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ALGEBRAIC IDENTITIES AND COMMUTATIVITY IN 3-PRIME NEARRINGS

Mohammad Mueenul Hasnain

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Abstract The aim of this paper is to investigate commutativity of nearrings in which derivations satisfy certain algebraic conditions. Moreover, we give some examples to prove that the hypothesis of 3-primeness is necessary.

1 Introduction

Nearring are one of the generalized structures of ring. The study and research on nearring is very systematic and continuous. Nearring are being used since the development of Calculus, but the key ideas important to nearrings were formalized in 1905 by Dickson who defined the near fields. It is well known fact that there are several results asserting that prime nearrings with certain constrained derivations have ring like behavior (see [1, 8, 10], where further references can be found).

Throughout this paper N will denote a zero symmetric left nearrings with multiplicative centre $Z(N)$; and usually N will be 3-prime, that is, will have the property that $xNy = \{0\}$ for all $x, y \in N$ implies $x = 0$ or $y = 0$. N is said to be 2-torsion free if $2x = 0$ implies $x = 0$. Unless otherwise specified, we will use the word nearring to mean zero symmetric right nearring. For any $x, y \in N$, as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well known Lie and Jordan product respectively. An additive mapping $D : N \longrightarrow N$ said to be a derivation if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$, or equivalently, as noted in [12] that $D(xy) = D(x)y + xD(y)$ for all $x, y \in N$.

In 1987, the relationship between commutativity of a prime nearrings and the behaviour of a derivation on prime nearrings was initiated by Bell and Mason in [6]. In [9] Hogan generalizes some results of Bell Mason by assuming that the commutativity condition is imposed on an ideal rather than on the wall nearring. In view of these results many authors have investigated commutativity of prime nearrings satisfying certain polynomial identities involving derivations(see [2, 7, 5, 11] for detail). Inspire by this work, our intent is to go further step in this direction and investigate conditions for 3-prime nearrings to be commutative.

2 Algebraic Identities and commutativity

We begin with the following lemmas which are essential for developing the proofs of our main results.

Lemma 2.1. *Let N be a 3-prime nearring and D a nonzero derivation.*

(i) [4, Theorem 2.1] If $D(N) \subset Z(N)$, then N is commutative.

(ii) [6, Theorem 4.1] If N is also 2-torsion free, then $D^2 \neq 0$.

Lemma 2.2. *[6, Lemma 1]Let D be a arbitrary derivation on a nearring N. Then N satisfies the following partial distributive law:*

$$(xD(y) + D(x)y)z = xD(y)z + D(x)yz, for\ all\ x, y, z \in N.$$

Theorem 2.3. *Let N be a 2-torsion free 3-prime nearring. If N admits a nonzero derivation D such that $D(xy \pm yx) = [D(x), y]$ for all $x, y \in N$, then N is commutative.*

Proof. From the hypothesis, we have

$$D(xy \pm yx) = [D(x), y], \quad \text{for all } \ x, y \in N. \tag{2.1}$$

The substitution xy for y in (2.1) gives because of (2.1)

$$xD(x)y \pm D(x)yx = 0, \quad \text{for all } \ x, y \in N. \tag{2.2}$$

The substitution yz for y in 2.2 and use it to obtain

$$D(x)y[x, z] = 0, \quad \text{for all } \ x, y, z \in N,$$

which leads to

$$D(x)N[x, z] = \{0\}, \quad \text{for all } \ x, t \in N,$$

Replace x by $D(y)$ in the above relation we have

$$D^2(y)N[D(y), z] = \{0\}, \quad \text{for all } \ y, z \in N. \tag{2.3}$$

From the 3-primeness of N, we obtain that $D(y) \in Z(N) \quad or \quad D^2(y) = 0$, for each $y \in N$. In the latter case, $D^2(y) = 0$ for all $y \in N$, so this case cannot be occur by Lemma 2.1(ii). Again, by using an application of Lemma 2.1(i) assures that N is commutative. □

Theorem 2.4. *Let N be a 2-torsion free 3-prime nearring. If N admits a nonzero derivation D such that $D[x, y] = D(x)y \pm yD(x)$ for all $x, y \in N$, then N is commutative.*

Proof. Given that

$$D[x, y] = D(x)y \pm yD(x), \quad \text{for all } \ x, y \in N. \tag{2.4}$$

The substitution xy for y in (2.4) gives because of (2.4)

$$xD(y)y = yxD(x), \quad \text{for all } \ x, y \in N.$$

From the last relation we can easily obtain that

$$D(y)N[y, z] = \{0\}, \quad \text{for all } \ y, z \in N,$$

The substitution $D(x)$ for y in the above relation gives

$$D^2(x)N[D(x), z] = \{0\}, \quad \text{for all } \ x, z \in N,$$

The last identity is same as the identity (2.3) in the proof of Theorem 2.3. Therefore, we can easily conclude the desired result here by using the same techniques as used in the proof of Theorem 2.3. Hence, the proof is completed. □

The following example demonstrates that the 3-primeness of N in the above theorems can not be omitted.

Example 2.5. Let $N = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in s \right\}$, where S is a 2-torsion free left nearring. Define $D : N \longrightarrow N$ by $D \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be seen easily that N is a zero-symmetric left nearring which is not 3-prime and the maps d is a non derivation on N satisfying all the requirements of Theorems 2.3 & 2.4. However, N is not a commutative ring.

Theorem 2.6. *Let N be a 2-torsion 3-prime nearrings. Then N admits no nonzero derivation D such that $D(x) \circ y = 0$ for all $x, y \in N$.*

Proof. Assume that D is a nonzero derivation in R. Then, from the hypothesis we have

$$D(x) \circ y = 0, \text{ for all } x, y \in N. \tag{2.5}$$

The substitution xy for x in (2.5) gives because of Lemma 2.2

$$xD(y)y + yxD(y) = 0, \text{ for all } x, y \in N. \tag{2.6}$$

Replace x with xz in (2.6) and using (2.6) to obtain

$$yzxD(y) - zyxD(y) = 0, \text{ for all } x, y, z \in N.$$

It follows from Lemma 2.2 that

$$[y, z]xD(y) = 0, \text{ for all } x, y, z \in N,$$

which leads to

$$[y, z]ND(y) = \{0\}, \text{ for all } y, z \in N,$$

Again replacing y by $D(x)$ in the last relation, we have

$$[D(x), z]ND^2(x) = \{0\}, \text{ for all } y, z \in N,$$

The last equality is same as (2.3) in the proof of Theorem 2.3. Hence, an application of Theorem 2.3 assures that N is commutative. In this situation, our hypothesis becomes

$$2D(x)y = 0, \text{ for all } x, y \in N.$$

Since N is 2-torsion free, we have

$$D(x)y = 0, \text{ for all } x, y \in N.,$$

which leads to

$$D(x)zy = 0, \text{ for all } x, y, z \in N.$$

Also,

$$D(x)Ny = \{0\}, \text{ for all } x, y \in N.$$

In the light of 3-primeness of N, we have that $N = \{0\}$, a contradiction. Hence, this completes the proof. □

Proceeding along the same line with necessary variation, we can prove the following theorem:

Theorem 2.7. *Let N be a 2-torsion 3-prime nearrings. Then N admits no nonzero derivation D such that $D(x) \circ D(y) = 0$ for all $x, y \in N$.*

Theorem 2.8. *Let N be a 2-torsion 3-prime nearrings. If N admits a nonzero derivation D such that $D(x) \circ y = x \circ y$ for all $x, y \in N$, then $Z(N) = 0$ and N is not a ring.*

Proof. We have that

$$D(x) \circ y = x \circ y, \text{ for all } x, y \in N. \tag{2.7}$$

The substitution xz for x in (2.7), where $z \in Z(N)$ gives because of Lemma 2.2

$$yxD(z) + xD(z)y = 0, \quad \text{for all} \quad x, y \in N.$$

Since $D(Z(N)) \subset Z(N)$ and from hypothesis, we have

$$D(z)N(D(x) \circ y) = \{0\}, \text{ for all } x, y \in N. \tag{2.8}$$

In view of 3-primeness of N, the relation (2.8) gives us

$$D(Z(N)) = 0 \ \ or \ \ D(x) \circ y = 0, \text{ for all } x, y \in N.$$

In the latter case, $D(x) \circ y = 0$ for all $x, y \in N$,so this case cannot be occur by Theorem 2.6. Hence, this completes the proof. □

The conclusion of Theorem 2.8 remains valid if we add the condition $Z(N) \neq 0$. In fact, we obtain the following result.

Theorem 2.9. *Let N be a 2-torsion 3-prime nearrings with $Z(N) \neq \{0\}$. Then N admits no nonzero derivation D such that $D(x) \circ y = x \circ y$ for all $x, y \in N$.*

Using similar techniques to those used in Theorem 2.6, we can prove the following.

Theorem 2.10. *Let N be a 2-torsion 3-prime nearrings with $Z(N) \neq \{0\}$. Then N admits no nonzero derivation D such that*

(i) $D(x) \circ y = x \circ D(y)$ *for all* $x, y \in N$.

(ii) $D(x) \circ D(y) = D(x) \circ y$ *for all* $x, y \in N$.

(ii) $D(x) \circ D(y) = x \circ y$ *for all* $x, y \in N$.

For completeness of we conclude this section by giving an example which shows that the restrictions in the speculations of above results are not superfluous.

Example 2.11. Let $N = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in S \right\}$, where S is a 2-torsion free left nearring. We define the following map:

$D\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be seen easily that D is a nonzero derivation on N. It is straightforward to check that D accomplishes all the requirements of Theorem 2.6, 2.7, 2.8, 2.9 and Theorem 2.10, but neither $Z(N) = 0$ nor $D = 0$.

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Author information

Mohammad Mueenul Hasnain, Department of Mathematics, Kamala Nehru College, University of Delhi, New Delhi-110 049, INDIA.
E-mail: mhamu786@gmail.com

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