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A new class of finite integral operators involving a product of generalized Bessel function and Jacobi polynomial

Waseem A. Khan, M. Ghayasuddin and Divesh Srivastava

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Abstract. In the present research paper, authors introduce a new class of integrals involving a product of generalized Bessel function and Jacobi polynomial, whose explicit representations are given in terms of Kampé de Fériet and Srivastava and Daoust functions. Furthermore, we derive some other interesting integrals as special cases of our main results.

1 Introduction

In recent years, a number of researchers have introduced a variety of integral operators involving different kind of special functions (see [1]- [10]). Such integrals play a very important role in many diverse field of engineering and sciences. In a continuation of such type of works, in this paper we introduce two new integrals involving a product of generalized Bessel function $w^u_{\delta,v}(z)$ and Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, which are expressed in terms of Kampé de Fériet and Srivastava and Daoust functions.
In order to derive our main results, we recall here the following definitions of some well known special functions:

The generalized Bessel function $w^u_{\delta,v}(z)$ is defined by (see [9]):

$$w^u_{\delta,v}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m v^m (z/2)^{\delta+2m}}{m!\Gamma(\delta+m+\frac{1+u}{2})}, \tag{1.1}$$

which have the following relations with sine and cosine functions:

$$w^u_{1-\frac{u}{2},-v^2}(z)=\left(\frac{2}{z}\right)^{\frac{u}{2}}\frac{\sin vz}{\sqrt{\pi}} \tag{1.2}$$

and

$$w^u_{-\frac{u}{2},v^2}(z)=\left(\frac{2}{z}\right)^{\frac{u}{2}}\frac{\cos vz}{\sqrt{\pi}}. \tag{1.3}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is defined by (see [8], [12]):

$$P_n^{(\alpha,\beta)}(z)=\frac{(1+\alpha)_n}{n!}{}_2F_1\left[\begin{array}{ll}-n,1+\alpha+\beta+n; & \frac{1-z}{2}\\ 1+\alpha; & \end{array}\right], \tag{1.4}$$

or, equivalently (see [8], [12]):

$$P_n^{(\alpha,\beta)}(z)=\sum_{k=0}^{n}\frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!\ (n-k!\ (1+\alpha)_k(1+\alpha+\beta)_n)}\left(\frac{z-1}{2}\right)^k. \tag{1.5}$$

From equation (1.4) and (1.5), we have

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}, \tag{1.6}$$

where $P_n^{(\alpha,\beta)}(z)$ is a polynomial of degree n.
By setting $\beta = \alpha$ in (1.4) the polynomial $P_n^{(\alpha,\alpha)}(z)$ is called the ultraspherical polynomial and further on setting $\alpha = \beta = \xi - \frac{1}{2}$ in (1.4), we have

$$P_n^{(\xi-\frac{1}{2},\xi-\frac{1}{2})}(z) = \frac{(\xi+\frac{1}{2})_n}{(2\xi)_n} C_n^{\xi}(z), \tag{1.7}$$

where $C_n^{\xi}(z)$ is the Gegenbauer polynomial (see [8], [12]).
Again setting $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$ respectively in equations (1.4) we get

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(z) = \frac{\left(\frac{1}{2}\right)_n}{n!} T_n(z) \tag{1.8}$$

and

$$P_n^{(\frac{1}{2},\frac{1}{2})}(z) = \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} U_n(z), \tag{1.9}$$

where $T_n(z)$ and $U_n(z)$ are Tchebicheff polynomials of first kind and second kind respectively (see [8], [12]).
Also, for $\alpha = \beta = 0$ in equation (1.4), we have

$$P_n^{(0,0)}(z) = P_n(z), \tag{1.10}$$

where $P_n(z)$ is the Legendre polynomial (see [8], [12]).

The Kampé de Fériet function (see [8], [12]) is defined by:

$$F_{l:m;n}^{p:q;k}\left[\begin{array}{ll} (a_p):(b_q);(c_k); & \\ & x,y \\ (\alpha_l):(\beta_m);(\gamma_n); & \end{array}\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r y^s}{r!\ \ s!}, \tag{1.11}$$

where, for convergence,

$$(i)\ p+q < l+m+1,\ p+k < l+n+1,\ |x| < \infty,\ |y| < \infty,\ \ or$$

$$(ii)\ p+q = l+m+1,\ p+k = l+n+1,$$
and

$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-1}} < 1; & if p > l \\ \max\{|x|,|y|\} < 1; & if p \leq l. \end{cases}$$

The Srivastava and Daoust multivariable hypergeometric function (see [8], [12]) is given as follows:

$$F_{l:q_1;\ldots;q_s}^{p:m_1;\ldots;m_s}\left[\begin{array}{ll} (a_j:\alpha_j^1,\ldots\alpha_j^{(s)})_{1,p}:(c_j^1,r_j^1)_{1,q_1};\ldots;(c_j^{(s)},r_j^{(s)})_{1,q_s}; & \\ & x_1,x_2,\ldots x_s \\ (b_j:\beta_j^1,\ldots\beta_j^{(s)})_{1,l}:(d_j^1,\delta_j^1)_{1,m_1};\ldots;(d_j^{(s)},\delta_j^{(s)})_{1,m_s}; & \end{array}\right]$$

$$= \sum_{n_1,n_2,\ldots n_s=0}^{\infty} \frac{\prod_{j=i}^{p}(a_j)_{n_1 a_j^1+\ldots n_s a_j^{(s)}}\prod_{j=i}^{q_1}(c_j^{(1)})_{n_1 r_j^{(1)}}\cdots\prod_{j=i}^{q_s}(c_j^{(s)})_{n_s r_j^{(s)}}}{\prod_{j=i}^{l}(b_j)_{n_1 b_j^1+\ldots n_s b_j^{(s)}}\prod_{j=i}^{m_1}(d_j^{(1)})_{n_1\delta_j^{(1)}}\cdots\prod_{j=i}^{m_s}(d_j^{(s)})_{n_s\delta_j^{(s)}}}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_s^{n_s}}{n_s!}, \tag{1.12}$$

where the multiple hypergeometric series converges absolutely under the parametric and variable constraints and $(\lambda)_\nu$ denotes the well known Pochhammer symbol (see [8], [12]).

For our present investigation, we also recall here the following interesting and useful result of MacRobert [10]:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}dx = \frac{1}{a^\alpha b^\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.13}$$

provided $\Re(\alpha) > 0$, $\Re(\beta) > 0$, a, b are non-zero constants and the expression $ax + b(1-x)$, where $0 \le x \le 1$, is non-zero.

2 Main Results

In this section, we establish two interesting integrals involving the product of generalized Bessel function of first kind $w^u_{\delta,v}(z)$ and Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, which are obtain in terms of Kampé de Fériet and Srivastava and Daoust functions.

Theorem 2.1. *For $\Re(\xi) > 0$, $\Re(\delta) > 0$, $\Re(\lambda) > 0$, $\Re(\delta) > -\left(\frac{1+u}{2}\right)$, p and q are non zero constants and the expression $px+q(1-x)$, (where $0 \le x \le 1$) is non zero, the following integral formula (in terms of Kampé de Fériet) holds true:*

$$\begin{aligned}
&\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^v_{\delta,u}\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right] \\
&\times P_n^{(a,b)}\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx \\
&= \frac{1}{p^\xi q^\lambda}\frac{(1+a)_n\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})n!}\left\{\frac{1}{\Gamma(\xi+\lambda+2\delta)}\right. \\
&\times F^{4:0;4}_{4:1;3}\left[\begin{matrix}\triangle\,(2;\xi+\delta),\triangle\,(2;\lambda+\delta):-;\triangle\,(2;1+a+b+n),\triangle\,(2;-n);\\ \triangle\,(4,\xi+\lambda+2\delta):\delta+\frac{1+u}{2};\triangle\,(2;1+a),\frac{1}{2};\end{matrix}\; -v, r^2\right] \\
&+\frac{p(\xi+\delta)(\lambda+\delta)(1+a+b+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(1+a)} \\
&\times F^{4:0;4}_{4:1;3}\left[\begin{matrix}\triangle\,(2;\xi+\delta+1), & \triangle\,(2;\lambda+\delta+1): & -; \\ \triangle\,(4,\xi+\lambda+2\delta+2): & \delta+\frac{1+u}{2}; & \triangle\,(2;2+a),\end{matrix}\right. \\
&\left.\left.\begin{matrix}\triangle\,(2;2+a+b+n), & \triangle\,(2;-n+1); \\ \frac{3}{2};\end{matrix}\; -v,-r^2\right]\right\}.
\end{aligned} \tag{2.1}$$

where $\triangle\,(m;l)$ abbreviates the array of m parameters $\frac{l}{m}, \frac{l+1}{m}, \cdots, \frac{l+m-1}{m}, m \ge 1$ and $F^{p:q;r}_{l:m;n}$ is the Kampé de Fériet function given in (1.11).

Proof. We denote the left hand side of (2.1) by I_1, using (1.1) and (1.5) in the integrand of (2.1) and then interchanging the order of integration and summation (which is true under the given conditions), we have

$$\begin{aligned}
I_1 \;=\; &\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^m v^m r^k(-n)_k(1+\alpha)_n(1+\alpha+\beta+n)_k}{m!\;n!\;k!\;\Gamma\left(\delta+\frac{1+u}{2}\right)(1+\alpha)_k(\delta+(1+u)/2)_m} \\
\times &\int_0^1 x^{\xi+\delta+2m+k-1}(1-x)^{\lambda+\delta+2m+k-1}[px+q(1-x)]^{-(\xi+\lambda+2\delta+4m+2k)}dx.
\end{aligned} \tag{2.2}$$

On using (1.13) in the above equation and after little simplification, we get

$$I_1 = \frac{1}{(p)^{\xi}(q)^{\lambda}} \frac{(1+\alpha)_n}{\Gamma\left(\delta+\frac{1+u}{2}\right)\ n!}$$
$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^m v^m r^k (-n)_k (1+\alpha+\beta+n)_k \Gamma(\xi+\delta+2m+k)\Gamma(\lambda+\delta+2m+k)}{m!\ k!\ \left(\delta+\frac{1+u}{2}\right)_m (1+\alpha)_k \Gamma(\xi+\lambda+2\delta+4m+2k)}. \quad (2.3)$$

Now separating k-series into even and odd terms and then using the result $(B)_{m+n} = (B)_m (B+m)_n$, in the second term of the given expression, we get

$$I_1 = \frac{1}{(p)^{\xi}(q)^{\lambda}} \frac{(1+\alpha)_n\ \Gamma(\xi+\delta)\ \Gamma(\lambda+\delta)}{\Gamma\left(\delta+\frac{1+u}{2}\right) n!} \left[\frac{1}{\Gamma(\xi+\lambda+2\delta)}\right.$$
$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\xi+\delta)_{2(m+k)}(\lambda+\delta)_{2(m+k)}(1+\alpha+\beta+n)_{2k}(-n)_{2k}(-v)^m\ (r)^{2k}}{m!\ k!\ 2^{2k}\left(\frac{1}{2}\right)_k \left(\delta+\frac{1+u}{2}\right)_m (1+\alpha)_{2k}\ (\xi+\lambda+2\delta)_{4(m+k)}}$$
$$+\frac{r(\xi+\delta)(\lambda+\delta)(1+\alpha+\beta+n)(-n)}{(1+\alpha)\Gamma(\xi+\lambda+2\delta+2)}$$
$$\left.\sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\xi+\delta+1)_{2(m+k)}(\lambda+\delta+1)_{2(m+k)}(2+\alpha+\beta+n)_{2k}(-n+1)_{2k}(-v)^m (r)^{2k}}{m!\ k!\ 2^{2k}\left(\frac{3}{2}\right)_k \left(\delta+\frac{1+u}{2}\right)_m (2+\alpha)_{2k}\ (\xi+\lambda+2\delta+2)_{4(m+k)}}\right]. \quad (2.4)$$

On using the result [12, Page 23, eq. 26] in (2.4), and after arranging the resulting expression into Kampé de Fériet function, we get our desired result.

Theorem 2.2. *For $\Re(\xi)>0, \Re(\delta)>0, \Re(\lambda)>0, \Re(\delta)>-\left(\frac{1+u}{2}\right)$, p and q are non zero constants and the expression $px+q(1-x)$, (where $0\le x\le 1$) is non zero, the following integral formula (in terms of Srivastava and Daoust function) holds true:*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^{v}_{\delta,u}\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]$$
$$\times P_n^{(a,b)}\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right] dx$$
$$= \frac{1}{p^{\xi}q^{\lambda}} \frac{\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})\Gamma(\xi+\lambda+2\delta)}$$
$$\times F^{4:0;0}_{4:0;1}\left[\begin{array}{l} (1+a:1,1),(1+a+b:1,2),(\xi+\delta:2,3),\\ (1:1,1),(\delta+\frac{1+u}{2}:1,1),(1+a+b:1,1),\end{array}\right.$$
$$\left.\begin{array}{l}(\lambda+\delta:2,3):\ -\ ;\ -\ ;\\ (\xi+\lambda+2\delta:4,6):-;(1+a:1);\end{array} -v, r^2\right]. \quad (2.5)$$

Proof. We denote the left hand side of (2.5) by I_2. Applying (1.1) and (1.5) in the integral (2.5), interchanging the order of integration and summation (which is guaranteed under the given conditions), and after using the following Lemma (see, [11], [12]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k),$$

we get

$$I_2 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(-v)^n(r)^{2k}(pq)^{\delta+2n+3k}}{(n+k)!\Gamma(n+k+\delta+\frac{1+u}{2})n!\ k!\ (1+\alpha)_k}$$

$$\times \int_0^1 x^{\xi+\delta+2n+3k-1}(1-x)^{\lambda+\delta+2n+3k-1}[px+q(1-x)]^{-(\xi+\lambda+2\delta+4n+6k)}dx. \tag{2.6}$$

Now by using (1.13) and after little simplification, the equation (2.6) reduces to

$$I_2 = \frac{\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{(p)^{\xi}(q)^{\lambda}\Gamma\left(\delta+\frac{1+u}{2}\right)\Gamma(\xi+\lambda+2\delta)}$$

$$\times \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\xi+\delta)_{2n+3k}(\lambda+\delta)_{2n+3k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}v^n r^{2k}(-1)^n}{(\xi+\lambda+2\delta)_{4n+6k}(1)_{n+k}(1+\alpha)_k(1+\alpha+\beta)_{n+k}\, n!\, k!\left(\delta+\frac{1+u}{2}\right)_{n+k}}. \tag{2.7}$$

Summing up the series (2.7) with the help of (1.12), leads us to the right hand side of (2.5).

Remark 1. For $u = v = 1$ in Theorem 2.1 and Theorem 2.2, the resulting identities reduces to a known result of Khan et al. [8].

3 Special Cases

In this section, we obtain some integrals with the help of Theorem 2.1, which involve the product of generalized Bessel function with some orthogonal polynomials, sine and cosine functions in the form of corollaries.

Corollary 3.1. *By taking $\beta = \alpha$ in 2.1, we get the following integral formula*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^v_{\delta,u}\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]$$
$$\times P_n^{(a,a)}\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx$$
$$= \frac{1}{p^{\xi}q^{\lambda}}\frac{(1+a)_n\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})n!}\left\{\frac{1}{\Gamma(\xi+\lambda+2\delta)}\right.$$
$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l} \triangle\,(2;\xi+\delta),\triangle\,(2;\lambda+\delta):-;\triangle\,(2;1+2a+n),\triangle\,(2;-n); \\ \triangle\,(4,\xi+\lambda+2\delta):\delta+\frac{1+u}{2};\triangle\,(2;1+a),\frac{1}{2}; \end{array} -v, r^2\right]$$
$$+\frac{r(\xi+\delta)(\lambda+\delta)(1+2a+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(1+a)}$$
$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l} \triangle\,(2;\xi+\delta+1),\triangle\,(2;\lambda+\delta+1):-;\triangle\,(2;2+2a+n),\triangle\,(2;-n+1); \\ \triangle\,(4,\xi+\lambda+2\delta+2):\delta+\frac{1+u}{2};\triangle\,(2;2+a),\frac{3}{2}; \end{array} -v, r^2\right]\Bigg\}, \tag{3.1}$$

where $\Re(\delta) > -(\frac{1+u}{2})$, $\Re(\xi) > 0, \Re(\lambda) > 0$ and $P_n^{(\alpha,\alpha)}(z)$ is the ultraspherical polynomial (see [12]).

Corollary 3.2. *If we assign $\beta = \alpha = l - \frac{1}{2}$ in (2.1), then by using (1.7), we have*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^v_{\delta,u}\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]$$
$$\times C_n^l\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx$$

$$= \frac{1}{p^{\xi}q^{\lambda}} \frac{(2l)_n \Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})n!} \left\{ \frac{1}{\Gamma(\xi+\lambda+2\delta)} \right.$$

$$\times F^{4:0;4}_{4:1;3} \left[\begin{array}{l} \triangle\ (2;\xi+\delta), \triangle\ (2;\lambda+\delta) : -; \triangle\ (2;2l+n), \triangle\ (2;-n); \\ \\ \triangle\ (4,\xi+\lambda+2\delta) : \delta+\frac{1+u}{2}; \triangle\ (2;l+\frac{1}{2}), \frac{1}{2}; \end{array} - v, r^2 \right]$$

$$+ \frac{r(\xi+\delta)(\lambda+\delta)(2l+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(l+\frac{1}{2})}$$

$$\times F^{4:0;4}_{4:1;3} \left[\begin{array}{l} \triangle\ (2;\xi+\delta+1), \triangle\ (2;\lambda+\delta+1) : -; \triangle\ (2;1+2l+n), \\ \\ \triangle\ (4,\xi+\lambda+2\delta+2) : \delta+\frac{1+u}{2}; \triangle\ (2;l+\frac{3}{2}), \end{array} \right.$$

$$\left. \begin{array}{l} \triangle\ (2;-n+1); \\ \frac{3}{2}; \end{array} - v, r^2 \right] \Bigg\}, \tag{3.2}$$

where $\Re(\delta) > -(\frac{1+u}{2})$, $\Re(\xi) > 0, \Re(\lambda) > 0$ *and* $C^l_n(z)$ *is the Gegenbauer polynomial (see [12]).*

Corollary 3.3. *Assuming* $\beta = \alpha = -\frac{1}{2}$ *in (2.1) and by using (1.8), the following integral formula holds true*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^v_{\delta,u} \left[\frac{2pqx(1-x)}{[px+q(1-x)]^2} \right]$$

$$\times T_n \left[1 - \frac{2pqrx(1-x)}{[px+q(1-x)]^2} \right] dx$$

$$= \frac{1}{p^{\xi}q^{\lambda}} \frac{\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})}$$

$$\times \left\{ \frac{1}{\Gamma(\xi+\lambda+2\delta)} F^{4:0;4}_{4:1;3} \left[\begin{array}{l} \triangle(2;\xi+\delta), \triangle(2;\lambda+\delta) : -; \triangle(2;n), \triangle(2;-n); \\ \\ \triangle(4;\xi+\lambda+2\delta) : \delta+\frac{1+u}{2}; \triangle(2;\frac{1}{2}), \frac{1}{2}; \end{array} - v, r^2 \right] \right.$$

$$+ \frac{r(\xi+\delta)(\lambda+\delta)(n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(\frac{1}{2})}$$

$$\left. \times F^{4:0;4}_{4:1;3} \left[\begin{array}{l} \triangle(2;\xi+\delta+1), \triangle(2;\lambda+\delta+1) : -; \triangle(2;n+1), \triangle(2;-n+1); \\ \\ \triangle(4;\xi+\lambda+2\delta+2) : \delta+\frac{1+u}{2}; \quad \triangle(2;\frac{3}{2}), \quad \frac{3}{2}; \end{array} - v, r^2 \right] \right\}, \tag{3.3}$$

where $\Re(\delta) > -(\frac{1+u}{2})$, $\Re(\xi) > 0, \Re(\lambda) > 0$ *and* $T_n(z)$ *is the Tchebicheff polynomial of first kind (see [12]).*

Corollary 3.4. *Setting* $\beta = \alpha = \frac{1}{2}$ *in (2.1) and then by using (1.9), we have*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda} w^v_{\delta,u} \left[\frac{2pqx(1-x)}{[px+q(1-x)]^2} \right]$$

$$\times U_n \left[1 - \frac{2pqrx(1-x)}{[px+q(1-x)]^2} \right] dx$$

$$= \frac{1}{p^{\xi}q^{\lambda}} \frac{(n+1)\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})} \left\{ \frac{1}{\Gamma(\xi+\lambda+2\delta)} \right.$$

$$\times F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\xi+\delta),\triangle(2;\lambda+\delta):-;\triangle(2;2+n),\triangle(2;-n);\\ \\ \triangle(4;\xi+\lambda+2\delta):\delta+\frac{1+u}{2};\triangle(2;\frac{3}{2}),\frac{1}{2};\end{array} -v,r^2\right]$$

$$+\frac{r(\xi+\delta)(\lambda+\delta)(2+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(\frac{3}{2})}$$

$$\times F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\xi+\delta+1),\triangle(2;\lambda+\delta+1):-;\triangle(2;n+3),\triangle(2;-n+1);\\ \\ \triangle(4;\xi+\lambda+2\delta+2):\delta+\frac{1+u}{2};\triangle(2;\frac{5}{2}),\frac{3}{2};\end{array} -v,r^2\right]\Bigg\},\tag{3.4}$$

where $\Re(\delta)>-(\frac{1+u}{2}),\ \Re(\xi)>0,\Re(\lambda)>0$ *and* $U_n(z)$ *is the Tchebicheff polynomial of second kind (see [12]).*

Corollary 3.5. *On setting* $\beta=\alpha=0$ *in (2.1) and then by using (1.10), we get*

$$\int_0^1 x^{\xi-1}(1-x)^{\lambda-1}[px+q(1-x)]^{-\xi-\lambda}w_{\delta,u}^{v}\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]$$

$$\times P_n\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx$$

$$=\frac{1}{p^{\xi}q^{\lambda}}\frac{\Gamma(\xi+\delta)\Gamma(\lambda+\delta)}{\Gamma(\delta+\frac{1+u}{2})}$$

$$\Bigg\{\frac{1}{\Gamma(\xi+\lambda+2\delta)}F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\xi+\delta),\triangle(2;\lambda+\delta):-;\triangle(2;2+n),\triangle(2;-n);\\ \\ \triangle(4;\xi+\lambda+2\delta):\delta+\frac{1+u}{2};\triangle(2;1),\frac{1}{2};\end{array} -v,r^2\right]$$

$$+\frac{r(\xi+\delta)(\lambda+\delta)(1+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(1)}$$

$$\times F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\xi+\delta+1),\triangle(2;\lambda+\delta+1):-;\triangle(2;n+2),\triangle(2;-n+1);\\ \\ \triangle(4;\xi+\lambda+2\delta+2):\delta+\frac{1+u}{2};\triangle(2;2),\frac{3}{2};\end{array} -v,r^2\right]\Bigg\},\tag{3.5}$$

where $\Re(\delta)>-(\frac{1+u}{2}),\ \Re(\xi)>0,\Re(\lambda)>0$ *and* $P_n(z)$ *is the Legendre polynomial (see [12]).*

Corollary 3.6. *For* $\delta=1/2$ *in (2.1) then by using (1.2), the following integral formula holds true*

$$\int_0^1 x^{\xi-3/2}(1-x)^{\lambda-3/2}[px+q(1-x)]^{-\xi-\lambda+1}\sin\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]$$

$$\times P_n^{(a,b)}\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx$$

$$=\frac{1}{p^{\xi-1/2}q^{\lambda-1/2}}\frac{\Gamma(\xi+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})(1+a)_n\Gamma(\frac{1}{2})}{n!}\Bigg\{\frac{1}{\Gamma(\xi+\lambda-1)}$$

$$\times F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\xi+\frac{1}{2}),\triangle(2;\lambda+\frac{1}{2}):-;\triangle(2;1+a+b+n),\triangle(2;-n);\\ \\ \triangle(4;\xi+\lambda+1):\frac{3}{2};\triangle(2;(1+a)),\frac{1}{2};\end{array} -v,r^2\right]$$

$$+\frac{r(\xi+\frac{1}{2})(\lambda+\frac{1}{2})(1+a+b+n)(-n)}{\Gamma(\xi+\lambda+1)(\frac{1}{2})}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\xi+\frac{3}{2}),\triangle(2;\lambda+\frac{3}{2}):-;\triangle(2;-n+1),\triangle(2;2+a+b+n);\\ \\ \triangle(4;\xi+\lambda+2):\frac{5}{2};\triangle(2;2+a),\frac{3}{2};\end{array} -v,r^2\right]\Bigg\}, \tag{3.6}$$

where $\Re(\xi)>0,\Re(\lambda)>0$.

Corollary 3.7. *By letting* $\delta=-1/2$ *in (2.1), and by using (1.3), we have*

$$\begin{aligned}&\int_0^1 x^{\xi-3/2}(1-x)^{\lambda-3/2}[px+q(1-x)]^{-\xi-\lambda+1}\cos\left[\frac{2pqx(1-x)}{[px+q(1-x)]^2}\right]\\ &\times P_n^{(a,b)}\left[1-\frac{2pqrx(1-x)}{[px+q(1-x)]^2}\right]dx\\ &=\frac{1}{p^{\xi-1/2}q^{\lambda-1/2}}\frac{\Gamma(\xi+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})(1+a)_n\Gamma(\frac{1}{2})}{n!}\Bigg\{\frac{1}{\Gamma(\xi+\lambda-1)}\\ &\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\xi-\frac{1}{2}),\triangle(2;\lambda-\frac{1}{2}):-;\triangle(2;1+a+b+n),\triangle(2;-n);\\ \\ \triangle(4;\xi+\lambda-1):\frac{1}{2};\triangle(2;1+a),\frac{1}{2};\end{array} -v,r^2\right]\\ &+\frac{r(\xi-\frac{1}{2})(\lambda-\frac{1}{2})(1+a+b+n)(-n)}{\Gamma(\xi+\lambda+1)()1+a}\\ &\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\xi+\frac{1}{2}),\triangle(2;\lambda+\frac{1}{2}):-;\triangle(2;-n+1),\triangle(2;2+a+b+n);\\ \\ \triangle(4;\xi+\lambda+1):\frac{1}{2};\triangle(2;2+a),\frac{3}{2};\end{array} -v,r^2\right]\Bigg\},\end{aligned} \tag{3.7}$$

where $\Re(\xi)>0,\Re(\lambda)>0$.

Remark 2. It is notice that, we can derive similar integral formulas (given in the above corollaries) for Theorem 2.2, in terms of Srivastava and Daoust function by arranging the same values to the parameters.

4 Relation between Kampé de Fériet and Srivastava and Daoust functions

In this section, we establish an interesting relation between Kampé de Fériet function and Srivastava and Daoust function as follows:

$$\begin{aligned}&\Bigg\{\frac{1}{\Gamma(\xi+\lambda+2\delta)}\\ &\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\xi+\delta),\triangle\,(2;\lambda+\delta):-;\triangle\,(2;1+a+b+n),\triangle\,(2;-n);\\ \\ \triangle\,(4,\xi+\lambda+2\delta):\delta+\frac{1+u}{2};\triangle\,(2;1+a),\frac{1}{2};\end{array} -v,r^2\right]\\ &+\frac{p(\xi+\delta)(\lambda+\delta)(1+a+b+n)(-n)}{\Gamma(\xi+\lambda+2\delta+2)(1+a)}\\ &\times F^{4:0;4}_{4:1;3}\left[\begin{array}{lll}\triangle\,(2;\xi+\delta+1), & \triangle\,(2;\lambda+\delta+1): & -;\\ \triangle\,(4,\xi+\lambda+2\delta+2): & \delta+\frac{1+u}{2}; & \triangle\,(2;2+a),\end{array}\right.\\ &\qquad\left.\begin{array}{ll}\triangle\,(2;2+a+b+n), & \triangle\,(2;-n+1);\\ \\ \frac{3}{2}; & \end{array} -v,-r^2\right]\Bigg\}\end{aligned}$$

$$= \frac{n!}{(1+a)_n\Gamma(\xi+\lambda+2\delta)} F^{4:0;0}_{4:0;1} \left[\begin{array}{l} (1+a:1,1),(1+a+b:1,2),(\xi+\delta:2,3), \\ (1:1,1),(\delta+\frac{1+u}{2}:1,1),(1+a+b:1,1), \end{array} \right.$$

$$\left. \begin{array}{l} (\lambda+\delta:2,3):\ -\ ;\ -\ ; \\ (\xi+\lambda+2\delta:4,6):-;(1+a:1); \end{array} -v,r^2 \right]. \tag{4.1}$$

References

[1] S. Ali, *On some new unified integrals*, Adv. Comput. Math. Appl., **1**(3), 151-153 (2012).

[2] A. Y. Brychkov, *Handbook of special functions: derivatives, integral, series and other formulae*, CRC Press, Boca Raton, (2008).

[3] J. Choi and P. Agarwal, *Certain unified integrals involving a product of Bessel functions of first kind*, Honam Math. J., **(35)**(4), 667-677 (2013), .

[4] J. Choi and P. Agarwal, S. Mathur and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc., **51**, 995-1003 (2014).

[5] J. Choi, A. Hasnov, H. M. Srivastava and M. Turaev, *Integral representation for Srivastava's triple hypergeometric functions*, Taiwanese J. Math., **51** 2751-2762, (2011), .

[6] M. Ghayasuddin, N. U. Khan and S. W. Khan, *Some finite integrals involving the product of Bessel function with Jacobi and Laguerre polynomials*, Commun. Korean Math. Soc., (2017). (Accepted)

[7] N. U. Khan, M. Ghayasuddin and T. Usman, *On certain integral foumulas involving the product of Bessel function and Jacobi polynomial*, Tamkang J. Math., **47**, 339-349 (2016).

[8] N. U. Khan, M. Ghayasuddin, W. A. Khan and Sarvat Zia, *Study of the unified integrals associated with Bessel function and Jacobi polynomial*. (communicated)

[9] N. U. Khan and M. Ghayasuddin, *A new class of integral formulas associatede with generalized Bessel functions*, Sohag J. Math., **3**(2), 1-4 (2016).

[10] T. M. Macrobert, *Beta functions formulae and integrals involving E-Function*, Math. Annalen, **142**, 450-452 (1961).

[11] E. D. Rainville, *Special functions*, The Macmillan Company, New York, (1960).

[12] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1984).

Author information

Waseem A. Khan, Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India.
E-mail: `waseem08_khan@rediffmail.com`

M. Ghayasuddin, Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India.
E-mail: `ghayas.maths@gmail.com`

Divesh Srivastava, Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India.
E-mail: `divesh2712@gmail.com`

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