example 42-32 in Zeckendorf representation.

a	1	0	1	0	0	0	0	1	= 42
b	1	0	0	0	0	1	0	0	= 32
subtract bit by bit			1	0	0	-1	0	1	= 10
rewrite 1000				1	1	-1	0	1	= 10
rewrite 0110, cancelling -1				1	0	0	1	1	= 10
consecutive 1				1	0	1	0	0	= 10
becomes				1	0	1	0	0	= 10

Table 5. Example of subtraction (42 – 32)

6 Multiplication

Using the following results (propositions 1,2,3,4) and section 3 above, one can derive a multiplication method of integers in Zackendorf representation.

Proposition 6.1. *If* $n \geq 3$, *then*

$$L_k L_{k+n} = \begin{cases} F_{n-1} + F_{n+1} + F_{2k+n\pm1} & (k \quad even), \\ F_{n-2} + F_{n+1} + F_{2k+n+1} + \sum_{j=1}^{k-2} F_{2j+n+2} & (k \geq 3, odd). \end{cases}$$

Proposition 6.2. *If* $n \geq 5$, *then*

$$2L_k L_{k+n} = \begin{cases} F_{n\pm3} + F_{2k+n\pm3} & (k \geq 4, even), \\ F_{n-4} + F_{2k+n+3} + \sum_{j=1}^{3} F_{2j+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, odd). \end{cases}$$

Proposition 6.3. *If* $n \geq 5$, *then*

$$3L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{2j+n-5} + F_{2j+2k+n-5}) & (k \geq 4, even), \\ F_{n-4} + F_{n+3} + \sum_{j=1}^{3} F_{2j+2k+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, odd). \end{cases}$$

Proposition 6.4. *If* $n \geq 6$, *then*

$$4L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{3j+n-8} + F_{3j+2k+n-8}) & (k \geq 6, even) \\ F_{n-4} + F_{n-2} + F_{n+1} + \sum_{j=1}^{3} F_{3j+2k+n-5} + \sum_{j=1}^{k-5} F_{2j+n+4} & (k \geq 5, odd) \end{cases}$$

Proof: (see[5])

Theorem 6.5. *The complexity of the multiplication algorithm is* $O(a\, ln(a))$.

Proof: We have $ab = \underbrace{a + a + ...a}_{b\ times}$ the addition costs $O(ln(a))$, then the addition $\underbrace{a + a + ...a}_{b\ times}$ costs $O(b\, ln(a))$, but $b \leq a$, then $O(b\, ln(a)) = O(a\, ln(a))$, finally :

$$T(a) = O(a\, ln(a)).$$

This example shows how to compute 17×10 in Zeckendorf representation :

a						1	0	1	0	0	1	=17
b							1	0	1	0	0	=10
Multiple of Luca 17												
multiple L_1						1	0	1	0	0	1	=17
multiple L_2			1	0	0	0	0	1	0	0	0	=51
multiple L_3			1	0	1	0	0	0	1	0	0	=68
multiple L_4		1	0	1	0	1	0	0	1	0	0	=119
multiple L_5	1	0	1	0	0	1	0	1	0	0	1	=187
Accumulate appropriate multiples												
Add multiple of L_2			1	0	0	0	0	1	0	0	0	=51
Add multiple of L_4		1	0	1	0	1	0	0	1	0	0	=119
$L_2 + L_4$=		1	1	1	0	1	0	1	1	0	0	=170
Eliminate 1 consecutive	1	0	0	1	0	1	1	0	0	0	0	=170
Eliminate 1 consecutive	1	0	0	1	1	0	0	0	0	0	0	=170
becomes=	1	0	1	0	0	0	0	0	0	0	0	=170

Table 6. Example of Zeckendorf multiplication (17×10)

7 Division

Using the following proposition 5 and section 4 above, one can derive a division method of integers in Zackendorf representation.

Proposition 7.1. *First,for $k = 4m$ and n odd, we obtain*

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{m} \left(L_{(k-4r+3)n} + L_{(k-4r+1)n}\right),$$

and thus

$$\frac{F_{kn}}{F_n} = S_{k,n},$$

say, where
$S_{k,n} = \sum_{r=0}^{[k/4]-1} (F_{(k-4r-1)n+1} + (\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s}) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2})$.

We similarly work through the other cases, where n is odd and $k \equiv 1, 2$ and $3 \mod 4$. In each case, the "most significant" part of the Zeckendorf form is $S_{k,n}$. The precise Zackendorf form is

$$\frac{F_{kn}}{F_n} = S_{k,n} + e_{k,n},$$

where the least significant part of the Zeckendorf sum is

$$e_{k,n} = \begin{cases} 0, & k \equiv 0 \mod 4), \\ F_2, & k \equiv 1 \mod 4), \\ F_{n+1} + F_{n-1}, & k \equiv 2 \mod 4), \\ F_{2n+1} + \sum_{r=1}^{n-1} F_{2n-2r}, & k \equiv 3 \mod 4). \end{cases}$$

Proof:(see[3])

Theorem 7.2. *The complexity of the Division algorithm is* $O(a\, ln(a))$.

Proof: (as in **Theorem 4**)

This example shows how to compute $250 \div 17$ in Zeckendorf representation:

a		1	0	1	0	1	0	0	0	1	0	0	=250
b							1	0	1	0	0	1	=17
Make lucas Multiples of divisor													
multiple L_1							1	0	1	0	0	1	=17
multiple L_2				1	0	0	0	0	1	0	0	0	=51
multiple L_3				1	0	1	0	0	0	1	0	0	=68
multiple L_4			1	0	1	0	1	0	0	1	0	0	=119
multiple L_5		1	0	1	0	0	1	0	1	0	0	1	=187
multiple L_6	1	0	1	0	1	0	0	0	0	0	0	1	=306
Trial subtraction													
$L5$ residue=				1	0	0	1	0	1	0	1	0	=63
$L2$ residue=							1	0	0	0	1	0	=12
quotient=							1	0	1	0	0	1	=17
remainder =							1	0	0	0	1	0	=12

Table 7. Example of Zeckendorf division $(250 \div 17)$

8 Conclusion

Although we have highlighted the main arithmetic operations on integers Zeckendorf, this arithmetic should not stay more than a curiosity. In future research, we plan to study the applications of our results to other areas of mathematics such as error correcting codes.

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