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ISOMETRIC IMMERSION OF 3-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. In this paper we study a three dimensional normal almost contact metric manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1.

1 Introduction

Let M be an almost contact metric manifold and (ϕ, ξ, η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and ϕ, ξ, η are tensor fields on M of types $(1, 1)$, $(1, 0)$ and $(0, 1)$ respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{1.1}$$

Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt}), \tag{1.2}$$

where the pair $(X, \lambda\frac{d}{dt})$ denotes a tangent vector on $M \times \mathbb{R}$, X and $\lambda\frac{d}{dt}$ being tangent to M and $\mathbb{R}$ respectively.

M and (ϕ, ξ, η) are said to be normal if the structure J is integrable ([1],[2]).
The necessary and sufficient condition for (ϕ, ξ, η) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{1.3}$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{1.4}$$

for any $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on M.

We say that the form η has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta\wedge(d\eta)^s = 0$ and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say r is rank of the structure (ϕ, ξ, η).

A Riemannian metric g on M satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

for any $X, Y \in \chi(M)$, is said to be compatible with the structure (ϕ, ξ, η). If g is such a metric, then the quadruple (ϕ, ξ, η, g) is called an almost contact metric structure on M and M is an almost contact metric manifold. On such a manifold we also have

$$\eta(X) = g(X, \xi) \tag{1.6}$$

for any $X \in \chi(M)$ and we can always define the 2-form Φ by

$$\Phi(Y, Z) = g(Y, \phi Z), \tag{1.7}$$

where $X, Y \in \chi(M)$.

It is no hard to see that if dim $M = 3$, then two Riemannian metric g and g' are compatible with the same almost contact structure (ϕ, ξ, η) on M if and only if

$$g' = \sigma g + (1 - \sigma)\eta \otimes \eta$$

for a certain positive function σ on M.

A normal almost contact metric structure (ϕ, ξ, η, g) satisfying additionally the condition $d\eta = \phi$ is called Sasakian. Of course, any such structure on M has rank 3. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [3]. In the paper [10], Olszak studied the curvature properties of normal almost contact manifold of dimension three with several examples. Also in [6], U. C. De and A. K. Mondal studied three dimensional normal almost contact metric manifolds satisfying certain curvature conditions.

T. Takahashi and S. Tanno [11] introduced the notion of isometric immersion on K-contact manifolds. D. E. Blair, T. Koufogiorgos and R. Sharma [4] studied isometric immersion for three dimensional contact manifolds satisfying $Q\phi = \phi Q$. In [7], U. C. De, A. Yildiz and A. Sarkar studied isometric immersion of three dimensional quasi-Sasakian manifolds. The quasi-Sasakian manifolds are particular types of normal almost contact metric manifolds. So in this paper we like to study isometric immersion on three-dimensional normal almost contact metric manifold.

The object of the present paper is to study a three-dimensional normal almost contact metric manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1.

The present paper is organized as follows:
Section 1 is the introductory section. In section 2 we give some preliminary notion of three dimensional normal almost contact metric manifolds. In section 3 we derive some results of three-dimensional normal almost contact metric manifolds isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1. In this section we also prove a necessary and sufficient condition for the immersion to be minimal. Finally in section 4 we construct an example of three-dimensional normal almost contact metric manifold which illustrates some results obtained in section 3.

2 Preliminaries

For a normal almost contact metric structure (ϕ, ξ, η, g) on M, we have [10]

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y) - \eta(Y)\phi \nabla_X \xi, \tag{2.1}$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] - \beta \phi X, \tag{2.2}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergent of ξ defined by $div\xi = trace\{X \longrightarrow \nabla_X \xi\}$ and $tr(\phi\nabla\xi) = trace\{X \longrightarrow \phi\nabla_X \xi\}$. Using (2.2) in (2.1) we get

$$(\nabla_X \phi)(Y) = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X]. \tag{2.3}$$

Also in this manifold the following relations hold:

$$\begin{aligned} R(X,Y)\xi &= [Y\alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X \\ &\quad -[X\alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y \\ &\quad +[Y\beta + 2\alpha\beta\eta(Y)]\phi X \\ &\quad -[X\beta + 2\alpha\beta\eta(X)]\phi Y, \end{aligned} \tag{2.4}$$

$$\begin{aligned} S(X,\xi) &= -X\alpha - (\phi X)\beta \\ &\quad -[\xi\alpha + 2(\alpha^2 - \beta^2)]\eta(X), \end{aligned} \tag{2.5}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.6}$$

where R denotes the curvature tensor and S is the Ricci tensor.

On the other hand, the curvature tensor in three dimensional Riemannian manifold always satisfies

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.7}$$

where r is the scalar curvature of the manifold.
By (2.4), (2.5) and (2.8) we can derive

$$\begin{aligned} S(Y,Z) &= (\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) \\ &\quad -\eta(Y)(Z\alpha + (\phi Z)\beta) - \eta(Z)(Y\alpha + (\phi Y)\beta) \\ &\quad -2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned} \tag{2.8}$$

From (2.6) it follows that if α, β = constant, then the manifold is either β-Sasakian or α-Kenmotsu [9] or cosympletic [1].
Also we have a 3-dimensional normal almost contact metric manifold is quasi-Sasakian if and only if $\alpha = 0$ [10].

Now we prove a Lemma here:

Lemma 2.1. *A three dimensional compact normal almost contact metric manifold satisfying $\phi^2(grad\alpha) + \phi(grad\beta) = 0$ is a quasi-Sasakian manifold, provided β is a non-zero constant.*

Proof. From the relation $\phi^2(grad\alpha) + \phi(grad\beta) = 0$, we obtain

$$-grad\alpha + (\xi\alpha)\xi + \phi(grad\beta) = 0. \tag{2.9}$$

Taking inner product with X in (2.9) yields

$$g(grad\beta, \phi X) + d\alpha(X) - (\xi\alpha)\eta(X) = 0. \tag{2.10}$$

Differentiating (2.10) covariantly with respect to Y, we obtain

$$\begin{aligned} g(\nabla_Y grad\beta, \phi X) &+ g(grad\beta, (\nabla_Y \phi)X) + (\nabla_Y d\alpha)X \\ &- g(\nabla_Y grad\alpha, \xi)\eta(X) - (\xi\alpha)(\nabla_Y \eta)(X) = 0. \end{aligned} \tag{2.11}$$

Interchanging X and Y in (2.11), we get

$$\begin{aligned} g(\nabla_X grad\beta, \phi Y) &+ g(grad\beta, (\nabla_X \phi)Y) + (\nabla_X d\alpha)Y \\ &- g(\nabla_X grad\alpha, \xi)\eta(Y) - (\xi\alpha)(\nabla_X \eta)(Y) = 0. \end{aligned} \tag{2.12}$$

Subtracting (2.11) from (2.12), we have

$$\begin{aligned} & g(\nabla_X grad\beta, \phi Y) - g(\nabla_Y grad\beta, \phi X) + g(grad\beta, (\nabla_X \phi)Y) \\ - & g(grad\beta, (\nabla_Y \phi)X) - g(\nabla_X grad\alpha, \xi)\eta(Y) + g(\nabla_Y grad\alpha, \xi)\eta(X) \\ - & (\xi\alpha)[(\nabla_X \eta)Y - (\nabla_Y \eta)X] = 0. \end{aligned} \tag{2.13}$$

Let $\{E_0, E_1, E_2\}$ be a ϕ-basis on the manofold where $E_0 = \xi$ and $\phi E_1 = E_2$. Taking $X = E_1, Y = E_2$ in (2.13) and using $(\nabla_{E_1}\phi)E_2 = \alpha\xi$, $(\nabla_{E_2}\phi)E_1 = -\alpha\xi$ and $(\nabla_{E_1}\eta)E_2 = -\beta$, we obtain

$$g(\nabla_{E_1} grad\beta, E_1) + g(\nabla_{E_2} grad\beta, E_2) = 2\alpha(\xi\beta) + 2(\xi\alpha)\beta. \tag{2.14}$$

Differentiating (2.6) covariantly with respectively ξ, we get

$$g(\nabla_\xi grad\beta, \xi) = -2\beta(\xi\alpha) - 2\alpha(\xi\beta). \tag{2.15}$$

Adding (2.14) and (2.15), we obtain

$$\Delta\beta = 0. \tag{2.16}$$

Since the manifold is compact we have β is constant. If β is a non-zero constant then we easily obtain from (2.6) that $\alpha = 0$ and hence the manifold becomes a quasi-Sasakian manifold. This proves the lemma.□ □

3 Isometric immersion of three-dimensional normal almost contact metric manifolds

Definition 3.1. Let M and M' be smooth manifolds of dimension m and m' respectively. If $f : M \to M'$ is a smooth map and $f_{*x} : T_x(M) \to T_{f(x)}(M')$ is the tangential map at $x \in M$ then f is said to be an immersion if f_{*x} is injective for each $x \in M$.

Let M and M' be two Riemannian manifolds with Riemannian metric g and g' respectively. A map $f : M \longrightarrow M'$ is called isometric at a point x of M if $g(X,Y) = g'(f_* X, f_* Y)$, for al $X, Y \in T_x(M)$.
An immersion f which is isometric at every point of M is called an isometric immersion [12].

If X and Y are two vector fields on a manifold M which is immersed in a Riemannian manifold M' then we know that [12] $B(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where B is the second fundamental form and $\tilde{\nabla}$ and ∇ denote the covariant differentiation with respect to the Levi-Civita connection in M and M' respectively.

We consider a three-dimensional normal almost contact metric manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1. Then we can write the Gauss and Codazzi equations as [5]

$$R(X,Y) = X\Lambda Y + AX\Lambda AY, \tag{3.1}$$

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY, \tag{3.2}$$

$$(\nabla_X A)(Y) = (\nabla_Y A)(X), \tag{3.3}$$

where A is a $(1,1)$ tensor field associated with second fundamental form B given by $B(X,Y) = g(AX,Y)$. A is symmetric with respect to g. If the trace of A vanishes then the immersion is called minimal. The type number of the immersion is equal to the rank of A. From (3.2) it follows that

$$\begin{aligned} g(R(X,Y)Z,U) &= g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \\ &\quad + g(AY,Z)g(AX,U) - g(AX,Z)g(AY,U). \end{aligned}$$

In the above equation putting $X = U = e_i$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at each point of the manifold M and taking summation over i, we get

$$S(Y,Z) = 2g(Y,Z) + g(AY,Z)\theta - g(AAY,Z), \tag{3.4}$$

where θ is the trace of A. Replacing Z by ξ we get from (3.4)

$$S(Y,\xi) = 2g(Y,\xi) + g(AY,\xi)\theta - g(AAY,\xi). \tag{3.5}$$

In view of (2.5) we get from (3.5)

$$\begin{aligned} -(Y\alpha) - (\phi Y)\beta \quad - \quad & \{\xi\alpha + 2(\alpha^2 - \beta^2) + 2\}\eta(Y) \\ & = g(AY,\xi) - g(AAY,\xi). \end{aligned} \tag{3.6}$$

For $g(gradf, X) = df(X)$, symmetry of A and skew-symetry of ϕ, the equation (3.6) implies

$$\begin{aligned} -grad\alpha + \phi(grad\beta) \quad - \quad & \{\xi\alpha + 2(\alpha^2 - \beta^2) + 2\}\xi \\ & = \theta(A\xi) - AA\xi. \end{aligned} \tag{3.7}$$

If $\theta = 0$ the equation (3.7) reduces to

$$-grad\alpha + \phi(grad\beta) - \{\xi\alpha + 2(\alpha^2 - \beta^2) + 2\}\xi + AA\xi = 0. \tag{3.8}$$

Thus we can state the following:

Theorem 3.1. If a three dimensional normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1 and if the immersion is minimal then (3.8) holds.

We now suppose that the relation (3.8) holds. Then in view of (3.7) we have $\theta A\xi = 0$. Therefore either $\theta = 0$ or $A\xi = 0$. If $A\xi = 0$, then from (3.7) we get

$$-grad\alpha + \phi(grad\beta) - \{\xi\alpha + 2(\alpha^2 - \beta^2) + 2\}\xi = 0. \tag{3.9}$$

Applying ϕ on both sides of (3.9), we obtain

$$\phi^2(grad\beta) - \phi(grad\alpha) = 0. \tag{3.10}$$

In view of the Lemma 2.1. we state the following:

Theorem 3.2. If a three dimensional compact normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1 and if (3.8) holds then either the immersion is minimal or the manifold is a quasi-Sasakian manifold, β being a non-zero constant.

By virtue of (1.1) and (2.6) we obtain from (2.8)

$$S(\phi Y, \phi Z) = (\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z). \tag{3.11}$$

From (3.4) we also have

$$S(\phi Y, \phi Z) = 2g(\phi Y, \phi Z) + g(A\phi Y, \phi Z)\theta - g(AA\phi Y, \phi Z). \tag{3.12}$$

From (3.11) and (3.12), we get

$$(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2 - 2)g(\phi^2 Y, Z) - g(\phi A\phi Y, Z)\theta + g(\phi AA\phi Y, Z) = 0. \tag{3.13}$$

We obtain from (3.13)

$$(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2 - 2)\phi^2 - \theta\phi A\phi + \phi AA\phi = 0. \tag{3.14}$$

If $\theta = 0$, then (3.14) reduces to

$$(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2 - 2)\phi^2 + \phi AA\phi = 0. \tag{3.15}$$

Thus we can state the following:

Theorem 3.3. If a three dimensional normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1 and if the immersion is minimal then (3.15) holds.

Next let (3.15) holds. Then from (3.14) we get $\theta\phi A\phi = 0$. Hence either $\theta = 0$ or $\phi A\phi = 0$. Hence we can state the following:

Theorem 3.4. If a three dimensional normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1 and if (3.15) holds then either the immersion is minimal or $\phi A\phi = 0$.

Combining Theorem 3.3 and Theorem 3.4 we get a necessary and sufficient condition for the immersion is minimal as the following:

Theorem 3.5. If a three dimensional normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1, then the immersion is minimal if and only if (3.13) holds, provided $\phi A\phi \neq 0$.

Putting $Z = \xi$ in (3.2) and using (2.4), we obtain

$$\begin{aligned} & [Y\alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X \\ & -[X\alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y \\ & +[Y\beta + 2\alpha\beta\eta(Y)]\phi X \\ & -[X\beta + 2\alpha\beta\eta(X)]\phi Y \\ = \; & \eta(Y)X - \eta(X)Y + \eta(AY)AX - \eta(AX)AY. \end{aligned} \tag{3.16}$$

Putting $Y = \xi$ in (3.16) and using (1.1), (2.6) yields

$$(\xi\alpha + \alpha^2 - \beta^2 + 1)[X - \eta(X)\xi] + \eta(A\xi)AX - \eta(AX)A\xi = 0. \tag{3.17}$$

Now $g(AX, Y) = B(X, Y)$ and we know that $B(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$. Hence

$$g(A\xi, \xi) = B(\xi, \xi) = \tilde{\nabla}_\xi \xi - \nabla_\xi \xi, \tag{3.18}$$

and

$$g(AX, \xi) = B(X, \xi) = \tilde{\nabla}_X \xi - \nabla_X \xi. \tag{3.19}$$

Using (3.18), (3.19) in (3.17), we obtain

$$(\xi\alpha + \alpha^2 - \beta^2 + 1)[X - \eta(X)\xi] - (\tilde{\nabla}_X \xi - \nabla_X \xi)A\xi = 0. \tag{3.20}$$

From [8] we know that $2\tilde{\nabla}_X X = gradf$, where $f = g(X, X)$ is a smooth function on a Riemannian manifold endowed with a metric g. Then for $X = \xi$ and $g(\xi, \xi) = 1$, we get $\tilde{\nabla}_\xi \xi = 0$, since $grad1 = 0$. Also from (2.1) it follows that $\nabla_\xi \xi = 0$. Hence applying ϕ on both sides of (4.20) we obtain

$$(\xi\alpha + \alpha^2 - \beta^2 + 1)\phi X = 0. \tag{3.21}$$

Since $\phi X \neq 0$, unless $X = \xi$, we have

$$(\xi\alpha + \alpha^2 - \beta^2 + 1) = 0. \tag{3.22}$$

Therefore we can state the following:

Theorem 3.6. If a three dimensional normal almost contact metric manifold is isometrically immersed in a four dimensional Riemannian manifold of constant curvature 1, then the manifold satisfies the relation (3.22).

4 Example

We consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard coordinate of $\mathbb{R}^3$.
The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.
Let g be a Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

that is, the form of the matrix becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.
Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \qquad \phi(e_2) = e_1, \qquad \phi(e_3) = 0.$$

Then using the identity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.
Then for $e_3 = \xi$,the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$\begin{aligned} [e_1, e_3] &= e_1 e_3 - e_3 e_1 \\ &= z\frac{\partial}{\partial x}(z\frac{\partial}{\partial z}) - z\frac{\partial}{\partial z}(z\frac{\partial}{\partial x}) \\ &= -e_1. \end{aligned}$$

Similarly

$$[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{4.1}$$

which is known as Koszul's formula. Using (4.1) we have

$$\begin{aligned} 2g(\nabla_{e_1}e_3, e_1) &= -2g(e_1, e_1) \\ &= 2g(-e_1, e_1). \end{aligned} \tag{4.2}$$

Again by (4.1)

$$2g(\nabla_{e_1}e_3, e_2) = 0 = 2g(-e_1, e_2) \tag{4.3}$$

and

$$2g(\nabla_{e_1}e_3, e_3) = 0 = 2g(-e_1, e_3). \tag{4.4}$$

From (4.2), (4.3) and (4.4) we obtain

$$2g(\nabla_{e_1}e_3, X) = 2g(-e_1, X)$$

for all $X \in \chi(M)$.

Thus

$$\nabla_{e_1}e_3 = -e_1.$$

Therefore, (4.1) futher yields

$$\begin{aligned} &\nabla_{e_1}e_3 = -e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = -e_3 \\ &\nabla_{e_2}e_3 = -e_2, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_1 = 0, \\ &\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0. \end{aligned} \tag{4.5}$$

(4.5) tells us that the manifold satisfies (2.2) for $\alpha = -1$ and $\beta = 0$ and $\xi = e_3$. Hence the manifold is a normal almost contact metric manifold with $\alpha, \beta =$constants.

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