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UNITS IN $\mathbb{Z}(C_n \times C_5)$

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Abstract Let G be a group. Characterization of units in integral group ring $\mathbb{Z}G$ is a classical open problem for various groups explicitly. In this work, we shall introduce a subgroup of unit group in the integral group ring of the direct product which is defined as

$$C_n \times C_5 = \langle a, x : a^n = x^5 = 1, ax = xa \rangle$$

in terms of the unit group in integral group ring of C_n.

1 Introduction

Let $\mathcal{U}(\mathbb{Z}G)$ denote the unit group of the integral group ring of the group G over integers. For many years, expression of $\mathcal{U}(\mathbb{Z}G)$ as a set of generators of finite index has become a classical hard problem for various types of G. In this study, we describe the subgroups of the unit group of integral group ring $\mathbb{Z}(C_n \times C_5)$ where

$$C_n \times C_5 = \langle a, x : a^n = x^5 = 1, ax = xa \rangle$$

by using the known unit group $\mathcal{U}(\mathbb{Z}C_n)$. One can notice that if G is a finite group, then the center $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is a finitely generated abelian group of the form $\pm\mathcal{Z}(G) \times F$ where F is a free $\mathbb{Z}$-module with rank $\frac{1}{2}(|G| + n_2 + 1 - 2l)$ [13]. Here, n_2 is the number of elements of order 2 of G and l is the number of all the distinct cyclic subgroups of G. We can achieve a such F for a few cases of the group G. F had been determined for the alternating groups A_5 and A_6 in [1] and [6]. Aleev also had introduced the unit groups of integral group rings of the cyclic groups C_7 and C_9 [8]. Hoechsmann had attained the set of generators of units in group rings for abelian groups [5]. Ferraz displayed that

$$\mathcal{U}(\mathbb{Z}[\theta]) = \left\langle -1, \theta, 1+\theta, ..., (1+\theta+...+\theta^{\frac{p-1}{2}}) \right\rangle$$

therefore $\mathcal{U}(\mathbb{Z}C_p) = \pm\langle g \rangle \times \langle S \rangle$ such that

$$S = \{(1+g^t+g^{2t}+...+g^{t(r-1)})(1+g^{t^i}+g^{2t^i}+...+g^{(t-1)t^i}) - k\hat{g} : i = 1, ..., \frac{p-3}{2}\}$$

where t is a positive integer such that $\mathcal{U}(\mathbb{Z}_p) = \langle t \rangle$, r is the least positive integer such that $tr \equiv 1(modp)$, $k = \frac{tr-1}{p}$, p is a prime between 5 and 67, θ is a pth primitive root of unity [9]. Ferraz and Marcuz also have considered the groups $G = C_p \times C_2$ and $G = C_p \times C_2 \times C_2$ where p is a prime between 5 and 67. They determined the unit groups of the integral group rings of these groups [10]. Li displayed that $\mathcal{U}(\mathbb{Z}[G \times C_2]) = K \rtimes D$ such that

$$K = \{u = 1 + \alpha(1-x) : \alpha \in \mathbb{Z}G, u \in \mathcal{U}(\mathbb{Z}[G \times C_2])\}$$

and

$$D = \mathcal{U}(\mathbb{Z}G) \subset \mathcal{U}(\mathbb{Z}[G \times C_2])$$

Moreover, any element which is of the form $1 + \alpha(1-x)$ is a unit in $\mathcal{U}(\mathbb{Z}[G \times C_2])$ if and only if $1 + 2\alpha \in \mathcal{U}(\mathbb{Z}G)$ [7]. Low effectuted the following split exact sequences for $\mathcal{U}(\mathbb{Z}[G \times C_p])$ where p is a prime:

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & \mathcal{U}(\mathbb{Z}[G\times C_p]) & \xrightarrow{\pi} & \mathcal{U}(\mathbb{Z}G) \\ \cong\downarrow & & \sigma\downarrow & & \rho\downarrow \\ M & \xrightarrow{\iota} & \mathcal{U}(\mathbb{Z}[\zeta]G) & \xrightarrow{\rho} & \mathcal{U}(\mathbb{Z}_2G). \end{array}$$

and stated that
$\mathcal{U}(\mathbb{Z}[G \times C_p]) = M \rtimes \mathcal{U}(\mathbb{Z}G)$. Since $M \subset \mathcal{U}(\mathbb{Z}[\zeta]G)$, it should be note that complete characterization of M depends on getting the set of unit generators of finite index in group rings whose coefficients are from complex integral domains [4]. He also had said that M could not be characterized explicitly [4].

Kelebek constructed the normalized unit group of $\mathbb{Z}[C_n \times K_4]$ for the group

$$C_n \times K_4 = \langle a, x, y : a^n = x^2 = y^2 = 1, ax = xa, ay = ya, xy = yx \rangle$$

as

$$\mathcal{U}_1(\mathbb{Z}[C_n \times K_4]) = \mathcal{U}_1(\mathbb{Z}C_n) \times \mathcal{U}_1(1 + K^x) \times \mathcal{U}_1(1 + K^y) \times \mathcal{U}_1(1 + K^{xy})$$

where

$$\begin{aligned} \mathcal{U}_1(1 + K^x) &= \{1 + P(x-1) : 1 - 2P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \\ \mathcal{U}_1(1 + K^y) &= \{1 + P(y-1) : 1 - 2P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \\ \mathcal{U}_1(1 + K^{xy}) &= \{1 + P(x-1)(y-1) : 1 + 4P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \end{aligned}$$

2 Structure Theorem

Let $C_n = \langle a : a^n = 1 \rangle$ and $C_5 = \langle x : x^5 = 1 \rangle$ be distinct cyclic groups. We can define the group epimorphism $\varphi : C_n \times C_5 \longrightarrow C_n$ by $\varphi(a, x) = a$ or $\varphi(x) = 1$. φ can be extend to the integral group rings as follows

$$\begin{array}{rcl} \varphi : \mathbb{Z}(C_n \times C_5) & \longrightarrow & \mathbb{Z}C_n \\ \sum_{j=0}^{4} A_j x^j & \mapsto & \sum_{j=0}^{4} A_j \end{array}$$

Let $\Delta_{\mathbb{Z}C_n}(C_5)$ denote the kernel of φ. Then we can rearrange the form of $\Delta_{\mathbb{Z}C_n}(C_5)$ as follows.

Proposition 2.1. $\Delta_{\mathbb{Z}C_n}(C_5) = \langle 1 - x \rangle \oplus \langle 1 - x^2 \rangle \oplus \langle 1 - x^3 \rangle \oplus \langle 1 - x^4 \rangle$ *over* $\mathbb{Z}C_n$.

Proof.

$$\begin{aligned} \Delta_{\mathbb{Z}C_n}(C_5) &= \{\textstyle\sum_{j=0}^{4} A_j x^j : \sum_{j=0}^{4} A_j = 0, A_j \in \mathbb{Z}C_n\} \\ &= \{\textstyle\sum_{j=0}^{4} A_j x^j : A_0 = -A_1 - A_2 - A_3 - A_4\} \\ &= \{-\textstyle\sum_{j=1}^{4} A_j (1 - x^j) : A_j \in \mathbb{Z}C_n\} \\ &= \langle 1 - x \rangle + \langle 1 - x^2 \rangle + \langle 1 - x^3 \rangle + \langle 1 - x^4 \rangle \end{aligned}$$

Let us show the sum is direct. Say $\sum_{j=1}^{4} A_j (1 - x^j) = \sum_{j=1}^{4} B_j (1 - x^j)$. Then $A_j = B_j$ for all $j = 1, 2, 3, 4$. Hence

$$\Delta_{\mathbb{Z}C_n}(C_5) = \langle 1 - x \rangle \oplus \langle 1 - x^2 \rangle \oplus \langle 1 - x^3 \rangle \oplus \langle 1 - x^4 \rangle$$

Hence we can write a split exact sequence as

$$\langle 1 - x \rangle \oplus \langle 1 - x^2 \rangle \oplus \langle 1 - x^3 \rangle \oplus \langle 1 - x^4 \rangle \xrightarrow{\iota} \mathbb{Z}(C_n \times C_5) \xrightarrow{\varphi} \mathbb{Z}C_n$$

Keeping in mind that $\mathbb{Z}(C_n \times C_5) = (\mathbb{Z}C_n)C_5 = (\mathbb{Z}C_5)C_n$, we can also define another group epimorphism $\psi : C_n \times C_5 \longrightarrow C_5$ by $\psi(a, x) = x$ or $\psi(a) = 1$. Then, extending ψ linearly to the integral group rings, we obtain

$$\begin{array}{rcl} \psi : \mathbb{Z}(C_n \times C_5) & \longrightarrow & \mathbb{Z}C_5 \\ \sum_{j=0}^{n-1} B_j a^j & \mapsto & \sum_{j=0}^{n-1} B_j \end{array}$$

Let $\Delta_{\mathbb{Z}C_5}(C_n)$ be the kernel of ψ. Then we can introduce the following proposition without giving the proof which is straightforward from the previous one.

Proposition 2.2. $\Delta_{\mathbb{Z}C_5}(C_n) = \langle 1-a\rangle \oplus \ldots \oplus \langle 1-a^{n-1}\rangle$ *over* $\mathbb{Z}C_5$.

Since

$$\psi(\Delta_{\mathbb{Z}C_n}(C_5)) = \Delta_{\mathbb{Z}}(C_5) = \langle 1-x\rangle_{\mathbb{Z}} \oplus \langle 1-x^2\rangle_{\mathbb{Z}} \oplus \langle 1-x^3\rangle_{\mathbb{Z}} \oplus \langle 1-x^4\rangle_{\mathbb{Z}}$$

and

$$\varphi(\Delta_{\mathbb{Z}C_5}(C_n)) = \Delta_{\mathbb{Z}}(C_n) = \langle 1-a\rangle_{\mathbb{Z}} \oplus \ldots \oplus \langle 1-a^{n-1}\rangle_{\mathbb{Z}}$$

it can be written that

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & \Delta_{\mathbb{Z}C_5}(C_n) & \xrightarrow{\varphi} & \Delta_{\mathbb{Z}}(C_n) \\
\iota\big\downarrow & & \iota\big\downarrow & & \iota\big\downarrow \\
\Delta_{\mathbb{Z}C_n}(C_5) & \xrightarrow{\iota} & \mathbb{Z}(C_n \times C_5) & \xrightarrow{\varphi} & \mathbb{Z}C_n \\
\psi\big\downarrow & & \psi\big\downarrow & & \psi\big\downarrow \\
\Delta_{\mathbb{Z}}(C_5) & \xrightarrow{\iota} & \mathbb{Z}C_5 & \xrightarrow{\varphi} & \mathbb{Z}
\end{array}$$

Let us determine the ideal K. As

$$\varphi(\sum_{j=1}^{n-1} A_j(1-a^j)) = \sum_{j=1}^{n-1} \varphi(A_j)(1-a^j)$$

Then for all $A_j \in \mathbb{Z}C_n$,

$$\varphi(A_j) = 0 \Longleftrightarrow A_j \in \langle 1-x\rangle_{\mathbb{Z}} \oplus \langle 1-x^2\rangle_{\mathbb{Z}} \oplus \langle 1-x^3\rangle_{\mathbb{Z}} \oplus \langle 1-x^4\rangle_{\mathbb{Z}}$$

Hence,

$$\begin{aligned}
Ker(\varphi)|_{\Delta_{\mathbb{Z}C_5}(C_n)} &= \{\textstyle\sum_{j=0}^{n-1} A_j(1-a^j) : \varphi(A_i) = 0, A_i \in \mathbb{Z}C_n\} \\
&= \{\textstyle\sum_{j=0}^{n-1} A_j(1-a^j) : A_i \in Ker(\varphi)\} \\
&= \{\textstyle\sum_{j=0}^{n-1} \sum_{k=0}^{4} \alpha_{jk}(1-a^j)(1-x^k) : \alpha_{jk} \in \mathbb{Z}\} \\
&= \langle (1-a^j)(1-x^k) : j = 1, ..., n-1; k = 1, ..., 4\rangle_{\mathbb{Z}}
\end{aligned}$$

If we move all the split exact sequences to unit level, we get the following sequences.

$$\begin{array}{ccccc}
\mathcal{U}(1+K) & \xrightarrow{\iota} & \mathcal{U}(1+\Delta_{\mathbb{Z}C_5}(C_n)) & \xrightarrow{\varphi} & \mathcal{U}(1+\Delta_{\mathbb{Z}}(C_n)) \\
\iota\big\downarrow & & \iota\big\downarrow & & \iota\big\downarrow \\
\mathcal{U}(1+\Delta_{\mathbb{Z}C_n}(C_5)) & \xrightarrow{\iota} & \mathcal{U}(\mathbb{Z}(C_n \times C_5)) & \xrightarrow{\varphi} & \mathcal{U}(\mathbb{Z}C_n) \\
\psi\big\downarrow & & \psi\big\downarrow & & \psi\big\downarrow \\
\mathcal{U}(1+\Delta_{\mathbb{Z}}(C_5)) & \xrightarrow{\iota} & \mathcal{U}(\mathbb{Z}C_5) & \xrightarrow{\varphi} & \mathcal{U}(\mathbb{Z})
\end{array}$$

As the embedding functions can be regarded as the reverse directions of φ and ψ, all these sequences split. This gives us the way on which we can state the unit group of $\mathbb{Z}(C_n \times C_5)$ as follows:

Corollary 2.3.

$$\mathcal{U}(\mathbb{Z}(C_n \times C_5)) = \mathcal{U}(\mathbb{Z}C_5) \times \mathcal{U}(1+\Delta_{\mathbb{Z}C_5}(C_n)) = \mathcal{U}(\mathbb{Z}C_n) \times \mathcal{U}(1+\Delta_{\mathbb{Z}C_n}(C_5))$$

Let $F(G)$ denote the torsion-free part of the unit group of the integral group ring $\mathbb{Z}G$. Since $\mathcal{U}(\mathbb{Z}) = \{\pm 1\}$, we obtain the following corollary:

Corollary 2.4.

$$F(C_n) \subseteq \mathcal{U}(1 + \Delta_{\mathbb{Z}}(C_n))$$

and

$$F(C_5) \subseteq \mathcal{U}(1 + \Delta_{\mathbb{Z}}(C_5))$$

Corollary 2.5.

$$\mathcal{U}(\mathbb{Z}(C_n \times C_5)) = (C_n \times C_5) \times F(C_n) \times F(C_5) \times \mathcal{U}(1 + K)$$

By splitting $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ into its subgroups, it is clear that the complete characterization of the unit group $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ depends on determining the subgroup $\mathcal{U}(1 + K) = \mathcal{U}(1 + \langle(1 - a^j)(1 - x^k)\rangle_{\mathbb{Z}})$. For some orders n, the rank of $\mathcal{U}(1 + K)$ can be calculated however we now need to give a very useful result of Tóth [12].

Proposition 2.6. *Let C_{n_1} and C_{n_2} be two cyclic groups have orders n_1 and n_2 respectively and ϕ be Euler's totient function. Then for every $n_1, n_2 \geq 1$ the number of cyclic subgroups of $C_{n_1} \times C_{n_2}$ is*

$$c(n_1, n_2) = \sum_{d_1|n_1, d_2|n_2} \phi(gcd(d_1, d_2))$$

Theorem 2.7. *Let $n = 5p^k$ where $p(\neq 5)$ is prime. Then, the rank of torsion-free part of the unit subgroup $\mathcal{U}(1 + K)$ is determined by the following formula:*

$$s(p, k) := 10p^k - 4k - 5.$$

Proof. We explain the proof with two cases:
Case 1. Let $p = 2$. Then, the rank of torsion-free part of the unit group $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ can easily be calculated by Ayoub and Ayoub [14]. It is trivial that the order of $C_n \times C_5$ is $25p^k$. We also need the number n_2 and l to complete the proof. These numbers can be seen at the table below:

$\lvert g\rvert$	1	5	p	p^2	$\dots$	p^k	$5p$	$5p^2$	$\dots$	$5p^k$
a^j	1	a^{p^k}	$a^{5p^{k-1}}$	$a^{5p^{k-2}}$	$\dots$	a^5	$a^{p^{k-1}}$	$a^{p^{k-2}}$	$\dots$	a
xa^j	–	x, xa^{p^k}	–	–	–	–	$xa^{p^{k-1}}$	$xa^{p^{k-2}}$	$\dots$	xa
x^2a^j	–	$x^2, x^2a^{p^k}$	–	–	–	–	$x^2a^{p^{k-1}}$	$x^2a^{p^{k-2}}$	$\dots$	x^2a
x^3a^j	–	$x^3, x^3a^{p^k}$	–	–	–	–	$x^3a^{p^{k-1}}$	$x^3a^{p^{k-2}}$	$\dots$	x^3a
x^4a^j	–	$x^4, x^4a^{p^k}$	–	–	–	–	$x^4a^{p^{k-1}}$	$x^4a^{p^{k-2}}$	$\dots$	x^4a

This table show us that $n_2 = 1$. We also have $6k + 10$ elements which satisfy $\langle x\rangle = \langle x^4\rangle$, $\langle x^2\rangle = \langle x^3\rangle$, $\left\langle xa^{p^{k-1}}\right\rangle = \left\langle x^4a^{p^{k-1}}\right\rangle$, $\left\langle x^2a^{p^{k-1}}\right\rangle = \left\langle x^3a^{p^{k-1}}\right\rangle$. This means there are $6k + 6$ distinct cyclic subgroups of the group $C_n \times C_5$. Actually, we can also calculate the number of cyclic subgroups of $C_n \times C_5$ from [12] since $i = 0, 1$ as follows

$$c(5p^k, 5) = \sum_{d_1|5p^k, d_2|5} \phi(gcd(d_1, d_2)) = \sum_{j=1}^{k} \phi(gcd(5^i p^j, 1)) + \phi(gcd(5^i p^j, 5)).$$

Thus, we confirm that $c(5p^k, 5) = (2k + 2)\phi(1) + (k + 1)\phi(5) = 6k + 6$. Hence, the rank of torsion-free part of the unit group $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ is obtained as $25p^{k-1} - 6k - 5$. Besides, it can be easily computed that the rank of the unit group $\mathcal{U}(\mathbb{Z}C_n)$ as $5p^{k-1} - 2k - 1$ and Karpilovsky displayed that the unit group $\mathcal{U}(\mathbb{Z}C_5)$ has a single generator. All the these parameters give us from Corollary 2.5. that the rank of $\mathcal{U}(1 + K)$ is $10p^k - 4k - 5$.
Case 2. Let $p \neq 2$. Then since the order of $C_n \times C_5$ is odd, the parameter n_2 is 0. We know also

that there are $6k+6$ distinct cyclic subgroups of $C_n \times C_5$. Hence, the rank of $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ is $\frac{25p^k-12k-11}{2}$ and then the rank of $\mathcal{U}(1+K)$ is obtained as $10p^k - 4k - 5$. ■

Example.

p	k	n	*Group*	*Rank of* $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$	$s(p,k)$
2	1	10	$C_{10} \times C_5$	14	11
2	2	20	$C_{20} \times C_5$	33	27
3	1	15	$C_{15} \times C_5$	26	21
3	2	45	$C_{45} \times C_5$	95	77
5	1	25	$C_{25} \times C_5$	51	41
5	2	125	$C_{125} \times C_5$	295	237
7	1	35	$C_{35} \times C_5$	83	61
7	2	245	$C_{245} \times C_5$	595	477

As we stated before, an explicit characterization of the unit group $\mathcal{U}(\mathbb{Z}(C_n \times C_5))$ can be introduced if $\mathcal{U}(1+K), \mathcal{U}(1+\Delta_{\mathbb{Z}C_5}(C_n))$ or $\mathcal{U}(1+\Delta_{\mathbb{Z}C_n}(C_5))$ can be expressed clearly. Now, let us state and prove our main result as follows:

Theorem 2.8. *Let* $C_n \times C_5 = \langle a, x : a^n = x^5 = 1, ax = xa \rangle$. *Then*

$$\mathcal{U}(\mathbb{Z}(C_n \times C_5)) = \mathcal{U}(\mathbb{Z}C_n) \times \{1 + \sum_{i=1}^{4} A_i(1-x^i) : A_i \in \mathbb{Z}C_n\}$$

if and only if the matrix

$$\begin{bmatrix} 1+A_1+\sum A_i & A_1-A_4 & A_1-A_3 & A_1-A_2 \\ -A_1+A_2 & 1+A_2+\sum A_i & A_2-A_4 & A_2-A_3 \\ -A_2+A_3 & -A_1+A_3 & 1+A_3+\sum A_i & A_3-A_4 \\ -A_3+A_4 & -A_2+A_4 & -A_1+A_4 & 1+A_4+\sum A_i \end{bmatrix}$$

is invertible in $\mathcal{M}_4(\mathbb{Z}C_n)$.

Proof. Let $v_i := 1 - x^i$. Then

$$\Delta_{\mathbb{Z}C_n}(C_5) = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \langle v_3 \rangle \oplus \langle v_4 \rangle$$

is a $\mathbb{Z}C_n$-algebra of the following multiplication:

$\cdot$	v_1	v_2	v_3	v_4
v_1	$2v_1 - v_2$	$v_1 + v_2 - v_3$	$v_1 + v_3 - v_4$	$v_1 + v_4$
v_2	$v_1 + v_2 - v_3$	$2v_2 - v_4$	$v_2 + v_3$	$-v_1 + v_2 + v_4$
v_3	$v_1 + v_3 - v_4$	$v_2 + v_3$	$-v_1 + 2v_3$	$-v_2 + v_3 + v_4$
v_4	$v_1 + v_4$	$-v_1 + v_2 + v_4$	$-v_2 + v_3 + v_4$	$-v_3 + 2v_4$

One can clearly see that $\Delta_{\mathbb{Z}C_n}(C_5)$ is also closed under addition and scalar multiplication. As

$$\mathcal{U}(1+\Delta_{\mathbb{Z}C_n}(C_5)) = [1+\Delta_{\mathbb{Z}C_n}(C_5)] \cap \mathcal{U}(\mathbb{Z}C_n)$$

we must investigate the units of the form $u = 1 + \sum_{i=1}^{4} A_i v_i$. An element of the form $u = 1 + \sum_{i=1}^{4} A_i v_i$ is a unit if and only if $\exists u^{-1} = 1 + \sum_{i=1}^{4} B_i v_i$ such that $A_i, B_i \in \mathbb{Z}C_n$ and

$uu^{-1} = 1$. By the above multiplication table, we can get

$$\begin{aligned} uu^{-1} = 1 \quad &+ \quad v_1[A_1 + B_1 + 2A_1B_1 + A_2B_1 + A_3B_1 \\ &\quad +A_4B_1 + A_1B_2 - A_4B_2 + A_1B_3 - A_3B_3 + A_1B_4 - A_2B_4] \\ &+ \quad v_2[A_2 + B_2 - A_1B_1 + A_2B_1 + A_1B_2 + 2A_2B_2 + A_3B_2 \\ &\quad +A_4B_2 + A_2B_3 - A_4B_3 + A_2B_4 - A_3B_4] \\ &+ \quad v_3[A_3 + B_3 - A_2B_1 + A_3B_1 - A_1B_2 + A_3B_2 + A_1B_3 \\ &\quad +A_2B_3 + 2A_3B_3 + A_4B_3 + A_3B_4 - A_4B_4] \\ &+ \quad v_4[A_4 + B_4 - A_3B_1 + A_4B_1 - A_2B_2 + A_4B_2 - A_1B_3 + A_4B_3 \\ &\quad +A_1B_4 + A_2B_4 + A_3B_4 + 2A_4B_4] = 1 \end{aligned}$$

It is clear that this equation is hold if and only if

$$i) \begin{array}{l} A_1 + B_1 + 2A_1B_1 + A_2B_1 + A_3B_1 + A_4B_1 + A_1B_2 \\ -A_4B_2 + A_1B_3 - A_3B_3 + A_1B_4 - A_2B_4 = 0 \end{array}$$

$$ii) \begin{array}{l} A_2 + B_2 - A_1B_1 + A_2B_1 + A_1B_2 + 2A_2B_2 + A_3B_2 \\ +A_4B_2 + A_2B_3 - A_4B_3 + A_2B_4 - A_3B_4 = 0 \end{array}$$

$$iii) \begin{array}{l} A_3 + B_3 - A_2B_1 + A_3B_1 - A_1B_2 + A_3B_2 + A_1B_3 \\ +A_2B_3 + 2A_3B_3 + A_4B_3 + A_3B_4 - A_4B_4 = 0 \end{array}$$

$$iv) \begin{array}{l} A_4 + B_4 - A_3B_1 + A_4B_1 - A_2B_2 + A_4B_2 - A_1B_3 \\ +A_4B_3 + A_1B_4 + A_2B_4 + A_3B_4 + 2A_4B_4 = 0 \end{array}$$

Therefore, since

$$\vec{X} := [A_1, A_2, A_3, A_4]^T, \vec{Y} := [B_1, B_2, B_3, B_4]^T$$

and

$$A := \begin{bmatrix} 1 + A_1 + \sum A_i & A_1 - A_4 & A_1 - A_3 & A_1 - A_2 \\ -A_1 + A_2 & 1 + A_2 + \sum A_i & A_2 - A_4 & A_2 - A_3 \\ -A_2 + A_3 & -A_1 + A_3 & 1 + A_3 + \sum A_i & A_3 - A_4 \\ -A_3 + A_4 & -A_2 + A_4 & -A_1 + A_4 & 1 + A_4 + \sum A_i \end{bmatrix}$$

we conclude from the uniqueness of the inverse of a unit that $A\vec{Y} = -\vec{X}$ has a unique solution in integral group ring $\mathbb{Z}C_n$. That is $A \in GL(4, \mathbb{Z}C_n)$. ■

The relation between the units in $\mathbb{Z}(C_n \times C_5)$ and the units in $\mathbb{Z}C_n$ comes from the determinant of this matrix which is very complicated. Hence, we consider some restrictions on the parameters $A_j's$.

Lemma 2.9.

$$S := \{\sum_{j=1}^{4} A_j v_j : A_1 = A_4, A_2 = A_3, \forall A_j \in \mathbb{Z}C_n\}$$

is a $\mathbb{Z}C_n$-subalgebra of $\Delta_{\mathbb{Z}C_n}(C_5)$.

Proof. Let $A_1 = A_4$ and $A_2 = A_3$ in $\Delta_{\mathbb{Z}C_n}(C_5)$. Then,

$$S = \langle v_1 + v_4 \rangle \oplus \langle v_2 + v_3 \rangle$$

and we attain the following multiplications:

.	$v_1 + v_4$	$v_2 + v_3$
$v_1 + v_4$	$4(v_1 + v_4) - (v_2 + v_3)$	$(v_1 + v_4) + (v_2 + v_3)$
$v_2 + v_3$	$(v_1 + v_4) + (v_2 + v_3)$	$4(v_2 + v_3) - (v_1 + v_4)$

It is clear that addition and scalar multiplication are also closed in S. ■
Since

$$\mathcal{U}(1+S) = (1+S) \cap \mathcal{U}(1+\Delta_{ZC_n}(C_5))$$

we need to get units of the form

$$1 + A_1(v_1+v_4) + A_2(v_2+v_3)$$

Then $u = 1 + A_1(v_1+v_4) + A_2(v_2+v_3)$ is a unit in $\mathcal{U}(1+S)$ if and only if there is an element $u^{-1} = 1 + B_1(v_1+v_4) + B_2(v_2+v_3)$ such that $uu^{-1} = 1$. Therefore,

$$\begin{aligned} uu^{-1} = 1 \quad &+ \quad (v_1+v_4)[A_1 + B_1 + 4A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2] \\ &+ \quad (v_2+v_3)[A_2 + B_2 - A_1B_1 + A_1B_2 + A_2B_1 + 4A_2B_2] = 1 \end{aligned}$$

Hence,

$$\begin{aligned} A_1 + B_1 + 4A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 = 0 \\ A_2 + B_2 - A_1B_1 + A_1B_2 + A_2B_1 + 4A_2B_2 = 0 \end{aligned}$$

and

$$\begin{bmatrix} 1+4A_1+A_2 & A_1-A_2 \\ -A_1+A_2 & 1+A_1+4A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -A_1 \\ -A_2 \end{bmatrix}$$

has a unique solution in $\mathbb{Z}C_n$. Thus,

$$1 + 5(A_1^2 + A_2^2 + 3A_1A_2 + A_1 + A_2) \in \mathcal{U}(\mathbb{Z}C_n)$$

If we also consider the conditions $A_1 = A_4$ and $A_2 = A_3$ in the matrix A, we get the LU decomposition of A by using a computer software as

$$L = (l_{ij})_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{A_1-A_2}{1+3A_1+2A_2} & 1 & 0 & 0 \\ 0 & -\frac{A_1-A_2}{1+3A_2+2A_1} & 1 & 0 \\ \frac{A_1-A_2}{1+3A_1+2A_2} & \frac{A_1-A_2}{1+3A_2+2A_1} & 0 & 1 \end{bmatrix}$$

and

$$U = (u_{ij})_{4\times 4} = \begin{bmatrix} 1+3A_1+2A_2 & 0 & A_1 - A_2 & A_1 - A_2 \\ 0 & 1+3A_2+2A_1 & -\frac{2A_1^2+A_1A_2-3A_2^2+A_1-A_2}{1+3A_1+2A_2} & \frac{(A_1-A_2)^2}{1+3A_1+2A_2} \\ 0 & 0 & -\frac{5A_1^2+15A_1A_2+5A_2^2+5A_1+5A_2}{1+3A_1+2A_2} & -\frac{5A_1^3+10A_1^2A_2-10A_1A_2^2-5A_2^3+5A_1^2-5A_2^2+A_1-A_2}{(1+3A_1+2A_2)(1+3A_2+2A_1)} \\ 0 & 0 & 0 & -\frac{5A_1^2+15A_1A_2+5A_2^2+5A_1+5A_2}{1+3A_2+2A_1} \end{bmatrix}$$

Since the entries l_{ij} and u_{ij} are elements in $\mathbb{Z}C_n$ for all $i, j \in \{1, 2, 3, 4\}$, we conclude that $1 + 3A_1 + 2A_2$ and $1 + 3A_2 + 2A_1$ must be units.

Corollary 2.10. *Let $A_1, A_2 \in \mathbb{Z}C_n$ such that*

$i)1 + 5(A_1^2 + A_2^2 + 3A_1A_2 + A_1 + A_2) \in \mathcal{U}(\mathbb{Z}C_n)$

$ii)1 + 3A_1 + 2A_2 \in \mathcal{U}(\mathbb{Z}C_n)$

$iii)1 + 3A_2 + 2A_1 \in \mathcal{U}(\mathbb{Z}C_n)$. *Then*

$$\mathcal{U}(1 + \Delta_{\mathbb{Z}C_n}(C_5)) \supset \mathcal{U}(1 + S) = \{1 + A_1(v_1 + v_4) + A_2(v_2 + v_3) : v_j = 1 - x^j\}$$

Remark. One can notice that if $u_1 = 1 + 3A_1 + 2A_2$, $u_2 = 1 + 3A_2 + 2A_1$ and $v = 1 + 5(A_1^2 + A_2^2 + 3A_1A_2 + A_1 + A_2)$ are units in $\mathbb{Z}C_n$,

$$u_1u_2 - (A_1 - A_2)^2 = v$$

Here, the term $-(A_1 - A_2)^2$ may not be a special element in $\mathbb{Z}C_n$. However, if we especially consider $-(A_1 - A_2)^2$ as a nilpotent element in $\mathbb{Z}C_n$, this last equality is satisfied since the sum of a unit and a nilpotent element is also a unit. Besides, we can say the nilpotent element is only 0 in $\mathbb{Z}C_n$ from Proposition 4 in [11]. Thus, if $-(A_1 - A_2)^2$ is a nilpotent element in $\mathbb{Z}C_n$, then $A_1 = A_2 = \alpha$. Let us define

$$\mathcal{U}(1 + S)_0 = \{1 + A_1(v_1 + v_4) + A_2(v_2 + v_3) : v_j = 1 - x^j, A_1 = A_2\}$$

Therefore we can illustrate to find generators of $\mathcal{U}(1 + S)_0 \subset \mathcal{U}(1 + S)$ satisfy the condition $1 + 5\alpha \in \mathcal{U}(\mathbb{Z}C_n)$ for some $n \in \mathbb{N}$.

Example Let $n = 8$. Then we know from [13] that $\mathcal{U}(\mathbb{Z}C_8) = \pm C_8 \times \langle u \rangle$ where $u = 2 + a - a^3 - a^4 - a^5 + a^7$. A straightforward computation gives us that

$$u^7 = 1 + 5(1960 + 1386a - 1386a^3 - 1960a^4 - 1386a^5 + 1386a^7)$$

Hence, by taking

$$\alpha = 1960 + 1386a - 1386a^3 - 1960a^4 - 1386a^5 + 1386a^7$$

we can say that $\mathcal{U}(1 + S)_0$ is generated by $1 + \alpha(v_1 + v_2 + v_3 + v_4)$. □
More examples can be introduced for $n \in \mathbb{N}$ for which the generators of $\mathcal{U}(\mathbb{Z}C_n)$ are obvious explicitly.

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