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Common fixed point theorems for generalized TAC contraction condition in b-metric spaces

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Abstract. In this paper we obtain common fixed point theorems for four self maps using generalized TAC contractive condition in b-metric spaces. These results generalize the results of Abbas and Doric[1], Roshan , Shobkolaei , Sedghi and Abbas[20] and extend the results of Babu and Dula[9] to four mappings. To support our results some illustrative examples are also furnished

1 Introduction

Now a days there are too many generalizations of metric spaces like b- metric spaces, quasi-metric space, quasi-b-metric space, dislocated metric space (or metric-like space), dislocated b-metric space (or b-metric-like space), dislocated quasi-metric space (or quasi- metric-like space), dislocated quasi-b-metric space (or quasi-b-metric-like space). For instance, we refe[2,7,13,16, 18-21, 23-26].

In 1997, Alber and Guerre-Delabrere[4] proved that a weakly contractive map defined on a Hilbert space is a Picard operator. Rhoades[22] extended this result considering the domain of the mapping a complete metric space. Dutta and Choudhury[14] introduced (ψ, φ)-weakly contractive maps and proved fixed point theorems in complete metric spaces. In continuation , in 2010 Abbas and Doric[1] proved a common fixed point theorem for four maps for a generalized (ψ, φ)-weakly contractive map. Recently, Chandok, Tas and Ansari[12] introduced the concept of TAC- contractions and proved some fixed point theorems in the setting of metric spaces. In sequel, Babu and Dula[9] extended this result to b-metric spaces.

We start by recalling some definitions and properties of b-metric spaces and well known results.

Definition 1.1. [11] Let X be a non-empty set and $s \geq 1$ be a real number. A function $d : X \times X \to [0, \infty)$ is called a *b-metric on* X if it satisfies the following conditions:

(1) $d(x, y)$=0 if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.

Then the order pair (X, d) is said to be a b- metric space with $s \geq 1$.

Here we note that the class of b-metric spaces is larger class than the class of metric spaces, since (X, d) is a metric space when $s = 1$.

In the following we give examples of b-metric which are not metric spaces.

Example 1.2. Let $X = R^2$ and we define $d : X \times X \to R$ by $d(x, y) = |x_1 - y_1|^2 + |x_2 - y_2|^2$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then (X, d) is a b-metric space with $s = 3$.

Example 1.3. Let $X = \{0, 1, 2\}$. Define $d : X \times X \to R$ by $d(x, x) = 0$ for all $x \in X$, $d(0, 1) = d(1, 0) = 1$, $d(1, 2) = d(2, 1) = 2$, $d(0, 2) = d(2, 0) = 6$. Then clearly, d is a b-metric space with $s = 2$. But, (X, d) is not a metric space. For, let $x = 0,\ y = 2,\ z = 1$ then

$$d(0, 2) = 6 > d(0, 1) + d(1, 2) = 1 + 2.$$

Hence (X, d) is not a metric space.

Definition 1.4. [11] Let (X, d) be b-metric space.

(i) A sequence $\{x_n\}$ in X is called *b-convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. In this we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called *b-Cauchy* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) The b-metric space (X, d) is said to be *b-complete* if every b-Cauchy sequence in X is b-convergent.

(iv) A set $B \subseteq X$ is said to be *b-closed* if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$, we have $z \in B$.

Proposition 1.5. [11] *In a b-metric space (X, d) the following assertions hold:*

(i) *a b-convergent sequence has a unique limit*

(ii) *each b-convergent sequence is b-Cauchy*

(iii) *in general, a b-metric need not be continuous.*

Lemma 1.6. [3] *Let (X, d) be a b-metric space and* $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y$. *Then*

(i) $\frac{1}{s^2} d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x, y)$.
In particular, if $x = y$ then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

(ii) *For each $x \in X$*
$\frac{1}{s} d(x, z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le sd(x, z)$.

Lemma 1.7. [16] *Let (X, d) be a b-metric space with $s \ge 1$ and $\{x_n\}$ be a sequence in (X, d). Then the following are equivalent.*

(i) $\{x_n\}$ *is a b-Cauchy sequence in* (X, d)

(ii) $\{x_{2n}\}$ *is a b-Cacuchy sequence in* (X, d) *and* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Definition 1.8. [16] Let A and B be nonempty subsets of X. A mapping $f : A \cup B \to A \cup B$ is said to be *cyclic* if $f(A) \subset B$ and $f(B) \subset A$.

Definition 1.9. [5] Let X be a nonempty set, f be a selfmap of X and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that f is a *cyclic* (α, β)*-admissible mapping* if

(i) for any $x \in X$ with $\alpha(x) \ge 1 \Rightarrow \beta(fx) \ge 1$, and

(ii) for any $y \in X$ with $\beta(y) \ge 1 \Rightarrow \alpha(fy) \ge 1$.

In 2016, Hussain , Isik and Abbas[17] extended the definition of cyclic (α, β)-admissible mapping two pair of maps as follows.

Definition 1.10. Let f, g, S and T be selfmaps of a nonempty set X and $\alpha,\ \beta : X \to R^+$. Then the pair (f, g) is called *cyclic* (α, β)*-admissible with respect to* (S, T) (briefly, (f, g) is cyclic (α, β) (S,T)-admissible pair) if

(i) $\alpha(Sx) \ge 1$ for some $x \in X$ implies $\beta(fx) \ge 1$,

(ii) $\beta(Tx) \ge 1$ for some $x \in X$ implies $\alpha(gx) \ge 1$.

If we take $S = T = IX$(identity mapping on X) and $f = g$, then Definition 1.10 reduces to Definition 1.9.

Recently, Ansari [6] defined the concept of C-class functions in the following.

Definition 1.11. [6] A mapping $F : R^+XR^+ \to R^+$ is called $\mathcal{C}$-*class function* if it is continuous and satisfies following conditions:

(1) $F(s,t) \leq s$;

(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in R^+$.

Here we note that $F(0,0) = 0$.

We denote the set of C-class functions as $\mathcal{C}$.

Example 1.12. [6] The following functions $F : R^+XR^+ \to R^+$ are elements of $\mathcal{C}$, for all $s,t \in R^+$:

$$(1)F(s,t) = \begin{cases} s-t & if\ s \geq t \\ 0 & otherwise \end{cases}$$

(2) $F(s,t) = ks$ for $0 < k < 1$, if $F(s,t) = s$ then $s = 0$;

(3) $F(s,t) = \frac{k}{r}s$ for $0 < k < 1$ and $r \in (1,\infty)$ if $F(s,t) = s$ then $s = 0$;

(4) $F(s,t) = \frac{s}{1+t}$ then if $F(s,t) = s$ then either $s = 0$ or $t = 0$.

For more literature on $\mathcal{C}$ class functions we refer [8, 15].

Notation: Throught this paper we denote:

$\Psi = \{\psi : [0,\infty) \to [0,\infty) | \psi\ is\ continuous,\ nondecreasing\ and\ \psi^{-1}(0) = 0\}$,

$\Phi = \{\phi : [0,\infty) \to [0,\infty) | \lim_{n\to\infty} \phi(t_n) \to 0 \Rightarrow \lim_{n\to\infty} t_n = 0\}$,

Here we observe that if $\phi \in \Phi$, then $t = 0$ implies $\phi(t) = 0$

$\Phi_1 = \{\phi : [0,\infty) \to [0,\infty) | \phi$ is lower semicontinuous, $\phi(t) > 0$ for all $t > 0$, $\phi(0) = 0\}$,

$C(f,g)$: set of all common fixed points of f and g and $W = \{0,1,2,3,.....\}$.

The following theorem was proved by Abbas and Doric[1] in complete metric spaces.

Theorem 1.13. [1] *Let f,g,S and T be selfmaps of a complete metric space (X,d) . Suppose that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible. If*

$$\psi(d(fx,gy)) \leq \psi(M(x,y)) - \phi(M(x,y)) \tag{1.13.1}$$

for all $x,y \in X$, where $\psi \in \Psi, \phi \in \Phi_1$ and

$M(x,y) = max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\}$

then f,g,S and T have a unique fixed point in X provided one of the ranges $f(X), g(X), S(X)$ and $T(X)$ is closed.

The following TAC type contractive definition is due to Chandok and Ansari[12].

Definition 1.14. [12] Let (X,d) be a metric space and let $\alpha,\beta : X \to [0,\infty)$ be two given mappings. We say that $T : X \to X$ is a *TAC-contractive mapping* if $for\ all\ x,y \in X$ with

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq F(\psi(d(x,y), \phi(d(x,y))) \tag{1.14.1}$$

where $\psi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{C}$.

Recently, Babu and Dula [9] introduced the notion of generalized TAC-contractive map in b-metric space setting and proved fixed point theorems.

Definition 1.15. [9] Let (X, d) be a b- metric space and let $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that $T : X \to X$ is a *generalized TAC-contractive map* if there exist $\psi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{C}$ such that $for\ all\ x, y \in X$ with

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(s^3 d(Tx, Ty)) \leq F(\psi(M_s(x, y), \phi(M_s(x, y)), \tag{1.15.1}$$

where $M_s(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx)+d(y,Ty)}{2s}\}$.

Babu and Dula[9] established the following result.

Theorem 1.16. [9] *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a seflmap of X. Assume that there exist two mappings $\alpha, \beta : X \to [0, \infty)$, $\psi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{C}$ such that T is a generalized TAC-contractive mapping. Further, suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, T is a cyclic (α, β)- admissible mapping and either of the following conditions hold:*

(i) *T is continuous,*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ $\alpha(x_n) > 1$ and $\beta(x_n) > 1$ for all n, then $\alpha(z) \geq 1$ and $\beta(z) \geq 1$.*

Then T has a fixed point in X. Moreover, if $\alpha(u) \geq 1$ and $\beta(u) \geq 1$ whenever $Tu = u$. Then T has a unique fixed point in X.

The following theorem was proved by Roshan, Shobkolaei, Sedghi and Abbas[20].

Theorem 1.17. [20] *Suppose that f, g, S and T are self mappings on a complete b-metric space (X, d) with $s \geq 1$ such that:*

(i) $f(X) \subseteq T(X), g(X) \subseteq S(X)$.

$$(ii)\ d(fx, gy) \leq \frac{q}{s^4} max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \tag{1.17.1}$$

holds for each $x, y \in X$ with $0 < q < 1$. Then f, g, S and T have a unique common fixed point in X provided that S and T are continuous and pairs (f, S) and (g, T) are compatible.

The aim of the paper is to extend Theorem 1.16 to four mappings and generalize Theorem 1.13 and Theorem 1.17. To support our results examples are also furnished.

2 Fixed point theorems for generalized TAC-contractive map for four selfmaps

In this section, first we define a generalized TAC-contractive map for four selfmaps.

Definition 2.1. Let (X, d) be a b- metric space with coefficient $s \geq 1$. Let $\alpha, \beta : X \to [0, \infty)$ be two given maps and f, g, S and T be four seflmaps on X. Suppose there exist $\psi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x, y \in X\ with$

$$\alpha(Sx)\beta(Ty) \geq 1 \Rightarrow \psi(s^3 d(fx, gy)) \leq F(\psi(M_s(x, y)), \phi(M_s(x, y))), \tag{2.1.1}$$

where $M_s(x, y) = max\{d(Sx, Ty), d(fx, Sx), d(Ty, gy), \frac{d(Sx,gy)+d(fx,Ty)}{2s}\}$.

Then the pair (f, g) is said to be *generalized TAC-(S, T) contractive map* in b- metric spaces.

Here we note that if we choose $f = g$ and $S = T = I$, the identity map on X then Definition 2.1 reduces to Definition 1.15.

Theorem 2.2. *Let (X, d) be a complete b- metric space with coefficient $s \geq 1$ and f, g, S and T be four seflmaps on X. Assume that there exist two mappings $\alpha, \beta : X \to [0, \infty)$, $\psi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{C}$ such that (f, g) is a generalized TAC-(S, T) contractive mapping with respect to F. Assume that:*

(i) $fX \subseteq TX$ and $gX \subseteq SX$

(ii) there exists $x_0 \in X$ such that $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$.

(iii) If $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all n, then $\alpha(x) \geq 1, \beta(x) \geq 1$.

(iv) one of the ranges fX, gX, TX, SX is b-closed .

Then $C(f,S) \neq \phi$ and $C(g,T) \neq \phi$.

Proof. Let $x_0 \in X$ as in (ii). By condition (i), we define a sequence $\{y_n\} \in X$ by

$$y_{2n} = fx_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+2} = gx_{2n+1}. \tag{2.2.1}$$

First we show that $\{y_n\}$ is a Cauchy sequence in X.

Since $\alpha(Sx_0) \geq 1$ and (f,g) is cyclic (α, β) -admissible with respect to (S,T), we have $\beta(fx_0) \geq 1 \Rightarrow \beta(Tx_1) \geq 1$, $\alpha(gx_1) \geq 1$ and $\beta(Sx_2) \geq 1$.

On continuing this process, we have

$$\alpha(Sx_{2n}) \geq 1 \; and \; \beta(Tx_{2n+1}) \geq 1 \; for \; all \; n \in W. \tag{2.2.2}$$

Similarly, $\beta(Tx_0) \geq 1$, we have

$$\beta(Tx_{2n}) \geq 1 \; and \; \alpha(Sx_{2n+1}) \geq 1 \; for \; all \; n \in W. \tag{2.2.3}$$

Thus from (2.2.2) and (2.2.3), we have

$$\alpha(Sx_n) \geq 1 \; and \; \beta(Tx_n) \geq 1 \; for \; all \; n \in W. \tag{2.2.4}$$

.

If $y_{2n} = y_{2n+1}$ for some $n \in W$ then we have

$$\begin{aligned} M_s(x_{2n+2}, x_{2n+1}) &= max\{d(Sx_{2n+2}, Tx_{2n+1}), d(fx_{2n+2}, Sx_{2n+2}), d(Tx_{2n+1}, gx_{2n+1}), \\ &\qquad \tfrac{1}{2s}[d(Sx_{2n+2}, gx_{2n+1}) + d(fx_{2n+2}, Tx_{2n+1})]\} \\ &= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), \tfrac{1}{2s} d(y_{2n+2}, y_{2n})\} \\ &\leq \; max \; \{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n}), \\ &\qquad \tfrac{s}{2s}[d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n+2})]\} \\ &\leq max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\} \; . \end{aligned}$$

Therefore $M_s(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$.

Now from (2.1.1) and (2.2.4), we have

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &\leq \psi(s^3 d(y_{2n+1}, y_{2n+2})) = \psi(s^3 d(fx_{2n+2}, gx_{2n+1})) \\ &\leq F(\psi(M_s(x_{2n+2}, x_{2n+1})), \phi(M_s(x_{2n+2}, x_{2n+1}))) \\ &\leq F(\psi(d(y_{2n+2}, y_{2n+1})), \phi(d(y_{2n+2}, y_{2n+1}))) \\ &= \psi(d(y_{2n+2}, y_{2n+1})) \end{aligned}$$

which implies $F(\psi(d(y_{2n+2}, y_{2n+1})), \phi(d(y_{2n+2}, y_{2n+1}))) = \psi(d(y_{2n+2}, y_{2n+1}))$.

Due to the property of F, we have $\psi(d(y_{2n+2}, y_{2n+1})) = 0$ or $\phi(d(y_{2n+2}, y_{2n+1})) = 0$, in any case $d(y_{2n+1}, y_{2n+2}) = 0$ this implies $y_{2n+1} = y_{2n} = y_{2n+2}$. On continuing this process we can prove that $y_{2n} = y_{2n+1} = y_{2n+2} = y_{2n+3} = ...$. Thus $y_{2n+1} = y_{2n}$ for all $n \in W$. Thus $\{y_k\}_{k \geq 2n}$ is a constant sequence hence it is convergent. Hence without loss of generality, assume that $y_{2n} \neq y_{2n+1}$ for all $n \in W$. First, we show that $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

In view of condition (2.2.4), we have $\alpha(Sx_{2n}) \geq 1$ and $\beta(Tx_{2n+1}) \geq 1$ implies $\alpha(Sx_{2n})\beta(Tx_{2n+1}) \geq 1$ for all $n \in W$.

Now on using inequality (2.1.1) with $x = x_{2n}$, $y = x_{2n+1}$, we have

$$\psi((d(y_{2n}, y_{2n+1})) \leq \psi(s^3 d(fx_{2n}, gx_{2n+1})) \leq F(\psi(M_s(x_{2n}, x_{2n+1})), \phi(M_s(x_{2n}, x_{2n+1}))). \quad (2.2.5)$$

Now, $M_s(x_{2n}, x_{2n+1}) = max\{d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(Tx_{2n+1}, gx_{2n+1}),$

$$\frac{1}{2s}[d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})]\}$$

$$= max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \tfrac{1}{2s} d(y_{2n-1}, y_{2n+1})\}$$

$$\leq max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}. \quad (2.2.6)$$

If $d(y_{2n}, y_{2n+1}) > d(y_{2n}, y_{2n-1})$ then from (2.2.6), we have

$$\psi((d(y_{2n}, y_{2n+1})) \leq F(\psi(d(y_{2n}, y_{2n+1}), \phi(d(y_{2n}, y_{2n+1})) \leq \psi((d(y_{2n}, y_{2n+1})). \quad (2.2.7)$$

Hence $F(\psi(d(y_{2n}, y_{2n+1}), \phi(d(y_{2n}, y_{2n+1})) = \psi((d(y_{2n}, y_{2n+1}))$.

Owing to the property of F, we have $\psi(d(y_{2n}, y_{2n+1})) = 0$ or $\phi(d(y_{2n}, y_{2n+1})) = 0$. In any case we have $y_{2n} = y_{2n+1}$, a contradiction to our assumption. Hence

$$d(y_{2n}, y_{2n+1}) < d(y_{2n}, y_{2n-1}) \ for \ all \ n \in W. \quad (2.2.8)$$

Therefore $\{d(y_{2n}, y_{2n+1})\}$ is a decreasing sequence of reals and hence it converges to $r \geq 0$. Suppose that $r > 0$. From (2.2.7), we have

$$\psi(d(y_{2n}, y_{2n+1})) \leq F(\psi(d(y_{2n}, y_{2n+1})), \phi(d(y_{2n}, y_{2n+1}))).$$

On letting $n \to \infty$, using continuity of ψ and F, we have

$$\psi(r) \leq F(\psi(r), lim_{n\to\infty}\phi(d(y_{2n}, y_{2n+1})) \leq \psi(r)$$

which implies $F(\psi(r), lim_{n\to\infty}\phi((d(y_{2n}, y_{2n+1})) = \psi(r)$. Thus, by the property of F, we have $\psi(r) = 0$ or $lim_{n\to\infty}\phi(d(y_{2n}, y_{2n+1})) = 0$, this implies $r = 0$. Hence

$$lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \quad (2.2.9)$$

We now show that $\{y_{2n}\}$ is a b-Cauchy sequence . If $\{y_{2n}\}$ is not a b-Cauchy sequence then by lemma 1.7 , there exist $\epsilon > 0$, and subsequences $\{y_{2m(k)}\}$, $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ where $m(k)$ is smallest integer such that $m(k) > n(k) \geq k$ and

$$d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon \ and \ d(y_{2m(k)-2}, y_{2n(k)}) < \epsilon. \quad (2.2.10)$$

Now from (2.2.10), we have

$$\epsilon \leq d(y_{2m(k)}, y_{2n(k)})$$
$$\leq sd(y_{2m(k)}, y_{2m(k)-2}) + sd(y_{2m(k)-2}, y_{2n(k)})$$
$$< s^2 d(y_{2m(k)}, y_{2m(k)-1}) + s^2 d(y_{2m(k)-1}, y_{2m(k)-2}) + s\epsilon.$$

Taking upper limit as $k \to \infty$, using (2.2.9), we get

$$\epsilon \leq \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) \leq s\epsilon. \quad (2.2.11)$$

Again,

$$d(y_{2m(k)}, y_{2n(k)+1}) \leq sd(y_{2m(k)}, y_{2n(k)}) + sd(y_{2n(k)+1}, y_{2n(k)}).$$

Taking upper limit as $k \to \infty$, using (2.2.9) and (2.2.11), we get

$$\limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)+1}) \leq s^2\epsilon. \quad (2.2.12)$$

Also, we have

$$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq sd(y_{2m(k)}, y_{2n(k)+1}) + sd(y_{2n(k)+1}, y_{2n(k)}).$$

Taking the upper limit as $k \to \infty$ and using (2.2.9), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)+1}). \quad (2.2.13)$$

Hence, from (2.2.12) and (2.2.13), it follows that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)+1}) \leq s^2\epsilon. \tag{2.2.14}$$

We also have

$d(y_{2m(k)-1}, y_{2n(k)}) \leq sd(y_{2m(k)-1}, y_{2m(k)}) + sd(y_{2m(k)}, y_{2n(k)})$.

On taking upper limit as $k \to \infty$ using (2.2.9) and (2.2.11), we get

$\limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) \leq s^2\epsilon.$ (2.2.15)

Also, we have

$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq sd(y_{2m(k)}, y_{2m(k)-1}) + sd(y_{2m(k)-1}, y_{2n(k)})$.

On taking the upper limit as $k \to \infty$ and using (2.2.9), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}). \tag{2.2.16}$$

Now, on combining (2.2.15) and (2.2.16), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) \leq s^2\epsilon. \tag{2.2.17}$$

In view of triangle inequality, we have

$$\begin{aligned} d(y_{2n(k)+1}, y_{2m(k)-1}) &\leq s[d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)-1})] \\ &\leq s[d(y_{2n(k)+1}, y_{2n(k)}) + sd(y_{2n(k)}, y_{2m(k)}) + sd(y_{2m(k)}, y_{2m(k)-1})]. \end{aligned}$$

Letting upper limit as $k \to \infty$, using (2.2.9) and (2.2.11), we get

$$\limsup_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) \leq s^3\epsilon. \tag{2.2.18}$$

Again,

$$\begin{aligned} \epsilon \leq d(y_{2m(k)}, y_{2n(k)}) &\leq sd(y_{2m(k)}, y_{2m(k)-1}) + sd(y_{2m(k)-1}, y_{2n(k)}) \\ &\leq sd(y_{2m(k)}, y_{2m(k)-1}) + s^2d(y_{2m(k)-1}, y_{2n(k)+1}) + s^2d(y_{2n(k)+1}, y_{2n(k)}). \end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.2.9) and (2.2.13), we get

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}). \tag{2.2.19}$$

Thus, from (2.2.20) and (2.2.21), we get

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) \leq s^3\epsilon. \tag{2.2.20}$$

From the condition (2.2.4),we have $\alpha(Sx_{2m(k)}) \geq 1$ and $\beta(Tx_{2n(k)+1}) \geq 1$, thus $\alpha(Sx_{2m(k)})\beta(Tx_{2n(k)+1}) \geq 1$, therefore from (2.1.1) , we have

$$\begin{aligned} \psi(s^3d(y_{2m(k)}, y_{2n(k)+1})) &= \psi(s^3d(fx_{2m(k)}, gx_{2n(k)+1})) \\ &\leq F(\psi(M_s(x_{2m(k)}, x_{2n(k)+1})), \phi(M_s(x_{2m(k)}, x_{2n(k)+1}))), \end{aligned} \tag{2.2.21}$$

where

$$\begin{aligned} M_s(x_{2m(k)}, x_{2n(k)+1}) = max\{&d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), \\ &d(y_{2n(k)}, y_{2n(k)+1}), \tfrac{1}{2s}[d(y_{2m(k)-1}, y_{2n(k)+1}) + d(y_{2m(k)}, y_{2n(k)})]\}. \end{aligned} \tag{2.2.22}$$

Letting limit supremum as $k \to \infty$ and using (2.2.11), (2.2.17) and (2.2.20), we have

$$\limsup_{k\to\infty} M_s(x_{2m(k)}, x_{2n(k)+1}) \leq max\{s^2\epsilon, \frac{1}{2s}(s^3\epsilon + s\epsilon)\} = s^2\epsilon. \tag{2.2.23}$$

Now from in (2.2.21), using (2.2.14) and (2.2.23), we have

$$\psi(s^2\epsilon) = \psi(s^3 \tfrac{\epsilon}{s}) \leq \psi(s^3 limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)+1}))$$
$$\leq F(\psi(\limsup_{k\to\infty} M_s(x_{2m(k)}, x_{2n(k)+1})), \phi(\limsup_{k\to\infty} M_s(x_{2m(k)}, x_{2n(k)+1})))$$
$$\leq F(\psi(s^2\epsilon), \limsup_{k\to\infty} \phi(M_s(x_{2m(k)}, x_{2n(k)+1})) \leq \psi(s^2\epsilon)$$

this implies that $F(\psi(s^2\epsilon), \limsup_{k\to\infty} \phi(M_s(x_{2m(k)}, x_{2n(k)+1})) = \psi(s^2\epsilon)$.

Hence by the property of F, we have either $\psi(s^2\epsilon) = 0$ or $\limsup_{k\to\infty} \phi(M_s(x_{2m(k)}, x_{2n(k)+1})) = 0$, this implies $s^2\epsilon = 0$ or $\limsup_{n\to\infty} M_s(x_{2m(k)}, x_{2n(k)+1}) = 0$. In both the cases we have $\epsilon = 0$ which is a contraction.Hence $\{y_{2n}\}$ is a b-Cauchy sequence in X. Thus by Lemma 1.7, we conclude that $\{y_n\}$ is a b-Cauchy sequence in X. Since (X, d) is b-complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_{2n} = z$. Therefore

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = z. \tag{2.2.24}$$

Case(i): Suppose SX is closed.

In view of (2.2.24), we have $z \in SX$, there exists $u \in X$ such that $z = Su$. From our assumption (iii) and (2.2.4), we have $\alpha(Su) \geq 1$ and $\beta(Tx_{2n+1}) \geq 1$. Now on using inequality (2.1.1), we have

$d(fu, z) \leq s[d(fu, gx_{2n+1}) + d(gx_{2n+1}, z)]$.

On taking upper limit as $n \to \infty$ in the above inequality and using (2.2.24), we have

$\frac{1}{s} d(fu, z) \leq \limsup_{n\to\infty} d(fu, gx_{2n+1})$.

Also, $d(fu, gx_{2n+1}) \leq s[d(fu, z) + d(z, gx_{2n+1})]$.

Taking limit supremum as $n \to \infty$ and again using (2.2.24), we get

$\limsup_{n\to\infty} d(fu, gx_{2n+1}) \leq s^2 d(fu, z)$.

Therefore

$$\psi(d(fu, z)) \leq \psi(s^2 d(fu, z)) = \psi(s^3(\tfrac{1}{s} d(fu, z))$$
$$\leq \psi(s^3 \limsup_{n\to\infty} d(fu, gx_{2n+1})$$
$$\leq F(\limsup_{n\to\infty} \psi(M_s(u, x_{2n+1})), \limsup_{n\to\infty} \phi(M_s(u, x_{2n+1})). \tag{2.2.25}$$

Now,

$$M_s(u, x_{2n+1}) = \max\{d(Su, Tx_{2n+1}), d(fu, Su), d(Tx_{2n+1}, gux_{2n+1}), \frac{1}{2s}[d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})]\}.$$

On taking upper limits as $n \to \infty$ and using (2.2.24) we have

$$\limsup_{n\to\infty} M_s(x_{2n+1}, u)$$
$$= \limsup_{n\to\infty} \max\{d(Su, Tx_{2n+1}), d(fu, Su), d(Tx_{2n+1}, gux_{2n+1}), \frac{1}{2s}[d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})]\}$$
$$= d(fu, Su). \tag{2.2.26}$$

Thus from (2.2.25) and (2.2.26), we get

$\psi(d(fu, z)) \leq F(\psi(d(fu, z)), \limsup_{n\to\infty} \phi(M_s(u, x_{2n+1})) \leq \psi(d(fu, z)))$.

This implies $\psi(d(fu, z))) = 0$ *or* $\limsup_{n\to\infty} \phi(M_s(u, x_{2n+1}) = 0$, thus, $fu = z$. Hence

$$z = Su = fu. \tag{2.2.27}$$

Since $z = fu \in fX \subseteq TX$, We have $z \in TX$, there exists $v \in X$ such that

$$Tv = z. \tag{2.2.28}$$

We now show that $gv = Tv$.

Owing to the assumption (iii) and (2.2.4), we have

$$\alpha(Sx_{2n}) \geq 1 \; and \; \beta(Tv) \geq 1. \tag{2.2.29}$$

By triangle inequality, we have

$d(z, gv) \leq s[d(z, fx_{2n}) + d(fx_{2n}, gv)]$

Taking limit supremum as $n \to \infty$, we have

$$\frac{1}{s} d(z, gv) \leq limsup_{n\to\infty} d(fx_{2n}, gv). \tag{2.2.30}$$

Also,

$d(fx_{2n}, gv) \leq s[d(fx_{2n}, z) + d(z, gv)].$

Taking limit supremum as $n \to \infty$, we have

$\limsup_{n\to\infty} d(fx_{2n}, gv) \leq sd(z, gv).$ (2.2.31)

Thus, from (2.1.1), (2. 2.29) and (2.2.30), it follows that

$$\begin{aligned} \psi(d(z, gv)) &\leq \psi(s^2 d(z, gv)) \leq \psi(s^3(\tfrac{1}{s} d(z, gv)) \\ &\leq \psi(s^3 \limsup_{n\to\infty} d(fx_{2n}, gv)) \\ &\leq \limsup_{n\to\infty} F(\psi(M_s d(x_{2n}, v)), \phi(M_s d(x_{2n}, v)) \end{aligned} \tag{2.2.32}$$

Now,

$$M_s(x_{2n}, v) = \max\{d(Sx_{2n}, Tv), d(Sx_{2n}, fx_{2n}), d(Tv, gv), \tfrac{1}{2s}[d((Sx_{2n}, gv) + d(fx_{2n}, Tv)]\}. \tag{2.2.33}$$

Taking limit supremum as $n \to \infty$, using (2.2.24) and (2.2.28), we have

$$\begin{aligned} \limsup_{n\to\infty} M_s(x_{2n}, v) &= \max\{d(z, Tv), 0, d(z, gv), \tfrac{1}{2s}[d(z, gv) + d(z, gv)]\} \\ &= d(z, gv). \end{aligned} \tag{2.2.34}$$

Hence from (2.2.33) and (2.2.34), we have

$$\begin{aligned} \psi(d(z, gv)) &\leq F(\limsup_{n\to\infty} \psi(M_s(x_{2n}, v)), \limsup_{n\to\infty} \phi(M_s(x_{2n}, v))) \\ &\leq F(\psi(d(z, gv)), \limsup_{n\to\infty} \phi(M_s(x_{2n}, v))) \\ &\leq \psi(d(z, gv)) \end{aligned}$$

which implies that $\psi(d(z, gv)) = 0$ or $\limsup_{n\to\infty} \phi(M_s(x_{2n}, v))) = 0$.
In both cases we have $d(z, gv) = 0$ Hence $z = gv$. (2.2.35)
Thus, from (2.2.27) and (2.2.35), it follows that

$$fu = Su = gv = Tv = z. \tag{2.2.36}$$

Hence

$C(f, S) \neq \phi$ and $C(g, T) \neq \phi$.

Case (ii): Suppose that gX is closed. In this case $z \in gX$, since $gX \subseteq SX$, we have $z \in SX$ and hence we can choose $u \in X$ such that $z = Su$. Hence the proof follows. For the cases TX and fX closed, the proof runs in the same lines of case (i) and case (ii). □

Theorem 2.3. *In addition to the hypotheses of Theorem 2.2, suppose*

(i) (f, S) and (g, T) are weakly compatible and

(ii) $\alpha(Su) \geq 1$ and $\beta(Tv) \geq 1$ whenever u and v are coincident points of (f, S) and (g, T) respectively.

Then f, g, T and S have a unique common fixed point in X.

Proof. In the light of Theorem 2.1, we have $z = fu = Su = Tv = gv$. Since the pair (f, S) is weakly compatible, we have $fz = fSu = Sfu = Sz$. z is a coincidence point of (f, S). In view of hypotheses (ii), we have $\alpha(Sz) \geq 1$ and $\beta(Tv) \geq 1$ this implies $\alpha(Sz)\beta(Tv) \geq 1$. Now on using the inequality with $x = z$ and $y = v$, we have

$$\psi(d(fz, gv)) \leq \psi(s^3 d(fz, gv)) \leq F(\psi(M_s(z, v)), \phi(M_s(z, v))). \tag{2.3.1}$$

Now

$$\begin{aligned} M_s(z, v) &= max\{d(Sz, Tv), d(fz, Sz), d(Tv, gv), \tfrac{1}{2s}[d(Sz, gv) + d(fz, Tv)]\} \\ &= max\{d(fz, gv), 0, 0, \tfrac{1}{2s}[d(Sz, gv) + d(fz, gv)]\} \\ &= d(fz, gv). \end{aligned} \tag{2.3.2}$$

Therefore from (2.3.1) and (2.3.2), we have

$\psi(d(fz, gv)) \leq F(\psi(d(fz, gv)), \phi(d(fz, gv))) \leq \psi(d(fz, gv))$,

this implies

$F(\psi(d(fz, gv)), \phi(d(fz, gv))) = \psi(d(fz, gv))$,

which in turn implies $\psi(d(fz, gv)) = 0$ or $\phi(d(fz, gv)) = 0$, in either case we have $d(fz, gv) = 0$. Hence

$$fz = Sz = z. \tag{2.3.3}$$

Thus, z is a common fixed point of f and S.

Since (g, T) is weakly compatible, we have $Tz = Tgv = gTv = gz$. z is a coincidence point of (T, g). Again by our hypotheses (ii) we have, $\alpha(Su) \geq 1$ and $\beta(Tz) \geq 1$ this implies $\alpha(Su)\beta(Tz) \geq 1$. Now on using the inequality (2.1.1) with $x = u$ and $y = z$, we have

$$\psi(d(fu, gz)) \leq \psi(s^3 d(fu, gz)) \leq F(\psi(M_s(u, z)), \phi(M_s(u, z))). \tag{2.3.4}$$

Now

$$\begin{aligned} M_s(u, z) &= max\{d(Su, Tz), d(fu, Su), d(Tz, gz), \tfrac{1}{2s}[d(Su, gz) + d(fu, Tz)]\} \\ &= max\{d(z, gz), 0, 0, \tfrac{1}{2s}[d(Su, gz) + d(z, gz)]\} \\ &= d(z, gz). \end{aligned}$$

Therefore

$\psi(d(fu, gz)) \leq F(\psi(d(fu, gz)), \phi(d(fu, gz))) \leq \psi(d(fu, gz))$, this implies $F(\psi(d(fu, gz)), \phi(d(fu, gz))) = \psi(d(fu, gz))$, which in turn implies $\psi(d(fu, gz)) = 0$ or $\phi(d(fu, gz)) = 0$, in either case we have $d(fu, gz) = 0$. Hence

$$gz = T = z. \tag{2.3.5}$$

Thus, z is a common fixed point of T and g.

We now show that f, g, S and T have a unique common fixed point in X. Suppose that u and z are two fixed points of S, f , g and T. Hence

$$fz = Tz = Sz = gz = z. \tag{2.3.6}$$

and

$$fu = Tu = gu = Su = u. \tag{2.3.7}$$

By the hypotheses, we have $\alpha(Su) \geq 1$ and $\beta(Tz) \geq 1$ this implies

$$\psi(d(fu, gz)) \leq \psi(s^3 d(fu, gz)) \leq F(\psi(M_s(u, z), \phi(M_s(u, z))).$$

Now,

$M_s(u, z) = max\{d(Su, Tz), d(fz, Sz), d(Tz, gz), \frac{1}{2s}[d(Su, gz) + d(fu, Tz)]\}$

$= \max\{d(u,z), 0, 0, \frac{1}{2s}[d(u,z) + d(u,z)]\} = d(u,z).$

Therefore $\psi(d(fu,gz)) \leq F(\psi(d(fu,gz)), \phi(d(fu,gz)) \leq \psi(d(fu,gz)),$ this implies $F(\psi(d(fu,gz)), \phi(d(fu,gz))) = \psi(d(fu,gz))$, which implies $\psi(d(fu,gz)) = 0$ or $\phi(d(fu,gz)) = 0$.

Hence $u = z$. Thus f, g, S, T have a unique common fixed point in X. □

Theorem 2.4. *Let A and B be two nonempty closed subsets of a b-metric space (X,d) such that $A \cap B \neq \phi$ and let $f, g : A \cup B \to A \cup B$ be mappings with $fA \subset B$ and $gB \subset A$. Assume that thereexist $\psi \in \Psi$, $\phi \in \Phi$, $F \in \mathcal{C}$ such that*

$$\psi(s^3 d(fx,gy)) \leq F(\psi(M_s(x,y), \phi(M_s(x,y))) \; for \; all \; x \in A \; and \; y \in B \tag{2.4.1}$$

where $M_s(x,y) = max\{d(x,y), d(fx,x), d(y,gy), \frac{1}{s}d(x,gy), \frac{1}{s}d(fx,y)\}$.

Then f and g have a unique common fixed point $u \in A \cap B$.

Proof. Let us define $\alpha, \beta : A \cup B \to R^+$ by

$$\alpha(x) = \begin{cases} 1 & if \; x \in A \\ 0 & otherwise, \end{cases} \quad \text{and } \beta(x) = \begin{cases} 1 & if \; x \in B \\ 0 & otherwise, \end{cases}$$

For any $x, y \in A \cup B$ with $\alpha(x)\beta(y) \geq 1$, we have $\alpha(x) = 1$ $\beta(y) = 1$ and $x \in A$, $y \in B$. Hence, from (2.4.1), we have

$$\psi(d(fx,gy)) \leq F(\psi(M_s(x,y)), \phi(M_s(x,y))$$

for all $x \in A$ and $y \in B$. Suppose $x \in A \cup B$ with $\alpha(x) \geq 1$. Then $x \in A$ and $fx \in fA \subset B$ so that $\beta(fx) \geq 1$. Suppose that $y \in A \cup B$ with $\beta(y) \geq 1$. Then $y \in B$, so that $gy \in gB \subset A$ so that $\alpha(gy) \geq 1$. Therefore (f,g) is cyclic (α, β) admissible map. Since $A \cap B \neq \phi$, thereexist $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

If $\{x_n\}$ is a sequence in $A \cup B$ such that $x_n \to x$ and $\alpha(x_n) \geq 1$, $\beta(x_n) \geq 1$ for all n, then $x_n \in A$ and $x_n \in B$. Since A and B are closed, $x \in A$ and $x \in B$ implies $\alpha(x) \geq 1$ and $\beta(x) \geq 1$. By choosing $S = T = I$ on X in Theorem 2.3, f and g satisfy the hypotheses of Theorem 2.3. Hence f and g have a unique common fixed point say u and clearly, $u \in A \cap B$. □

3 Corollaries

Corollary 3.1. *Let (X,d) be a complete b-metric space with $s \geq 1$. Suppose that $\alpha, \beta : X \to [0, \infty)$ are two mappings. Let f, g, S and T be four seflmaps on X satisfying*

(i) *the pair (f,g) is cyclic (α, β) admissible mapping with respect to (S,T)*

(ii) $\alpha(Sx)\beta(Ty) \geq 1 \Rightarrow \psi(s^3 d(fx,gy)) \leq \psi(M(x,y)) - \phi(M(x,y))$
for all $x, y \in X$, where $\psi \in \Psi, \phi \in \Phi$ and

$$M(x,y) = max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\}.$$

(iii) $fX \subseteq TX$, $gX \subseteq SX$

(iv) *there exists $x_0 \in X$ such that $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$.*

(v) *If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all n, then $\alpha(x) \geq 1, \beta(x) \geq 1$.*

(vi) *one of the ranges fX, gX, TX, SX is b-closed .*

(vii) $\alpha(Su) \geq 1$ *and* $\beta(Tv) \geq 1$ *whenever u and v are coincidence points of (f,S) and (g,T) respectively.*

Then f, g, T and S have a unique common fixed point in X provided (f,S) and (g,T) are weakly compatible on X.

Proof. Proof follows from Theorem 2.3 by choosing

$$F(s,t) = \begin{cases} s-t & if\ s \geq t \\ \\ 0 & otherwise. \end{cases}$$

□

Corollary 3.2. *Let (X,d) be a complete b- metric space with $s \geq 1$. Suppose that f, g, S and T be four seflmaps on X satisfying*

(i) $fX \subseteq TX$, $gX \subseteq SX$

(ii) $\psi(s^3 d(fx,gy)) \leq F(\psi(M_s(x,y)), \phi(M_s(x,y)))$
for all $x, y \in X$, where $\psi \in \Psi, \phi \in \Phi$, $F \in \mathcal{C}$ and

$M_s(x,y) = max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2s}\}$.

(iii) *one of the ranges fX, gX, TX, SX is b-closed.*

Then f, g, T and S have a unique common fixed point in X provided (f,S) and (g,T) are weakly compatible on X.

Proof. Proof follows by choosing $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 2.3. □

Corollary 3.3. . *Let (X,d) be a complete b-metric space. Suppose that $\alpha, \beta : X \to [0,\infty)$ are two mappings. Let f, g, S and T be four seflmaps on X satisfying:*

(i) $fX \subseteq TX$, $gX \subseteq SX$

(ii) *the pair (f,g) is cyclic (α,β) admissible mapping with respect to (S,T)*

(iii) $\alpha(Sx)\beta(Ty)\psi(s^3 d(fx,gy)) \leq F(\psi(M_s(x,y)), \phi(M_s(x,y)))$ (3.3.1)
for all $x, y \in X$, where $\psi \in \Psi, \phi \in \Phi$, $F \in \mathcal{C}$ and

$M_s(x,y) = max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\}$.

(iv) *there exists $x_0 \in X$ such that $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$.*

(v) *If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all n, then $\alpha(x) \geq 1, \beta(x) \geq 1$.*

(vi) *one of the ranges fX, gX, TX, SX is b-closed .*

(vii) $\alpha(Su) \geq 1$ *and* $\beta(Tv) \geq 1$ *whenever u and v are coincident points of (f,S) and (g,T) respectively.*

Then f, g, T and S have a unique common fixed point in X provided (f,S) and (g,T) are weakly compatible on X.

Proof. Let $x, y \in X$ with $\alpha(Sx)\beta(Ty) \geq 1$. Then

$$\psi(s^3 d(fx,gy)) \leq \alpha(Sx)\beta(Ty)\psi(s^3 d(fx,gy)) \leq F(\psi(M_s(x,y)), \phi(M_s(x,y))).$$

Hence the conclusion this theorem follows from Theorem 2.3. □

Corollary 3.4. *Let f, g, S and T be selfmaps of a complete metric space (X,d). Suppose that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and the pairs (f,S) and (g,T) are weakly compatible. If*

$$\psi(d(fx,gy)) \leq \psi(M(x,y)) - \phi(M(x,y)) \tag{3.4.1}$$

for all $x, y \in X$, where $\psi \in \Psi, \phi \in \Phi$ and

$M(x,y) = max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\}$

then f, g, S and T have a unique fixed point in X provided one of the ranges $f(X), g(X), S(X)$ and $T(X)$ is closed.

Proof. The proof of this corollary follows from Corollary 3.2 by choosing $s = 1$ and

$$F(s,t) = \begin{cases} s - t & if\ s \geq t \\ \\ 0 & otherwise. \end{cases}$$

. □

Remark 3.5. By choosing $f = g = T$ and $S = T = I$, where I is the identity map on R^+, Theorem 1.15 follows as a Corollary to Theorem 2.3.

Corollary 3.6. *Suppose that f, g, S and T are self mappings on a complete b-metric space (X, d) with $s \geq 1$ such that:*

(i) $f(X) \subseteq T(X), g(X) \subseteq S(X)$.

(ii) $s^4 d(fx, gy) \leq qmax\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2s}(d(Sx, gy) + d(fx, Ty))\}$, *(3.6.1)*

holds for each $x, y \in X$ with $0 < q < 1$, then f, g, S and T have a unique common fixed point in X provided that S and T are continuous and and pairs f, S and g, T are compatible.

Proof. Proof follows by choosing $\alpha(x) = 1$,$\beta(x) = 1$ for all $x \in X$, $\psi(t) = t, \phi(t) = 1$, and $F(r, t) = \frac{q}{s}r$, where $s \in [0, \infty)$ and $0 < k < 1$ in Theorem 2.3. □

4 Examples

Example 4.1. Let $X = [0, 1]$ and we define $d : X \times X \to [0, \infty)$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b metric space with $s = 2$. We define f, g, S and T on X by

$$f(x) = \begin{cases} \frac{x^8}{2^8} & if\ x \in [0, \frac{1}{2}] \\ \\ \frac{1}{32} & if x \in (\frac{1}{2}, 1], \end{cases} \quad \text{and } g(x) = \begin{cases} \frac{x^4}{2^4} & if\ x \in [0, \frac{1}{2}] \\ \\ \frac{1}{16} & if x \in (\frac{1}{2}, 1], \end{cases}$$

$$S(x) = \begin{cases} \frac{x^2}{4} & if\ x \in [0, \frac{1}{2}] \\ \\ \frac{1}{6} & if x \in (\frac{1}{2}, 1], \end{cases} \quad T(x) = \frac{x^4}{2^4} \text{ for all } x \in [0, 1].$$

Clearly, $fX = [0, \frac{1}{2^8 \times 2^8}] \cup \{\frac{1}{32}\} \subseteq TX = [0, \frac{1}{2^4}]$ and $gX = [0, \frac{1}{2^4 \times 2^4}] \cup \{\frac{1}{16}\} \subseteq [0, \frac{1}{2^4}] = SX$. Clearly, TX is closed.

Also, the pairs (f, S) and (g, T) are weakly compatible. We now define α, β on X by

$$\alpha(x) = \begin{cases} \frac{x+5}{4} & if\ x \in [0, \frac{1}{2}] \\ \\ 0 & otherwise, \end{cases} \quad \text{and } \beta(x) = \begin{cases} e & if\ x \in [0, \frac{1}{2}] \\ \\ 0 & otherwise, \end{cases}$$

We now prove that (f, g) is cyclic (α, β) admissible mapping with respect to (S, T), indeed if

$\alpha(Sx) \geq 1 \Rightarrow x \in [0, \frac{1}{2}] \Rightarrow \beta(fx) = \frac{x^8}{2^8} = e \geq 1.$

Similarly, if

$\beta(Tx) \geq 1 \Rightarrow x \in [0, \frac{1}{2}] \Rightarrow \alpha(gx) = \alpha(\frac{x^4}{2^4}) = \frac{x^4}{2^6} + \frac{5}{4} \geq 1.$

Hence (f, g) is cyclic (α, β) admissible mapping with respect to (S, T). Also, at $x_0 = 0$, $\alpha(Sx_0) = \alpha(0) = \frac{5}{4} \geq 1$ and $\beta(x_0) = \beta(0) = e \geq 1$. Next we will show that, (f, g) is a generalized TAC-(S, T) contractive map with $\psi(t) = t$, $\phi(t) = \frac{20}{32}t$ and $f(s, t) = \frac{s}{1+t}$, $for\ all\ s, t \in [0, \infty)$. Clearly, $\phi \in \Phi$ and $\psi \in \Psi$. Now, if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in W$ then by the definition of α and β we have $x_n \in [0, \frac{1}{2}]$, therefore $\alpha(x) \geq 1, \beta(x) \geq 1$, hence we have

$\psi(s^3 d(fx, gy)) = (2^3[(\frac{x^8}{2^8}) - (\frac{y^4}{2^4})]^2) = (2^3[(\frac{x^4}{2^4})^2 - (\frac{y^2}{2^2})^2]^2]$

$$= (2^3[(\frac{x^4}{2^4})^2 + (\frac{y^2}{2^2})^2]^2[(\frac{x^4}{2^4})^2 - ((\frac{y^2}{2^2})^2]^2)$$
$$\leq (2^3[\frac{1}{2^4} + \frac{1}{2^2}]^2 d(Sx, Ty))$$
$$= \frac{5}{2^4} d(Sx, Ty)$$
$$\leq \frac{d(Sx,Ty)}{1+\frac{20}{32}d(Sx,Ty)}$$
$$\leq \frac{M_s(x,y)}{1+\frac{20}{32}M_s(x,y)}$$
$$= \frac{\psi(M_s(x,y))}{1+\phi(M_s(x,y))}$$
$$= f(\psi(M_s(x,y)), \phi(M_s(x,y)).$$

Hence (f, g) is a generalized TAC-(S, T) contractive map. Hence f, g, S and T satisfy all the conditions of Theorem 2.3 and 0 is the unique common fixed point of S, T, f and g. Here we note that the with the usual distance, the condition (1.13.1) fails to hold when $x \in (\frac{1}{2}, 1]$ and $y = 1$, for any $\phi \in \Phi$ and $\psi \in \Psi$, since

$$\psi(d(fx, gy)) = \psi(\frac{1}{16}) \neq \psi(M_s(x, y)) - \phi(M_s(x, y)) = \psi(\frac{1}{16}) - \phi(\frac{1}{16}).$$

Hence Theorem 1.13 is not applicable.

Also, we observe that the inequality (1.17.1) fails to hold for any $q \in [0, 1)$ since
$d(fx, gy) = \frac{1}{256} = \frac{q}{2^4}\frac{1}{256}$
$= \frac{q}{s^4} max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}.$
Hence Theorem 1.17 is not applicable.

Example 4.2. Let $X = \{1, 2, 3, 4\}$. We write

$S_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$S_2 = \{(1, 3), (3, 1)\}$ and $S_3 = \{(2, 3), (3, 2), (4, 3), (3, 4)\}$.

We define $d : X \times X \to R$ by

$$d(x, y) = \begin{cases} 0 \; if \; (x, y) \in S_1 \\ 1 \; if \; (x, y) \in S_2 \\ 32 \; if \; (x, y) \in S_3 \\ 16 \;\; otherwise \end{cases}$$

Then (X, d) is a complete b-metric space with $s = 2$. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. We define $f, g : A \cup B \to R^+$ by $f1 = 1, f2 = 3, f3 = 1, f4 = 2$ $g1 = 1, g2 = 3, g3 = 1, g4 = 3$. Clearly, $fA = f(\{1, 2, 3\}) = \{1, 2, 3\} \subseteq B$ and $gB = (\{1, 2, 3, 4\}) = \{1, 3\} \subseteq A$, $A \cap B = \{1, 2, 3\} \neq \phi$. We define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$, $\phi(t) = \frac{t}{16}, t \geq 0$ and $F : [0, \infty)^2 \to R$ by $F(a, t) = \frac{a}{1+t}$. Now we verify the inequality (2.4.1).

Case(i): If $(x, y) \in \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3)\}$. Then

$$\psi(8d(fx, gy)) = 0 \leq F(\psi(M_s(x, y)), \phi(M_s(x, y)))$$

Case(ii): If $(x, y) = \{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4)\}$. Then

$$\psi(8d(fx, gy)) = 8 \leq \frac{32}{3} = F(\psi(M_s(x, y)), \phi(M_s(x, y))).$$

Also, 1 is the unique fixed point of f and g.

Example 4.3. Let $X = \{1, 2, 3, 4\}$. We write A= $\{(1, 3), (3, 1)\}$, $B = \{(1, 2), (2, 1)\}$, $C = \{(2, 3), (3, 2)\}$ $D = \{(1, 4), (4, 1)\}$ and

$E = \{(3,4),(4,3),(2,4),(4,2)\}$. We define $d : X \times X \to R$ by

$$d(x,y) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; (x,y) \in A \\ 5 \; if \; (x,y) \in B \\ 11 \; if (x,y) \in C \\ 48 \; if \; (x,y) \in D \\ 96 \; if \; (x,y) \in E. \end{cases}$$

Then (X,d) is a complete b-metric space with $s = 2$.We now define f, g, S and T on X by

$f1 = 1, f2 = 1, f3 = 1, f4 = 2, g1 = 1, g2 = 3, g3 = 1, g4 = 3,$

$S1 = 1, S2 = 3, S3 = 2, S4 = 4$ and $T1 = 1, T2 = 2, T3 = 4, T4 = 4$. Clearly, $fX = \{1,2\} \subseteq TX = \{1,2,4\}$ $gX = \{1,3\} \subseteq SX = \{1,2,3,4\}$.

We define $\alpha, \beta : [0,\infty) \to [0,\infty)$ by $\alpha(x) = 1$ and $\beta(x) = 1$, $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{16}, t \geq 0$ and $F : [0,\infty)^2 \to R$ by

$$F(s,t) = \begin{cases} s - t & if \; s \geq t \\ 0 & if otherwise. \end{cases}$$

Now we verify the inequality (2.1.1)

Case(i): If $(x,y) \in \{(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}$. Then

$$\psi(s^3 d(fx,gy)) = 0 \leq F(\psi(M_s(x,y)), \phi(M_s(x,y)))$$

Case(ii): If $(x,y) = \{(1,2),(2,2),(3,2)\}$. Then

$$\psi(s^3 d(fx,gy)) = 8 \leq 11 - \frac{11}{16} = 10.3 = F(\psi(M_s(x,y)), \phi(M_s(x,y))).$$

Case(iii): If $(x,y) = \{(2,4),(1,4),(3,4)\}$. Then

$$\psi(s^3 d(fx,gy)) = 8 \leq 96 - \frac{96}{16} = 90 = f(\psi(M_s(x,y)), \phi(M_s(x,y))).$$

Case(iv): If $(x,y) = \{(4,2),(4,4)\}$. Then

$$\psi(s^3 d(fx,gy)) = 88 \leq 96 - \frac{96}{16} = 90 = F(\psi(M_s(x,y)), \phi(M_s(x,y))).$$

Case(v): If $(x,y) = \{(4,1),(4,3)\}$. Then

$$\psi(s^3 d(fx,gy)) = 40 \leq 96 - \frac{96}{16} = 90 = F(\psi(M_s(x,y)), \phi(M_s(x,y))).$$

Here we observe that with the usual distance the inequality (1.13.1) fails to hold at $x = 3$ and $y = 2$ for any $\phi \in \Phi$ and $\psi \in \Psi$ since

$$d(fx,gy) = 2 \neq \psi(M_s(x,y)) - \phi(M_s(x,y)) = 1.$$

Hence Theorem 1.13 is not applicable.

Also, the inequality (1.17.1) fails to hold at $x = 3$ and $y = 2$ for any $q < 1$ since

$$d(fx, gy) = 1 > \frac{q}{2^4}(M_s(x, y)) = \frac{11}{16}.$$

Hence Theorem 1.17 is not applicable.

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