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IDENTITIES INVOLVING PARTIAL DERIVATIVES OF BIVARIATE B-q BONACCI AND B-q LUCAS POLYNOMIALS

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Abstract. In this paper some identities of $(k, j)^{\text{th}}$ order partial derivatives of B-q bonacci and B-q Lucas polynomials with respect to x and y are introduced.

1 Introduction

The classical Fibonacci sequence is an unique and fascinating string of numbers with interesting properties. This sequence has been extended in many directions depending upon its recurrence relation as well as the seed values or initial values. In [2], we introduced two extensions of Fibonacci sequence and called them B-Fibonacci sequence and B-Tribonacci sequence. This is further extended to B-q bonacci sequence in [5]. Another interesting feature of study is to consider the polynomials associated with the Fibonacci sequence. In [6, 10, 11, 12], Fibonacci and Lucas polynomials in single variable are discussed, while in [8], bivariate Fibonacci and Lucas polynomials are introduced. In [1], we have extended these polynomials to bivariate B-Tribonacci polynomials and bivariate B-Tri Lucas polynomials defined respectively by

$$({}^tB)_{n+2}(x,y) = x^2({}^tB)_{n+1}(x,y) + 2xy({}^tB)_n(x,y) + y^2({}^tB)_{n-1}(x,y), \forall n \geq 1, \tag{1.1}$$

$$\text{with } ({}^tB)_0(x,y) = 0,\ ({}^tB)_1(x,y) = 0 \text{ and } ({}^tB)_2(x,y) = 1,$$

where the coefficients of the terms on right hand side of (1.1) are the terms of the binomial expansion of $(x+y)^2$ and $({}^tB)_n(x,y)$ is the n^{th} polynomial,

$$({}^tL)_{n+2}(x,y) = x^2({}^tL)_{n+1}(x,y) + 2xy({}^tL)_n(x,y) + y^2({}^tL)_{n-1}(x,y), \forall n \geq 1, \tag{1.2}$$

$$\text{with } ({}^tL)_0(x,y) = 0,\ ({}^tL)_1(x,y) = 2 \text{ and } ({}^tL)_2(x,y) = x^2,$$

where the coefficients of the terms on right hand side of (1.2) are the terms of the binomial expansion of $(x+y)^2$ and $({}^tL)_n(x,y)$ is the n^{th} polynomial. Further extensions of this idea can be seen in [3, 4].

In [7], the second derivative of Fibonacci and Lucas polynomials are introduced. The k^{th} derivative of Fibonacci and Lucas polynomials are discussed in [12], where as in [8], it is further extended to $(k, j)^{th}$ order derivative of bivariate Fibonacci polynomials and bivariate Lucas polynomials. The identities of $(k, j)^{th}$ order derivative of bivariate B-Tribonacci and B-Tri Lucas polynomials are studied in [1].

In this paper we extend (1.1) and (1.2) to bivariate B-q bonacci and B-q Lucas polynomials respectively, where $q \geq 2$ is any natural number. We also study the $(k, j)^{th}$ order partial derivatives of these polynomials with respect to x and y.

2 Bivariate B-q bonacci and Bivariate B-q Lucas Polynomials

In this section we define bivariate B-q bonacci polynomials and bivariate B-q Lucas polynomials and obtain some identities related to their first order partial derivatives with respect to x and y.

Definition 2.1. Let $q \geq 2$ be any natural number. The bivariate B-q bonacci polynomials are defined by

$$({}^{q}B)_{n+q-1}(x,y) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ x^{q-1-r}\ y^{r}\ ({}^{q}B)_{n+q-2-r}(x,y),\ \forall n \geq 1, \tag{2.1}$$

with $({}^{q}B)_{i}(x,y) = 0,\ i = 0,1,2,3,\cdots,q-2$ and $({}^{q}B)_{q-1}(x,y) = 1$.

Here the coefficients of the terms on the R.H.S. are the terms of the binomial expansion of $(x+y)^{q-1}$, $({}^{q}B)_{n}(x,y)$ is the n^{th} polynomial and $p^{\underline{s}}$ is p to the s falling factorial [9].

With $q=2$, (2.1) reduces to (1) of [8] and $q=3$, it reduces to (1.1) above.

Some polynomials defined by (2.1) are listed below:

$({}^{q}B)_{q-1}(x,y) = 1$, $({}^{q}B)_{q}(x,y) = x^{q-1}$, $({}^{q}B)_{q+1}(x,y) = x^{2(q-1)} + (q-1)x^{q-2}\ y$,

and $({}^{q}B)_{q+2}(x,y) = x^{3(q-1)} + 2(q-1)\ x^{2q-3}\ y + \frac{(q-1)(q-2)}{2}\ x^{q-3}y^{2}$.

Following equation gives the n^{th} term of (2.1). We state the result without proof.

Theorem 2.2. *The n^{th} term of (2.1) is given by*

$$({}^{q}B)_{n}(x,y) = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!}\ x^{(q-1)\left(n-(q-1)\right)-qr}\ y^{r}, \tag{2.2}$$

$\forall n \geq q-1$.

With $q=2$, (2.2) reduces to equation (4) of [8] and $q=3$, it gives the n^{th} term of (1.1) above. We list it below.

$$({}^{t}B)_{n}(x,y) = \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{\Big(2n-4-2r\Big)^{\underline{r}}}{r!}\ x^{2n-4-3r}\ y^{r},\ \forall n \geq 2.$$

For simplicity, let us denote $({}^{q}B)_{n}(x,y)$ by $({}^{q}B)_{n}$. We prove below the results related to first order partial derivatives of $({}^{q}B)_{n}$ with respect to x and y.

Theorem 2.3. *For all $n \geq 0$,*

(i) $qy\ \frac{\partial}{\partial x}[({}^{q}B)_{n}]\ + x\ \frac{\partial}{\partial x}[({}^{q}B)_{n+1}]\ =\ (q-1)(n-(q-2))({}^{q}B)_{n+1}$.

(ii) $\frac{\partial}{\partial x}[({}^{q}B)_{n}]\ =\ \frac{\partial}{\partial y}[({}^{q}B)_{n+1}]$.

(iii) $qy\ \frac{\partial}{\partial y}[({}^{q}B)_{n}]\ + x\ \frac{\partial}{\partial y}[({}^{q}B)_{n+1}]\ =\ (q-1)(n-(q-1))({}^{q}B)_{n}$.

(iv) $qy\ \frac{\partial}{\partial y}[({}^{q}B)_{n}]\ + x\ \frac{\partial}{\partial x}[({}^{q}B)_{n}]\ =\ (q-1)(n-(q-1))({}^{q}B)_{n}$.

Proof. (i) Note that equation (2.1) implies, for $0 \leq n \leq q-2$, L.H.S.= 0 = R.H.S.

Let $n \geq q-1$ and take $n = qm$. Using (2.2) and L.H.S. of (i), we have

$qy\ \frac{\partial}{\partial x}[({}^{q}B)_{qm}]\ + x\ \frac{\partial}{\partial x}[({}^{q}B)_{qm+1}]$

$= (q-1)\big(qm-(q-2)\big)x^{(q-1)(qm-(q-2))} + \sum_{r=1}^{(q-1)m-(q-2)} \Big[\ qr\frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r}}}{r!}$

$$+\frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r+1}}}{r!}\Big]x^{(q-1)(qm-(q-2)-r)-r}\ y^r$$

$$= (q-1)\big(qm-(q-2)\big)$$

$$\sum_{r=0}^{(q-1)m-(q-2)} \frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r}}}{r!}\ x^{(q-1)(qm-(q-2)-r)-r}y^r$$

$$= (q-1)(qm-(q-2))({}^qB)_{qm+1}.$$

Therefore, the result is true for $n = qm$.

Similarly, the result can be proved for $n = qm+1, \cdots, qm+q-1$. Hence (i) is proved.

Identity (ii) can be verified by differentiating $({}^qB)_n$ and $({}^qB)_{n+1}$ respectively with respect to x and with respect to y. Identity (iii) can be proved using Identities (i) and (ii). Identity (iv) can be deduced from (ii) and (iii).

Definition 2.4. Let $q \geq 2$, be any natural number. We define the bivariate B-q Lucas polynomials by

$$({}^qL)_{n+q-1}(x,y) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ \ x^{q-1-r}\ y^r\ ({}^qL)_{n+q-2-r}(x,y), \forall n \geq 1, \tag{2.3}$$

with $({}^qL)_i(x,y) = 0,\ i = 0,\ 1,\ 2,\ \cdots,\ q-3,\ ({}^qL)_{q-2}(x,y) = 2$ and $({}^qL)_{q-1}(x,y) = x^{q-1}$, where the coefficients of the terms on the R.H.S. are the terms of the binomial expansion of $(x+y)^{q-1}$ and $({}^qL)_n(x,y)$ is the n^{th} polynomial.

With $q = 2$, (2.3) reduces to (2) of [8] and $q = 3$, it reduces to (1.2) above.

We list below few polynomials defined by (2.3).

$({}^qL)_{q-2}(x,y) = 2, ({}^qL)_{q-1}(x,y) = x^{q-1}, ({}^qL)_q(x,y) = x^{2(q-1)} + 2(q-1)x^{q-2}\ y,$

and $({}^qL)_{q+1}(x,y) = x^{3(q-1)} + 3(q-1)x^{2q-3}\ y + (q-1)(q-2)x^{q-3}y^2.$

For simplicity, we denote $({}^qL)_n(x,y)$ by $({}^qL)_n$. We state the theorem related to n^{th} term of (2.3).

Theorem 2.5. *The n^{th} term of (2.3) is given by*

$$\begin{aligned}
&({}^qL)_n \\
&= \sum_{r=0}^{p}\Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)}\frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big]x^{(q-1)\big(n-(q-2)\big)-qr}y^r \\
&\quad -\sum_{r=2}^{p}\Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big]x^{(q-1)\big(n-(q-2)\big)-qr}y^r,
\end{aligned}\tag{2.4}$$

$\forall n \geq q-1$, *where* $p = \lfloor\frac{(q-1)(n-(q-2))}{q}\rfloor$.

In particular for $q = 3$, the n^{th} term of (2.3) defined above is given by
$({}^tL)_n$

$$= \sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor}\Big(\frac{(2n-2)}{(2n-2-2r)}\frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!}\Big)\ x^{2n-2-3r}y^r, \forall n \geq 2.$$

Note that for $q = 2$, (2.4) reduces to (5) of [8].

Following theorem gives the relation between bivariate B-q bonacci and bivariate B-q Lucas polynomials.

Theorem 2.6. *For all* $n \geq q-1$,

$$({}^qL)_n = ({}^qB)_{n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ x^{q-1-r} \ y^r \ ({}^qB)_{n-r}. \tag{2.5}$$

Proof. We prove the theorem by principle of mathematical induction on n. Note that (2.5) is true for $n = q-1$.

Assume that the result is true for $n \leq m$. Then,

$$\begin{aligned}({}^qL)_{m+1} &= \textstyle\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ x^{q-1-r} \ y^r \ ({}^qL)_{m-r} \\ &= \textstyle\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ x^{q-1-r} \ y^r \left[({}^qB)_{m+1-r} + \sum_{s=1}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \ x^{q-1-s} \ y^s ({}^qB)_{m-r-s}\right] \\ &= ({}^qB)_{m+2} + \textstyle\sum_{s=1}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} x^{q-1-s} \ y^s \ ({}^qB)_{m+1-s}.\end{aligned}$$

Hence the result follows.

Following result follows immediately.

Corollary 2.7.

$$({}^qL)_n = 2 \ ({}^qB)_{n+1} - x^{(q-1)} ({}^qB)_n, \forall \ n \geq q-2. \tag{2.6}$$

Proof. Note that $2 \ ({}^qB)_{q-1} - x^{(q-1)} ({}^qB)_{q-2} = 2 = ({}^qL)_{q-2}$. Hence, equation (2.6) is true for $n = q-2$. For $n \geq q-1$, the result can be proved using equation (2.1) and Theorem 2.6.

We prove below the identities related to first order partial derivatives of $({}^qL)_n$ with respect to x and y.

Theorem 2.8. *For all* $n \geq 0$,

(i) $qy \ \frac{\partial}{\partial x}[({}^qL)_n] \ + x \ \frac{\partial}{\partial x}[({}^qL)_{n+1}]$

$$= \ (q-1)(n-(q-3))({}^qL)_{n+1} - q(q-1)x^{q-2}y({}^qB)_n.$$

(ii) $\frac{\partial}{\partial x}[({}^qL)_n] \ = \ \frac{\partial}{\partial y}[({}^qL)_{n+1}] - (q-1)x^{q-2}({}^qB)_n.$

(iii) $qy \ \frac{\partial}{\partial y}[({}^qL)_n] \ + x \ \frac{\partial}{\partial y}[({}^qL)_{n+1}]$

$$= \ (q-1)(n-(q-1))({}^qL)_n + 2 \ (q-1)({}^qB)_{n+1}.$$

(iv) $qy \ \frac{\partial}{\partial y}[({}^qL)_n] \ + x \ \frac{\partial}{\partial x}[({}^qL)_n] = (q-1)(n-(q-2))({}^qL)_n.$

Proof. (i) Note that equation (2.3) implies, for $0 \leq n \leq q-3$, L.H.S.= 0 = R.H.S.

For $n \geq q-2$, we have $({}^qL)_n = 2 \ ({}^qB)_{n+1} - x^{(q-1)}({}^qB)_n$, (from Corollary 2.7).

Differentiating both sides with respect to x, we get

$$\frac{\partial}{\partial x}[({}^qL)_n] = 2 \ \frac{\partial}{\partial x}[({}^qB)_{n+1}] - x^{(q-1)}\frac{\partial}{\partial x}[({}^qB)_n] - (q-1)x^{q-2}({}^qB)_n.$$

Also,

$$\frac{\partial}{\partial x}[({}^qL)_{n+1}] = 2 \ \frac{\partial}{\partial x}[({}^qB)_{n+2}] - x^{(q-1)}\frac{\partial}{\partial x}[({}^qB)_{n+1}] - (q-1)x^{q-2}({}^qB)_{n+1}.$$

Thus, using (i) of Theorem 2.3, we get

$$qy\ \frac{\partial}{\partial x}[({}^{q}L)_n]\ +x\ \frac{\partial}{\partial x}[({}^{q}L)_{n+1}]$$

$$= 2(q-1)(n-(q-3))({}^{q}B)_{n+2} - x^{q-1}(q-1)(n-(q-3))({}^{q}B)_{n+1}$$

$$-x^{q-1}(q-1)({}^{q}B)_{n+1} - (q-1)x^{q-2}\big(qy\ ({}^{q}B)_n + x\ ({}^{q}B)_{n+1}\big)$$

$$= (q-1)(n-(q-3)({}^{q}L)_{n+1} - q(q-1)y\ x^{q-2}({}^{q}B)_n.$$

Similarly, the other identities can be proved.

3 Main Results

In this section, we prove some identities involving k^{th} order partial derivative with respect to x and j^{th} order partial derivative with respect y of bivariate polynomials $({}^{q}B)_n$ and $({}^{q}L)_n$, where $k, j \geq 0$.

Let $(.)^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}[(.)]$, $(.)^{(s,0)} = \frac{\partial^s}{\partial x^s}[(.)]$ and $(.)^{(0,p)} = \frac{\partial^p}{\partial y^p}[(.)]$.

We have the following identities.

Theorem 3.1. *For all $n \geq q-1$,*

(i) $({}^{q}L)_n^{(k,j)} = ({}^{q}B)_{n+1}^{(k,j)}$
$+\sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{p}}}{p!} (x^{q-1-r})^{(s,0)} (y^r)^{(0,p)} ({}^{q}B)_{n-r}^{(k-s,j-p)}.$

(ii) $({}^{q}B)_n^{(k,j)}$
$= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{t}}}{p!} (x^{q-1-r})^{(s,0)} (y^r)^{(0,p)} ({}^{q}B)_{n-1-r}^{(k-s,j-p)}.$

(iii) $({}^{q}L)_n^{(k,j)}$
$= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{p}}}{p!} (x^{q-1-r})^{(s,0)} (y^r)^{(0,p)} ({}^{q}L)_{n-1-r}^{(k-s,j-p)}.$

(iv) $(q-1)(n-(q-2))({}^{q}B)_{n+1}^{(k,j)}$
$= qy\ ({}^{q}B)_n^{(k+1,j)} + qj\ ({}^{q}B)_n^{(k+1,j-1)} + x\ ({}^{q}B)_{n+1}^{(k+1,j)} + k({}^{q}B)_{n+1}^{(k,j)}.$

(v) $(q-1)(n-(q-1))({}^{q}B)_n^{(k,j)}$
$= qy\ ({}^{q}B)_n^{(k,j+1)} + qj\ ({}^{q}B)_n^{(k,j)} + x\ ({}^{q}B)_{n+1}^{(k,j+1)} + k({}^{q}B)_{n+1}^{(k-1,j+1)}.$

(vi) $({}^{q}B)_n^{(k+1,j)} = ({}^{q}B)_{n+1}^{(k,j+1)}.$

(vii) $(q-1)(n-(q-1))({}^{q}B)_n^{(k,j)}$
$= qy\ ({}^{q}B)_n^{(k,j+1)} + x({}^{q}B)_n^{(k+1,j)} + (k+jq)({}^{q}B)_n^{(k,j)}.$

(viii) $(q-1)(n-(q-3))({}^{q}L)_{n+1}^{(k,j)}$
$= qy\ ({}^{q}L)_n^{(k+1,j)} + qj\ ({}^{q}L)_{n+1}^{(k+1,j-1)} + x\ ({}^{q}B)_n^{(k+1,j)} + k({}^{q}B)_{n+1}^{(k,j)}$
$+ q(q-1)\sum_{s=0}^{q-2} \frac{k^{\underline{s}}}{s!} (x^{q-2})^{(s,0)} \left[y({}^{q}B)_n^{(k-s,j)} + j({}^{q}B)_n^{(k-s,j-1)}\right].$

(ix) $(q-1)(n-(q-1))({}^qL)_n^{(k,j)} + 2(q-1)({}^qB)_n^{(k,j)}$
$= qy\ ({}^qL)_n^{(k,j+1)} + qj\ ({}^qL)_n^{(k,j)} + x\ ({}^qL)_{n+1}^{(k,j+1)} + k({}^qL)_{n+1}^{(k-1,j+1)}.$

(x) $({}^qL)_n^{(k+1,j)} = ({}^qL)_{n+1}^{(k,j+1)} - (q-1)\sum_{s=0}^{q-2} \frac{k^{\underline{s}}}{s!}\ (x^{q-2})^{(s,0)}\ ({}^qB)_n^{(k-s,j)}.$

(xi) $(q-1)(n-(q-2))({}^qL)_n^{(k,j)}$
$= qy\ ({}^qL)_n^{(k,j+1)} + x({}^qL)_n^{(k+1,j)} + (k+jq)({}^qL)_n^{(k,j)}.$

Proof.

(i) Note that $({}^qL)_n = ({}^qB)_{n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ x^{q-1-r}\ y^r\ ({}^qB)_{n-r}$.

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^qL)_n^{(k,j)} = ({}^qB)_{n+1}^{(k,j)} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \frac{\partial^k}{\partial x^k}\left(x^{q-1-r}\sum_{p=0}^{r} \frac{j^{\underline{p}}}{p!}\ (y^r)^{(0,p)}\ ({}^qB)_{n-r}^{(0,j-p)}\right)$$

$$= ({}^qB)_{n+1}^{(k,j)}$$

$$+ \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ \sum_{s-0}^{q-1-r} \frac{k^{\underline{s}}}{s!}\ \left(x^{q-1-r}\right)^{(s,0)}\ \sum_{p=0}^{r} \frac{j^{\underline{p}}}{p!}\ (y^r)^{(0,p)}\ ({}^qB)_{n-r}^{(k-s,j-p)}$$

$$= ({}^qB)_{n+1}^{(k,j)}$$

$$+ \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{p}}}{p!}\ \left(x^{q-1-r}\right)^{(s,0)}\ (y^r)^{(0,p)}\ ({}^qB)_{n-r}^{(k-s,j-p)}.$$

Hence (i) is proved.

(ii) We have from (2.1), $({}^qB)_n = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ x^{q-1-r}\ y^r\ ({}^qB)_{n-1-r}$.

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^qB)_n^{(k,j)} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \frac{\partial^k}{\partial x^k}\left(x^{q-1-r}\ \sum_{p=0}^{r} \frac{j^{\underline{p}}}{p!}\ (y^r)^{(0,p)}\ ({}^qB)_{n-1-r}^{(0,j-p)}\right)$$

$$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ \sum_{s=0}^{q-1-r} \frac{k^{\underline{s}}}{s!}\ \left(x^{q-1-r}\right)^{(s,0)}\ \sum_{p=0}^{r} \frac{j^{\underline{p}}}{p!}\ (y^r)^{(0,p)}\ ({}^qB)_{n-1-r}^{(k-s,j-p)}$$

$$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ \sum_{s-0}^{q-1-r} \sum_{p=0}^{r} \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{p}}}{p!}\ \left(x^{q-1-r}\right)^{(s,0)}\ (y^r)^{(0,p)}\ ({}^qB)_{n-1-r}^{(k-s,j-p)}.$$

Hence (ii) is proved.

Similarly, we can prove the identity (iii). Using Leibnitz theorem for derivatives, (iv), (v), (vi) and (vii) can be obtained by differentiating identities (i), (ii), (iii) and (iv) of Theorem 2.3 respectively on both sides, k times with respect to x and j times with respect to y. Identities (viii), (ix),(x) and (xi) can be proved using similar procedure.

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