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PERPETUAL AMERICAN POWER PUT OPTIONS WITH NON-DIVIDEND YIELD IN THE DOMAIN OF MELLIN TRANSFORMS

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Abstract In this paper, we present the Mellin transforms and its applications in perpetual American power put option valuation with non-dividend paying stock. We obtain the integral representations for the price and the free boundary of American power put option by means of the Mellin inversion formula and value-matching condition, respectively. We also extend our results to obtain the free boundary and the analytic valuation formula for perpetual American power put option. The main tool in this approach is the principle of smooth pasting conditions. We assume assets are driven by geometric wiener process. Numerical result shows that the value of a perpetual American power put option with $n = 1$ on a non-dividend paying stock coincides with the value of [6]. Hence, the Mellin transforms is a good alternative approach for the valuation of perpetual American power put option.

1 Introduction

Power option is a financial derivative in which the payoff at time to expiry is related to the n^{th} power of the underlying asset price. Because of the non-linear characteristics of these options, they are appropriate for hedging non-linear price risks. Power options preserve volatility exposure better than plain vanilla options if the underlying moves significantly in the same direction. These options offer flexibility to investors and of practical interest since many OTC-traded options exhibit such a payoff structure. For example, an option whose payoff is a polynomial function of the Nikkei level at the expiry was issued in Tokyo [3]. Bankers Trust in Germany has issued capped foreign-exchange power options with power exponent two [11], [13]. More examples on power options can be found in [4] and [10]. Power option comes in two forms namely power call option and power put option. A power call option is an option with non-linear payoff given by the difference between underlying asset price at expiry raised to a strictly positive power and the strike price. A power put option is an option with non-linear payoff given by the difference between the strike price and underlying asset price at expiry raised to a strictly positive power. For a power option on the underlying asset price S_T^n with strike price K and time to expiry T, the payoff for the power call option is given by

$$P_c^n(S_T^n, T) = \max(S_T^n - K, 0) = (S_T^n - K)^+ \tag{1.1}$$

and the payoff for the power put option is given by

$$P_p^n(S_T^n, T) = \max(K - S_T^n, 0) = (K - S_T^n)^+ \tag{1.2}$$

where n is some power $(n > 0)$. Power options can be classified as European or American. European power option can be exercised only at the expiry date while American power option can be exercised before or at the expiry date. The early exercise feature of the American power put option makes the valuation of the option mathematically challenging and therefore, creating a great field of research. Perpetual American power put option is a financial contract that grants its holder rights, but not obligation to sell an underlying stock in a fixed price at any time up until infinite future. In other words, this type of power option never expires. Obviously, in a special case of vanilla perpetual option, a plain perpetual American put on a non-dividend yield should at least satisfy [12]:

$$K - S_0 \leq P_\infty(S_0, K, r, \infty, \sigma) \leq K, \text{ for } 0 < t \leq \infty \tag{1.3}$$

and

$$P_A(S_0, K, r, \infty, \sigma) \leq P_\infty(S_0, K, r, \infty, \sigma), \text{ for } 0 < t \leq \infty \tag{1.4}$$

with the current stock price S_0, strike price K, risk-free interest rate r, expiration time ∞ and volatility σ. A closed form solution for the free boundary and price of the American put was derived by [5] and [9]. [6] proposed a closed-form solution for pricing a perpetual American put option. For the mathematical background of the Mellin transforms in derivatives valuation see [1], [2], [7], [8], [14], just to mention a few. In this paper, we focus on the Mellin transforms and its applications in perpetual American power put options valuation with non-dividend yield under geometric wiener process. We also assume that the underlying asset price follows lognormal distribution. The rest of the paper is structured as follows: In Section 2, we present American power put options in the domain of the Mellin transforms. Section 3 presents the Mellin transforms for the valuation of the perpetual American power put option. Section 4 presents the derivation of a closed-form solution for the free boundary and price of the perpetual American put option. In Section 5, we present two numerical examples and discussion of results. Section 6 concludes the paper.

2 American Power Put Option in the Domain of the Mellin Transforms

Analytical approximations and numerical techniques have been proposed for the valuation of plain American put option but there is no known closed-form solution for the price of American power put option. The following result gives the integral representation for the price of the American power put option and the integral equation to determine the free boundary of the option via the Mellin transforms for the case of non-dividend yield.

Theorem 2.1. *Let S_t^n be the price of underlying asset, n be the power of the option, K be the strike price, r be the risk-free interest rate, q be the dividend yield and T be the time to expiry. Assume S_t^n yields no dividend and follows a random process in*

$$dS_t^n = \left(nr + \frac{n(n-1)\sigma^2}{2}\right) S_t^n dt + n\sigma S_t^n dW_t \tag{2.1}$$

then the integral representation for the price of the American power put option $P_A^n(S_t^n, t)$ is given by

$$P_A^n(S_t^n, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} (S_t^n)^{-\omega} d\omega$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\hat{S}_t^n(y))^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy d\omega$$

Proof. Consider the non-homogeneous Black-Scholes partial differential equation for the price of American power put option with non-dividend yield given by

$$\frac{\partial P_A^n(S_t^n, t)}{\partial t} + n\left(\frac{1}{2}\sigma^2(n-1) + r\right) S_t^n \frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}$$

$$+\frac{1}{2}(\sigma n S_t^n)^2 \frac{\partial^2 P_A^n(S_t^n, t)}{\partial (S_t^n)^2} - rP_A^n(S_t^n, t) = f(S_t^n, t) \tag{2.2}$$

where the early exercise function $f(S_t^n, t)$ defined on $(0, \infty) \times (0, T)$ is given by

$$f(S_t^n, t) = \begin{cases} -rK, & \text{if } 0 < S_t^n \leq \hat{S}_t^n \\ 0, & \text{if } S_t^n > \hat{S}_t^n. \end{cases} \tag{2.3}$$

The final time condition is given by
$P^n_A(S^n_T, T) = \phi(S^n_T) = \max(K - S^n_T, 0) = (K - S^n_T)^+$ on $[0, \infty)$.
The other boundary conditions are given by

$$\lim_{S^n_t \to \infty} P^n_A(S^n_t, t) = 0 \quad \text{on} \quad [0, T) \tag{2.4}$$

$$\lim_{S^n_t \to 0} P^n_A(S^n_t, t) = K \quad \text{on} \quad [0, T) \tag{2.5}$$

The free boundary $\hat{S}^n_t$ is determined by the value-matching condition and super-contact condition given by

$$P^n_A(\hat{S}^n_t, t) = K - \hat{S}^n_t \tag{2.6}$$

and

$$\left.\frac{\partial P^n_A(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \hat{S}^n_t} = -1 \tag{2.7}$$

respectively. Equations (2.6) and (2.7) ensure that the price of the power option is continuous across the free boundary and the slope of the price is continuous across the free boundary, respectively. The two conditions are jointly referred to as the smooth pasting conditions. Let $\tilde{P}^n_A(\omega, t)$ be the Mellin transform of the American power put option which is defined by the relation

$$\mathcal{M}(P^n_A(S^n_t, t), \omega) = \tilde{P}^n_A(\omega, t) = \int_0^\infty P^n_A(S^n_t, t)(S^n_t)^{\omega-1} dS^n_t \tag{2.8}$$

where ω is a complex variable with $0 < \Re(\omega) < \infty$. Conversely the inversion formula for the Mellin transform in (2.8) is defined as

$$P^n_A(S^n_t, t) = \mathcal{M}^{-1}(\tilde{P}^n_A(\omega, t)) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{P}^n_A(\omega, t)(S^n_t)^{-\omega} d\omega \tag{2.9}$$

Taking the Mellin transform of (2.2) by means of (2.8), we have that

$$\frac{\partial \tilde{P}^n_A(\omega, t)}{\partial t} + \frac{n^2\sigma^2}{2}\left(\omega^2 + \omega\left(1 - \frac{n-1}{n} - \frac{2r}{n\sigma^2}\right) - \frac{2r}{n^2\sigma^2}\right)\tilde{P}^n_A(\omega, t) = \tilde{f}(\omega, t) \tag{2.10}$$

Setting $\alpha_1 = \left(1 - \frac{n-1}{n} - \frac{2r}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$. Then (2.10) becomes

$$\frac{\partial \tilde{P}^n_A(\omega, t)}{\partial t} + \frac{n^2\sigma^2}{2}(\omega^2 + \omega\alpha_1 - \alpha_2)\tilde{P}^n_A(\omega, t) = \tilde{f}(\omega, t) \tag{2.11}$$

Similarly, the Mellin transform of the early exercise function in (2.11) is obtained as

$$\begin{aligned} \tilde{f}(\omega, t) &= \int_0^\infty f(S^n_t, t)(S^n_t)^{\omega-1} dS^n_t \\ &= \int_0^{\hat{S}^n_t} -rK(S^n_t)^{\omega-1} dS^n_t \\ &= \frac{-rK(\hat{S}^n_t)^\omega}{\omega} \end{aligned} \tag{2.12}$$

Solving further and from the theory of differential equations, the particular solution of (2.11) is obtained as

$$\tilde{P}^n_A(\omega, t)_{(p.sol)} = \int_t^T \frac{rK(\hat{S}^n_t)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1\omega - \alpha_2)(y-t)} dy \tag{2.13}$$

Similarly, the complementary solution of the left hand side of (2.11) is obtained as

$$\tilde{P}^n_A(\omega, t)_{comp.sol} = c(\omega)e^{-\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1\omega - \alpha_2)t} \tag{2.14}$$

where $c(\omega)$ is the integration constant given by

$$c(\omega) = \tilde{\phi}(\omega, t)e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1\omega - \alpha_2)T} \tag{2.15}$$

$\tilde{\phi}(\omega, t)$ is the Mellin transform of the final time condition and is given by

$$\begin{aligned}\tilde{\phi}(\omega, t) &= \int_0^\infty (K - S_T^n)^+ (S_T^n)^{\omega-1} dS_T^n \\ &= \int_0^K (K - S_T^n)(S_T^n)^{\omega-1} dS_T^n \\ &= \frac{K^{\omega+1}}{\omega(\omega+1)}\end{aligned} \tag{2.16}$$

Using (2.15) and (2.16) in (2.14) yields

$$\tilde{P}_A^n(\omega, t)_{comp.sol} = \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} \tag{2.17}$$

Hence the general solution of (2.11) is given by

$$\begin{aligned}\tilde{P}_A^n(\omega, t) &= \tilde{P}_A^n(\omega, t)_{comp.sol} + \tilde{P}_A^n(\omega, t)_{(p.sol)} \\ &= \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} \\ &+ \int_t^T \frac{rK(\hat{S}_y^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy\end{aligned} \tag{2.18}$$

The Mellin inversion of (2.18) is obtained as

$$\begin{aligned}P_A^n(S_t^n, t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} (S_t^n)^{-\omega} d\omega \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\hat{S}_y^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy d\omega\end{aligned} \tag{2.19}$$

where $(S_t^n, t) \in \{(0, \infty) \times [0, T)\}, c \in (0, \infty)$ and $\{\omega \in \mathbb{C} | 0 < \Re(\omega) < \infty\}$. This completes the proof.

□

Remark 2.2. Equations (2.6) and (2.7) jointly ensure that the premature exercise of the American power put option on the endogenously determined early exercise boundary, $\hat{S}_t^n$, will be optimal and self-financing.

Remark 2.3. Equation (2.19) expresses the value of an American power put option as the sum of the value of a European power put option and the early exercise premium.

Remark 2.4. The first term in (2.19) is the integral representation for the price of the European power put option which pays no dividend yield (stems from the minimum guaranteed payoff of the American power put). The second term in (2.19) is called the early exercise premium (the value attributable to the right of exercising the option early) for the American power put option with non-dividend yield denoted by $e_p^n(S_t^n, t)$. Therefore (2.19) becomes

$$P_A^n(S_t^n, t) = P_E^n(S_t^n, t) + e_p^n(S_t^n, t) \tag{2.20}$$

where

$$P_E^n(S_t^n, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} (S_t^n)^{-\omega} d\omega$$

$$e_p^n(S_t^n, t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\hat{S}_y^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy d\omega$$

Remark 2.5. Using the value-matching condition given by (2.6), the integral representation for the free boundary of the American power put option with non-dividend yield is obtained as

$$\hat{S}_t^n = K - P_E^n(\hat{S}_t^n, t)$$

$$-\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\hat{S}_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega \tag{2.21}$$

where

$$P_E^n(\hat{S}_t^n, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}(\hat{S}_t^n)^{-\omega}d\omega$$

Remark 2.6. The American power put option $P_A^n(S_t^n, t)$ which pays no dividend yield satisfies the decomposition

$$P_A^n(S_t^n, t) = P_E^n(S_t^n, t)$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega$$

where $\alpha_1 = \left(1 - \frac{n-1}{n} - \frac{2r}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$, $(S_t^n, t) \in \{(0,\infty)\times[0,T)\}, c \in (0,\infty)$ and $\{\omega \in \mathbb{C} | 0 < \Re(\omega) < \infty\}$.

Remark 2.7. The upper and the lower bounds for an American power put option with $n = 1$ on a non-dividend yield is given by

$$(K - S_t) \le P_A(S_t, K, r, T, \sigma) \le C_A(S_t, K, r, T, \sigma) + (K - S_t) \le K$$

3 The Free Boundary and the Fundamental Analytic Valuation Formula for Perpetual American Power Put Option

Now, we apply the integral representations in (2.19) to power options which have no expiry date. The expressions for the free boundary and the fundamental analytic valuation formula of the perpetual American power put option with non-dividend yield, using the Mellin transforms are given by the following result.

Theorem 3.1. *Consider the perpetual American power put option with non-dividend yield. If $T \to \infty$ and $0 < \Re(\omega) < \omega_2$, then the free boundary of the perpetual American power put option is given by*

$$\hat{S}_\infty^n = \hat{S}_\infty^n(t) = K\frac{\alpha_2}{(\omega_2 - \omega_1)} \tag{3.1}$$

and the fundamental valuation formula of the perpetual American power put option becomes

$$P_\infty^n(S_t^n, t) = \frac{\alpha_2 K}{\omega_2(\omega_2 - \omega_1)}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega_2} \quad \text{for} \ \ \hat{S}_\infty^n < S_t^n \tag{3.2}$$

where

$$\alpha_2 = \frac{2r}{n^2\sigma^2} \tag{3.3}$$

Proof. The integral representation for the price of the American power put option which pays no dividend yield given by (2.19) can be expressed as

$$P_A^n(S_t^n, t) = P_E^n(S_t^n, t) + P_1^n(S_t^n, t) \tag{3.4}$$

where

$$\begin{aligned} P_E^n(S_t^n, t) &= Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) \\ &- S_t^n e^{\left(r(n-1)+\frac{1}{2}n(n-1)\sigma^2\right)(T-t)}\mathcal{N}(-d_{1,n}) \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} d_{1,n} &= \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}} \\ d_{2,n} &= \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - \frac{\sigma^2}{2}\right)(T-t)}{n\sigma\sqrt{T-t}} \end{aligned}$$

and

$$P_1^n(S_t^n, t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\hat{S}_y^n)^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy d\omega \tag{3.6}$$

For (3.4) to hold as $T \to \infty$, it is necessary that $\Re(\omega^2 + \alpha_1\omega - \alpha_2) < 0$, that is $0 < \Re(\omega) < \omega_2$, where ω_2 is one of the roots of $\omega^2 + \alpha_1\omega - \alpha_2 = 0$. Using the super-contact condition (2.7), the perpetual American power put option as $T \to \infty$ becomes

$$\left.\frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = \left.\frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} + \left.\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = -1 \tag{3.7}$$

where the free boundary $\hat{S}_t^n = \hat{S}_\infty^n$ is now independent of time. Now, Differentiating (3.5) with respect to S_t^n at $S_t^n = \hat{S}_\infty^n$ yields

$$\left.\frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = -e^{\left(r(n-1) + \frac{1}{2}n(n-1)\sigma^2\right)(T-t)} \mathcal{N}(-\hat{d}_{1,n}) \tag{3.8}$$

where

$$\hat{d}_{1,n} = \frac{\ln\left(\frac{\hat{S}_\infty^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}} \tag{3.9}$$

As $T \to \infty, \hat{d}_{1,n} \to \infty$ and therefore

$$\left.\frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} \to 0 \tag{3.10}$$

Also consider the $P_1^n(S_t^n, t)$ term,

$$\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n} = -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-1} \left(\int_t^T \left(\frac{S_t^n}{\hat{S}_y^n}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy \right) d\omega \tag{3.11}$$

Taking the limit of (3.11) as $T \to \infty$ yields

$$\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n} = -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-1} \left(\int_t^\infty \left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy \right) d\omega \tag{3.12}$$

Therefore,

$$\begin{aligned} \frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n} &= -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-1} \left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega} \left(\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}}{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)}\right|_t^\infty \right) d\omega \\ &= -\frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-1} \left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega} \left(\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}}{(\omega^2+\alpha_1\omega-\alpha_2)}\right|_t^\infty \right) d\omega \\ &= \frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-1} \left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega} \frac{d\omega}{(\omega^2+\alpha_1\omega-\alpha_2)} \end{aligned}$$

Thus,

$$\left.\frac{\partial P_1^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = \frac{K}{2\pi i}\frac{2r}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\hat{S}_\infty^n(\omega^2+\alpha_1\omega-\alpha_2)} \tag{3.13}$$

Since $\alpha_2 = \frac{2r}{n^2\sigma^2}$, (3.13) becomes

$$\left.\frac{\partial P_1^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = \frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\hat{S}_\infty^n(\omega^2+\alpha_1\omega-\alpha_2)} \tag{3.14}$$

But $\omega^2+\alpha_1\omega-\alpha_2 = (\omega-\omega_1)(\omega-\omega_2)$, where

$$\omega = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2+4\alpha_2}}{2} \tag{3.15}$$

$$\omega_1 = \frac{-\alpha_1 - \sqrt{\alpha_1^2+4\alpha_2}}{2} \tag{3.16}$$

$$\omega_2 = \frac{-\alpha_1 + \sqrt{\alpha_1^2+4\alpha_2}}{2} \tag{3.17}$$

The limiting cases ω_1 and ω_2 are the roots of $\omega^2+\alpha_1\omega-\alpha_2$. Hence (3.14) becomes

$$\left.\frac{\partial P_1^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = \frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\hat{S}_\infty^n(\omega-\omega_1)(\omega-\omega_2)} \tag{3.18}$$

By applying the residue theorem given by

$$\frac{1}{2\pi i}\int_{\delta\omega} f(\omega)d\omega = \sum_{j=0}^{k} Res(f,\omega_j), \omega \in \mathbb{C} \tag{3.19}$$

Therefore, (3.18) leads to a relation

$$\left.\frac{\partial P_1^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = \alpha_2\frac{K}{\hat{S}_\infty^n(\omega_1-\omega_2)} \tag{3.20}$$

Substituting (3.10) and (3.20) into (3.7) gives

$$\left.\frac{\partial P_A^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = 0 + \alpha_2\frac{K}{\hat{S}_\infty^n(\omega_1-\omega_2)} = -1$$

The free boundary of a perpetual American power put option is obtained as

$$\hat{S}_\infty^n = K\frac{\alpha_2}{(\omega_2-\omega_1)} \tag{3.21}$$

Next, use (3.21) to derive an expression for the price of perpetual American power put option $P_\infty^n(S_t^n,t)$. Note that the price of a perpetual European power put option is zero, since it can never be exercised. Therefore, taking the limit as $T\to\infty$ in (3.4), the price of perpetual American power put option for $S_t^n > \hat{S}_\infty^n$ is given by

$$P_\infty(S_t^n,t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega}\frac{1}{\omega}\left(\int_t^\infty e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy\right)d\omega \tag{3.22}$$

where $\Re(\omega^2+\alpha_1\omega-\alpha_2)<0$. Integrating the inner integral (that is, the time variable) in (3.22) leads to

$$P_\infty(S_t^n,t) = -\frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega}\frac{d\omega}{\omega(\omega-\omega_1)(\omega-\omega_2)} \tag{3.23}$$

Once again applying the residue theorem (3.19) to get

$$P_{\infty}^{n}(S_t^n, t) = \frac{\alpha_2 K}{\omega_2(\omega_2 - \omega_1)} \left(\frac{S_t^n}{\hat{S}_{\infty}^n}\right)^{-\omega_2} \quad \text{for } \hat{S}_{\infty}^n < S_t^n \tag{3.24}$$

Equation (3.24) is the fundamental valuation formula of perpetual American power put option. This completes the proof. □

Remark 3.2. Note that the price of a perpetual European power put option with non-dividend yield is zero, since it can never be exercised before expiration.

Remark 3.3. For $n = 1$, the free boundary of the perpetual American put option with non-dividend yield given by (46) coincides with the Merton's result [6] given by

$$S_{\infty}^{*} = \left(\frac{k_1}{k_1 + 1}\right) K$$

with $k_1 = \frac{2r}{\sigma_2}$

4 Derivation of a Closed-Form Solution for the Free Boundary and Price of the Perpetual American Put Option ([5] and [9])

In the special case of a perpetual option, a closed-form solution for the free boundary and price of the American put was derived by [5] and [9]. They derived the price P_{∞} as a solution of the time-independent homogeneous second order partial differential equation given by

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{\infty}}{\partial S^2} + rS\frac{\partial P_{\infty}}{\partial S} - rP_{\infty} = 0, \quad \text{for } S > S_{\infty}^{*} \tag{4.1}$$

with boundary conditions

$$P_{\infty} \to 0 \text{ as } S \to \infty \tag{4.2}$$

$$P_{\infty}(S_{\infty}^{*}) = K - S_{\infty}^{*} \tag{4.3}$$

$$\left.\frac{\partial P_{\infty}}{\partial S}\right|_{S=S_{\infty}^{*}} = -1 \tag{4.4}$$

Equation (4.1) has a solution of the form

$$P_{\infty}(S) = c_1 S^{\mu_+} + c_2 S^{\mu_-} \tag{4.5}$$

where μ_+ and μ_- are given by

$$\mu_+ = 1 \tag{4.6}$$

and

$$\mu_- = -\frac{2r}{\sigma^2} \tag{4.7}$$

respectively. Since P_{∞} vanishes as $S \to \infty$, $c_1 = 0$ then we have that

$$P_{\infty}(S) = c_2 S^{\mu_-} \tag{4.8}$$

Using (4.3), (4.4) and (4.8), we obtain

$$S_{\infty}^{*} = \frac{rK}{\frac{\sigma^2}{2} + r} \tag{4.9}$$

$$P_{\infty}(S) = (K - S_{\infty}^{*})\left(\frac{S}{S_{\infty}^{*}}\right)^{-\frac{2r}{\sigma^2}} \tag{4.10}$$

Equations (4.9) and (4.10) give the free boundary and the price of a perpetual put option respectively.

Remark 4.1. Equations (4.9) and (4.10) can be obtained by means of the Mellin transforms by setting $n = 1$ in (3.21) and (3.24), respectively.

The following result gives the probabilistic approach for the valuation of an American power put option with $n = 1$ on a non-dividend yield.

Theorem 4.2. *The value of an American power put option with $n = 1$; $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is equal to the expected value of the maximum option premium.*

$$P_A(S_0, K, r, T, \sigma) = \mathbf{E}^Q\{\max[P_E(S_0, K, r, T, \sigma), \textit{MaxPremium(early exercise)}]\} \tag{4.11}$$

Proof. According to the definition of an American power put option with $n = 1$, the holder has the right to exercise it at any time during its lifetime. As we know, when an American power put option with $n = 1$ is not early-exercised, the premium will be equal to its European counterpart.

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma) \tag{4.12}$$

The holder of an American put option should take an optimal exercise strategy to get maximum option premium. So the valuation of the option is such an optimization problem:

(i) When the maximum option premium of optimally early exercise is not less than $P_E(S_0, K, r, T, \sigma)$, the American power put option with $n = 1$ should be optimally early-exercised and get the max premium

$$P_A(S_0, K, r, T, \sigma) = \text{Max Premium(early exercise)} \tag{4.13}$$

(ii) Otherwise, the American power put option with $n = 1$ should not be early-exercised and get the same premium as its European counterpart:

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma) \tag{4.14}$$

Therefore,

$$P_A(S_0, K, r, T, \sigma) = \mathbf{E}^Q\{\max[P_E(S_0, K, r, T, \sigma), \text{MaxPremium(early exercise)}]\}$$

This completes the proof. □

The following result gives an alternative approach for the derivation of closed-form solution for the valuation of American put option of power one on a non-dividend yield.

Theorem 4.3. *The price of an American power put option with $n = 1$ on a non-dividend paying stock at current time $t = 0$ is given by*

$$\begin{aligned} P_A(S_0, K, r, T, \sigma) &= P_E(S_0, Ke^{rT}, r, T, \sigma)\mathcal{N}(-d_4) \\ &\quad + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]\mathcal{N}(d_4) \end{aligned} \tag{4.15}$$

where

$$P_E(S_0, K, r, T, \sigma) = Ke^{-rT}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \tag{4.16}$$

and

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = K\mathcal{N}(-d_4) - S\mathcal{N}(-d_3) \tag{4.17}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}} \tag{4.18}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - 0.5\sigma^2\right)T}{\sigma\sqrt{T}} \tag{4.19}$$

$$d_3 = \frac{\ln\left(\frac{S_0}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} \tag{4.20}$$

$$d_4 = \frac{\ln\left(\frac{S_0}{K}\right) - 0.5\sigma^2 T}{\sigma\sqrt{T}} \tag{4.21}$$

Proof. From Theorem 4.2, we have that

$$P_A(S_0, K, r, T, \sigma) = \mathbf{E}^Q\{\max[P_E(S_0, K, r, T, \sigma), \text{MaxPremium(early exercise)}]\}$$

Here, MaxPremium(early exercise) is either $P_E(S_0, K, r, T, \sigma)$ or $K - S_0$, with the probability $N(-d_4)$ or $N(d_4)$, respectively. Since

$$P_E(S_0, Ke^{rT}, r, T, \sigma) > P_E(S_0, K, r, T, \sigma)$$

Therefore,

$$\begin{aligned} P_A(S_0, K, r, T, \sigma) &= \mathbf{E}^Q\{\max[P_E(S_0, K, r, T, \sigma), \text{MaxPremium(early exercise)}]\} \\ &= P_E(S_0, Ke^{rT}, r, T, \sigma)\mathcal{N}(-d_4) \\ &+ \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]\mathcal{N}(-d_4) \end{aligned}$$

This completes the proof. □

The following result shows that the price of perpetual American put option of power one is equal to its strike price.

Proposition 4.4. *The price of a perpetual American put option of power one on a non-dividend paying stock is equal to its strike price*

$$P_\infty(S_0, K, r, \infty, \sigma) = K \tag{4.22}$$

Proof. A perpetual American put option of power one on a non-dividend paying stock whose maturity time is infinite is given by

$$P_\infty(S_0, K, r, \infty, \sigma) = P_A(S_0, K, r, T, \sigma), \ (\text{when } \ T \to \infty) \tag{4.23}$$

From Theorem 4.3, we know that

$$\begin{aligned} P_A(S_0, K, r, T, \sigma) = P_E(S_0, Ke^{rT}, r, T, \sigma)\mathcal{N}(-d_4) \\ + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]\mathcal{N}(d_4) \end{aligned}$$

where

$$P_E(S_0, K, r, T, \sigma) = Ke^{-rT}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1)$$

and

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = K\mathcal{N}(-d_4) - S\mathcal{N}(-d_3)$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}, d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_3 = \frac{\ln\left(\frac{S_0}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, d_4 = \frac{\ln\left(\frac{S_0}{K}\right) - 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

For perpetual American put option of power one,

$$d_3 = \lim_{T\to\infty} \frac{\ln\left(\frac{S_0}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \infty \tag{4.24}$$

$$d_4 = \lim_{T\to\infty} \frac{\ln\left(\frac{S_0}{K}\right) - 0.5\sigma^2 T}{\sigma\sqrt{T}} = -\infty \tag{4.25}$$

Therefore,

$$\lim_{T\to\infty} \mathcal{N}(d_4) = 0; \lim_{T\to\infty} \mathcal{N}(-d_4) = 1; \lim_{T\to\infty} N(-d_3) = 0; \tag{4.26}$$

$$\lim_{T\to\infty} P_E(S_0, Ke^{rT}, r, T, \sigma) = \lim_{T\to\infty} (K\mathcal{N}(-d_4) - S_0\mathcal{N}(-d_3)) = K \tag{4.27}$$

Thus,

$$\begin{aligned}\lim_{T\to\infty} P_A(S_0,K,r,T,\sigma) &= \lim_{T\to\infty} P_E(S_0,Ke^{rT},r,T,\sigma)\mathcal{N}(-d_4)\\ &+ \lim_{T\to\infty} \max[(K-S_0),P_E(S_0,K,r,T,\sigma)]\mathcal{N}(d_4)\\ &= K\end{aligned}$$

Hence,

$$P_\infty(S_0,K,r,\infty,\sigma) = K \tag{4.28}$$

This completes the proof. □

5 Numerical Examples and Discussion of Results

This section presents two numerical examples and discussion of results.

5.1 Example 1

By varying volatility, we consider the valuation of perpetual American power put option with non-dividend yield by means of (3.24) with the following parameters

$$n = 1, S_t = 30, K = 31, r = 0.01, T = \infty, t = 0$$

The price of perpetual American put option of power one is shown in Figure 1 below.

5.2 Example 2

We consider the valuation of the American power put option for $n = 1$ with the following parameters:

At The Money (ATM): $S_t = 100, K = 100, r = 0.03, \sigma = 0.6, t = 0$

In The Money (ITM): $S_t = 100, K = 150, r = 0.03, \sigma = 0.6, t = 0$

Out of The Money (OTM): $S_t = 100, K = 50, r = 0.03, \sigma = 0.6, t = 0$

The relationship between the price of the option (at the money, in the money, out of the money, respectively) and the maturity time T is shown in Figures 2, 3 and 4 below.

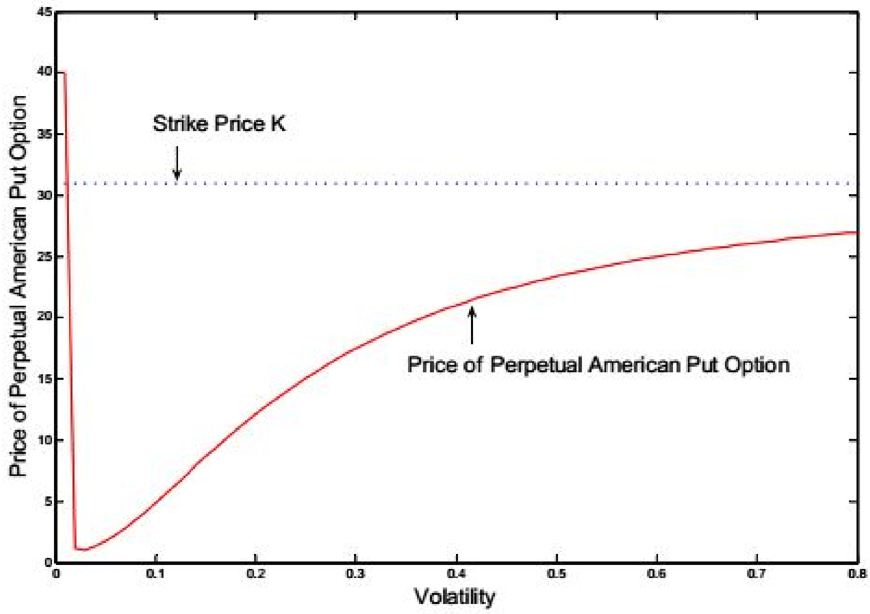

Figure 1. Price of perpetual American put option of power one at different volatility, σ.

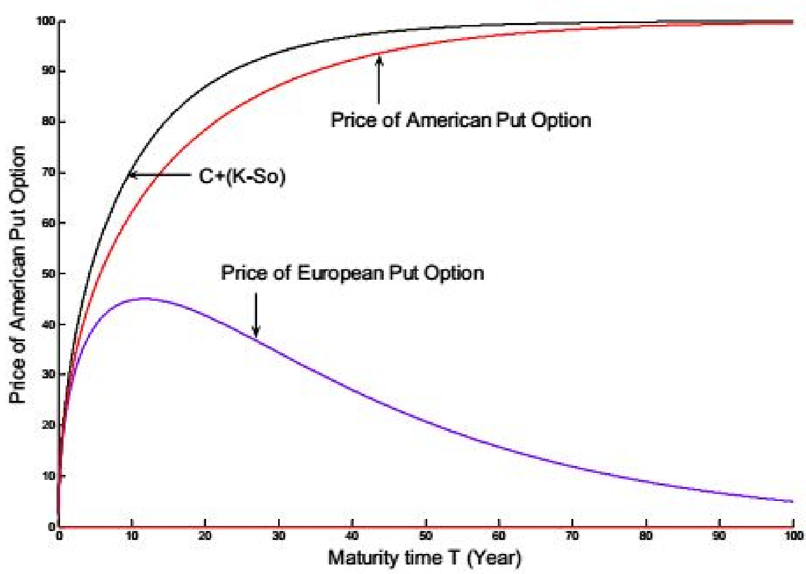

Figure 2. Price of American put option of power one at different maturity time, T for the case of ATM.

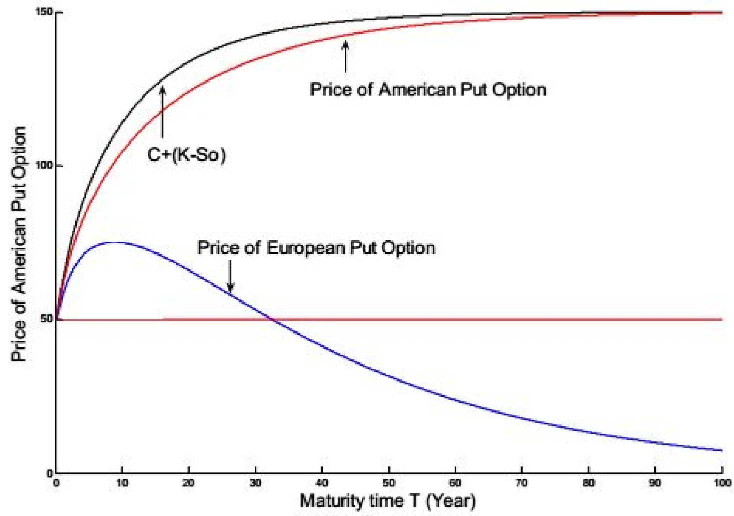

Figure 3. Price of American put option of power one at different maturity time, T for the case of ITM.

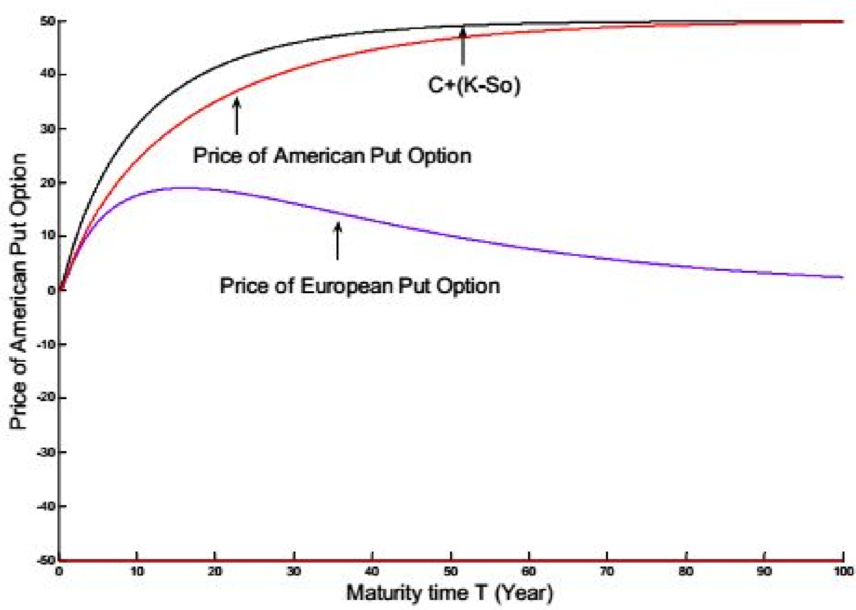

Figure 4. Price of American put option of power one at different maturity time, T for the case of OTM.

5.3 Discussion of Results

Figure 1 shows the results based on formula (3.24) for the price of perpetual American power put option for $n = 1$ at different volatility. It is observed that the result obtained satisfies the upper limit of perpetual American put option on a non-dividend yield given by (1.3). From Figures 2, 3 and 4, It is observed that the price of an American power put option for $n = 1$:

(i) increases as maturity time T increases.

(ii) tends to K as maturity time T is large.

(iii) satisfies the upper bound for American put option on a non-dividend yield.

6 Conclusion

In this paper, we considered the Mellin transforms and its applications in perpetual American power put option valuation. In option valuation, the Mellin transforms enables option equations to be solved directly in terms of market prices rather than log-prices, providing a more natural setting to the problem of valuation. The integral representations for the price and the free boundary of the American power put option was obtained. The integral representation for the price of the American power put option with non-dividend yield was used to obtain the free boundary and the fundamental valuation formula for perpetual American power put option. The main tool in this approach is the principle of smooth pasting condition. Our expression for the price of perpetual American power put option was derived as a steady-state solution[1] to the non-homogeneous Black-Scholes equation rather than as a solution to a 'static' problem[2]. We deduced that the value of the price of perpetual American power put option with $n = 1$ coincides with the value of [6]. We observed that the result obtained satisfies the upper limit of perpetual American put option on a non-dividend yield given by (1.3) as shown in Figure 1 above. We also showed that the value of a perpetual American put option of power one on a non-dividend yield is equal to its strike price. From Figures 2, 3 and 4, it is clearly seen that the price of American power put option with $n = 1$ increases as the maturity time increases and tends to K for large value of maturity time. Hence, Mellin transforms is a good approach for the valuation of American power put option with non-dividend yield.

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[1] That is, solving as a limiting case of the time to expiry tending to infinity

[2] Where the price is assumed to be independent of time

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