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Frictional contact problem between thermoelastic piezoelectric bodies with damage, adhesion and normal compliance

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Abstract. We study of a friction contact problem between two thermoelastic piezoelectric bodies with damage, adhesion and normal compliance. The evolution of the damage is described by an inclusion of parabolic type. The contact is modelled with a version of normal compliance condition and the associated Coulomb's law of friction in which the adhesion of contact surfaces is taken into account. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem.

1 Introduction

Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metal and rocks polymers. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field. See for examples [1, 15, 16, 17, 21, 25] for the case of hardening, temperature and other internal state variables and the references [3, 6, 10, 11, 12, 13, 17, 24] for the case of adhesion field and the damage field which is denoted in this paper by α^ℓ. The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [10, 11] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [8, 9]. Here we describe a variant of one of their models, and we note that other models for damage, based on different considerations, can be found in the engineering literature. The new idea of [10, 11] was the introduction of the *damage function* $\alpha^\ell = \alpha^\ell(x,t)$, which is the ratio between the elastic moduli of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y^ℓ be the Young modulus of the original material and E_{eff}^ℓ be the current modulus, then the damage function is defined by

$$\alpha^\ell = \alpha^\ell(x,t) = \frac{E_{eff}^\ell}{E_Y^\ell}.$$

Clearly, it follows from this definition that the damage function α^ℓ is restricted to have values between zero and one. When $\alpha^\ell = 1$, there is no damage in the material, when $\alpha^\ell = 0$, the material is completely damaged, when $0 < \alpha^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [10, 24, 26]. In this paper the inclusion used for the evolution of the damage field is

$$\dot{\alpha}^\ell - \kappa^\ell \Delta\alpha^\ell + \partial\psi_{K^\ell}(\alpha^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell(s), \alpha^\ell\big), \tag{1.1}$$

where K^ℓ denotes the set of admissible damage functions defined by

$$K^\ell = \{\xi \in H^1(\Omega^\ell);\ 0 \le \xi \le 1,\ \text{a.e. in } \Omega^\ell\}, \tag{1.2}$$

κ^ℓ is a positive coefficient, $\partial\psi_{K^\ell}$ represents the subdifferential of the indicator function of the set K^ℓ and ϕ^ℓ is a given constitutive function which describes the sources of the damage in the system.

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [18, 19] and recently in the monographs [20, 22]. Following [14], the bonding field satisfies the restriction $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

The aim of this paper is to study the quasistatic evolution of damage in thermo-electroelastic materials. For this, we use an thermo-electroelastic constitutive law with long-term memory and damage given by

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\big(\varepsilon(\boldsymbol{u}^\ell), \theta^\ell, \alpha^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s), \alpha^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \tag{1.3}$$

where $\boldsymbol{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\varepsilon(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively, θ^ℓ represents the absolute temperature and α^ℓ represents the damage field. Here $\mathcal{Q}^\ell$ is the relaxation operator, and $\mathcal{A}^\ell$ represents the thermo-elasticity operator with damage. $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, $\mathcal{E}^\ell$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In this paper we study a quasistatic Coulomb's frictional contact problem between two thermo-electroelastic bodies with long-term memory and damage. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution. The paper is organized as follows. In section 2 we describe the mathematical models for the frictional contact problem between two thermo-electroelastic bodies with long-term memory and damage. The contact is modelled with normal compliance and adhesion. We introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We prove in section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on a classical existence and uniqueness result on parabolic inequalities, evolutionary variational equalities, differential equations and fixed point arguments.

2 Statement of the Problem

Let us consider two electric-thermo-elastic bodies with long-term memory and damage, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0, T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies is in adhesive contact over the surface Γ_3. The mechanical problem may be formulated as follows.

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{S}^d$, a damage field $\alpha^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, an electric potential field $\varphi^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a temperature $\theta^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a bonding field $\beta : \Gamma_3 \times (0, T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\big(\varepsilon(\boldsymbol{u}^\ell), \theta^\ell, \alpha^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s), \alpha^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \quad \text{in } \Omega^\ell \times (0, T), \tag{2.1}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\varphi^\ell)\big), \quad \text{in } \Omega^\ell \times (0, T), \tag{2.2}$$

$$\dot{\alpha}^\ell - \kappa^\ell \Delta\alpha^\ell + \partial\psi_{K^\ell}(\alpha^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell, \alpha^\ell\big) \text{ in } \Omega^\ell \times (0,T), \tag{2.3}$$

$$\dot{\theta}^\ell - \kappa_0^\ell \Delta\theta^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell, \alpha^\ell\big) + \rho^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{2.4}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.5}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.6}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{2.7}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{2.8}$$

$$\dot{\beta} = H_{ad}(\beta, \xi_\beta, R_\nu([u_\nu]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \text{ where } \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| \le \mu p_\nu([u_\nu]), \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| < \mu p_\nu([u_\nu]) \Rightarrow [\boldsymbol{u}_\tau] = 0, \quad \text{on } \Gamma_3 \times (0,T), \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| = \mu p_\nu([u_\nu]) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) = -\lambda [\boldsymbol{u}_\tau] \end{cases} \tag{2.11}$$

$$\frac{\partial \alpha^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.12}$$

$$\kappa_0^\ell \frac{\partial^\ell \theta^\ell}{\partial \nu^\ell} + \lambda_0^\ell \theta^\ell = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.13}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.14}$$

$$\boldsymbol{D}^\ell . \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.15}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \ \theta^\ell(0) = \theta_0^\ell, \ \alpha^\ell(0) = \alpha_0^\ell \quad \text{in } \Omega^\ell, \tag{2.16}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.17}$$

Here and below $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$, whereas "." and $\|.\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively; ν^ℓ is the unit outer normal vector on Γ^ℓ, and $r_+ = max\{r, 0\}$ denotes the positive part of r, equations (2.1) and (2.2) represent the thermo-electroelastic constitutive law with long term-memory and damage. Inclusion (2.3) describes the evolution of the damage field. Equation (2.4) represents the energy conservation where Θ^ℓ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ^ℓ is a given volume heat source. Equations (2.5) and (2.6) are the equilibrium equations for the stress and electric-displacement fields, respectively. Next, the equations (2.7) and (2.8) represent the displacement and traction boundary condition, respectively. Condition (2.10) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient, p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is $[u_\nu]$ can be positive on Γ_3.

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad -L \le s \le 0, \\ 0 & \text{if} \quad s > 0. \end{cases} \qquad \boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases} \tag{2.18}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [23]). Condition (2.11) are a non local Coulomb's friction law conditions coupled with adhesive, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. The relation (2.12) represents a homogeneous Neumann boundary condition for the damage field on Γ^ℓ. The relation (2.13) represent a Fourier boundary condition for the temperature on Γ^ℓ. Equations (2.14) and (2.15) represent the electric boundary conditions. Equation(2.9) describes the evolution of the bonding field and it was already used in [4, 5], see also [26] for more details. The evolution of the adhesion field is assumed to depend generally on β, $[u_\nu]$ and $[\boldsymbol{u}_\tau]$. We do not impose sign restrictions on the process and, thus, cycles of debonding and rebonding may take place, as a result of imposed periodic forces or displacements. In addition, we include here the possibility that, as the cycles of bonding and debonding go on, there is a decrease in the bonding effectiveness. Therefore, the process is also assumed to depend on the time history of the bonding, which we denote by

$$\xi_\beta(x,t) = \int_0^t \beta(x,s)ds \quad on \quad \Gamma_3 \times (0,T). \tag{2.19}$$

The whole process is assumed to be governed by the differential equation,

$$\dot{\beta} = H_{ad}\big(\beta, \xi_\beta, R_\nu([u_\nu]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big),$$

where H_{ad} is a general function discussed below, which vanishes when its first argument vanishes. We use it in H_{ad}, since usually when the glue is stretched beyond the limit L it does not contribute more to the bond strength. An example of such a function, used in [4], the following form of the evolution of the bonding field was employed:

$$H_{ad}(\beta, \xi_\beta, R_1, \boldsymbol{R}_2) = -\beta_+ \gamma_n R_1^2,$$

where γ_n is the normal rate coefficient and $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide and $\beta_+ = \max(0,\beta)$. We note that in this case, only debonding is allowed. A different rate equation for the the evolution of the bonding field is

$$H_{ad}(\beta, \xi_\beta, R_1, \boldsymbol{R}_2) = -\Big(\beta\big(\gamma_n R_1^2 + \gamma_t \left|\boldsymbol{R}_2\right|^2\big) - \varepsilon_a\Big)_+,$$

see, e.g., [5, 12, 13]. Here, γ_t is the tangential rate coefficient, which may also be interpreted as the tangential stiffness coefficient of the interface when the adhesion is complete ($\beta = 1$). Another example, in which H_{ad} depends on all variables is

$$H_{ad}(\beta, \xi_\beta, R_1, \boldsymbol{R}_2) = -\gamma_n \beta_+ R_1^2 - \gamma_t \beta_+ |\boldsymbol{R}_2|^2 + \gamma_r \frac{\beta_+(1-\beta)_+}{1+\xi_\beta^2},$$

where γ_r is the rebonding rate coefficient. However, the bonding cannot exceed $\beta = 1$ and, moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1 + \xi_\beta^2$ in the denominator. Finally the functions $\boldsymbol{u}_0, \theta_0, \alpha_0$ and β_0 in (2.16)-(2.17) are the initial data.

We now proceed to obtain a variational formulation of Problem P. For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $H^\ell = L^2(\Omega^\ell)^d$, $H_1^\ell = H^1(\Omega^\ell)^d$, $\mathcal{H}^\ell = L^2(\Omega^\ell)_s^{d\times d}$, $\mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\tau}^\ell \in H^\ell\}$. The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell . \boldsymbol{v}^\ell dx,\ (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell . \boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla \boldsymbol{u}^\ell . \nabla \boldsymbol{v}^\ell dx,$$

$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell . \boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell . \boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\, \boldsymbol{\sigma}^\ell . \,\mathrm{Div}\, \boldsymbol{\tau}^\ell dx$$

and the associated norms $\|.\|_{H^\ell}$, $\|.\|_{H_1^\ell}$, $\|.\|_{\mathcal{H}^\ell}$, and $\|.\|_{\mathcal{H}_1^\ell}$ respectively.
We introduce the set of admissible bonding fields, defined by

$$\mathcal{Z} = \left\{\varsigma \in L^2(\Gamma_3);\ 0 \le \varsigma \le 1, \quad \text{a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \left\{\boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell\right\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds (see [20]) :

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{2.20}$$

Over the space V^ℓ we consider the inner product given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \tag{2.21}$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (2.20) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|.\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2.21), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{2.22}$$

We also introduce the spaces

$$\begin{aligned} &E_0^\ell = L^2(\Omega^\ell), \quad E_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \left\{\psi^\ell \in E_1^\ell;\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell\right\}, \\ &\mathcal{W}^\ell = \left\{\boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell)\right\}. \end{aligned}$$

Since $meas\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{W^\ell} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{2.23}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ. Over the space W^ℓ, we consider the inner product given by

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \textstyle\int_{\Omega^\ell} \nabla\varphi^\ell . \nabla\psi^\ell dx \tag{2.24}$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (2.23) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|.\|_{W^\ell})$ is areal Hilbert space. The space $\mathcal{W}^\ell$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Phi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell . \boldsymbol{\Phi}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell . \operatorname{div} \boldsymbol{\Phi}^\ell dx,$$

where $\operatorname{div} \boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|.\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\begin{aligned} &\boldsymbol{V} = V^1 \times V^2,\ H = H^1 \times H^2,\ H_1 = H_1^1 \times H_1^2,\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,\ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \\ &E_0 = E_0^1 \times E_0^2,\ E_1 = E_1^1 \times E_1^2,\ W = W^1 \times W^2,\ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{aligned}$$

The spaces $\boldsymbol{V}$, E_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{E_1}$, $(.,.)_W$ and $(.,.)_{\mathcal{W}}$.

In the study of the Problem **P**, we consider the following assumptions:

The *thermo-elasticity operator* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that } : \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\ \qquad |\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1, r_1, d_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2, r_2, d_2)| \leq L_{\mathcal{A}^\ell}\big(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + \\ \qquad |r_1 - r_2| + |d_1 - d_2|\big), \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}, r, d) \text{ is measurable in } \Omega^\ell, \quad \forall \boldsymbol{\xi} \in \mathbb{S}^d, r, d \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, 0, 0, 0) \text{ belongs to } \mathcal{H}^\ell. \end{cases} \tag{2.25}$$

The *relaxation function* $\mathcal{Q}^\ell : \Omega^\ell \times (0,T) \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{Q}^\ell} > 0 \text{ such that } : \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\ \qquad |\mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_1, r_1, d_1) - \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_2, r_2, d_2)| \le L_{\mathcal{Q}^\ell}\big(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + \\ \qquad |r_1 - r_2| + |d_1 - d_2|\big), \text{ for all } t \in (0,T), \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r, d) \text{ is measurable in } \Omega^\ell, \\ \qquad \text{for any } t \in (0,T),\ \boldsymbol{\xi} \in \mathbb{S}^d,\ \ r, d \in \mathbb{R}. \\ \text{(c) The mapping } t \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r, d) \text{ is continuous in } (0,T), \\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d,\ \ r, d \in \mathbb{R},\ \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, 0, 0, 0) \text{ belongs to } \mathcal{H}^\ell, \forall t \in (0,T). \end{cases} \tag{2.26}$$

The *energy function* $\Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\Theta^\ell} > 0 \text{ such that } : \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2, d_1, d_2 \in \mathbb{R}, \\ \quad |\Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1, \alpha_1, d_1) - \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2, \alpha_2, d_2)| \le L_{\Theta^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + \\ \quad |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2| + |d_1 - d_2|\big), \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha, d) \text{ is measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d \text{ and } \alpha, d \in \mathbb{R}, \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, 0, 0, 0, 0) \text{ belongs to } L^2(\Omega^\ell), \\ \text{(d) } \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha, d) \text{ is bounded for all } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d, \alpha, d \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.27}$$

The *damage source function* $\phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\phi^\ell} > 0 \text{ such that } : \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2, \\ \quad d_1, d_2 \in \mathbb{R},\ \ |\phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1, \alpha_1, d_1) - \phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2, \alpha_2, d_2)| \le \\ \quad L_{\phi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2| + |d_1 - d_2|\big), \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha, d) \text{ is measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d \text{ and } \alpha, d \in \mathbb{R}, \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, 0, 0, 0, 0) \text{ belongs to } L^2(\Omega^\ell), \\ \text{(d) } \phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha, d) \text{ is bounded, } \forall \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d, \alpha, d \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.28}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x}, \tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i, j, k \le d. \end{cases} \tag{2.29}$$

The *electric permittivity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$, satisfies:

$$\begin{cases} \text{(a) } \mathcal{G}^\ell(\boldsymbol{x}, \mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j),\ b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i, j \le d. \\ \text{(b) There exists } m_{\mathcal{G}^\ell} > 0 \text{ such that :} \\ \quad \mathcal{G}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{G}^\ell}|\mathbf{E}|^2,\ \forall \mathbf{E} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.30}$$

The *adhesion rate function* $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{ad} > 0 \text{ such that } : \forall \beta_1, \beta_2, \xi_1, \xi_2, r_1, r_2 \in \mathbb{R}, d_1, d_2 \in \mathbb{R}^{d-1}, \\ \quad |H_{ad}(\boldsymbol{x}, \beta_1, \xi_1, r_1, d_1) - H_{ad}(\boldsymbol{x}, \beta_2, \xi_2, r_2, d_2)| \le L_\nu|\beta_1 - \beta_2| + |\xi_1 - \xi_2| + \\ \quad |r_1 - r_2| + |d_1 - d_2|,\ \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x}, \beta, \xi, r, d) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } \beta, \xi, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1}, \\ \text{(c) The mapping } (\beta, \xi, r, d) \mapsto H_{ad}(\boldsymbol{x}, \beta, \xi, r, d) \text{ is continuous on} \\ \quad \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(d) } H_{ad}(\boldsymbol{x}, 0, \xi, r, d) = 0, \forall \xi, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(e) } H_{ad}(\boldsymbol{x}, \beta, \xi, r, d) \ge 0, \quad \forall \beta \le 0, \xi, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \text{ and} \\ \quad H_{ad}(\boldsymbol{x}, \beta, \xi, r, d) \le 0, \quad \forall \beta \ge 1, \xi, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{2.31}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that } : \forall r_1, r_2 \in \mathbb{R}, \\ \quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu |r_1 - r_2|, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0, \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \forall r \in \mathbb{R}. \\ \text{(d) } p_\nu(\boldsymbol{x}, r) = 0, \text{ for all } r \leq 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{2.32}$$

The *forces, tractions* have the regularity

$$\begin{gathered} \mathbf{f}_0^\ell \in C(0,T; L^2(\Omega^\ell)^d), \quad \mathbf{f}_2^\ell \in C(0,T; L^2(\Gamma_2^\ell)^d), \\ q_0^\ell \in C(0,T; L^2(\Omega^\ell)), \quad q_2^\ell \in C(0,T; L^2(\Gamma_b^\ell)), \quad \rho^\ell \in C(0,T; L^2(\Omega^\ell)), \end{gathered} \tag{2.33}$$

The adhesion coefficients γ_ν and γ_τ satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \gamma_\nu, \gamma_\tau \geq 0, \text{ a.e. on } \Gamma_3. \tag{2.34}$$

The energy coefficient κ_0^ℓ and the microcrack diffusion coefficient κ^ℓ satisfies :

$$\kappa_0^\ell > 0, \quad \kappa^\ell > 0. \tag{2.35}$$

Finally, the friction coefficient and the initial data satisfy:

$$\begin{gathered} \mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3, \\ \boldsymbol{u}_0^\ell \in \boldsymbol{V}^\ell, \ \alpha_0^\ell \in K^\ell, \ \theta_0^\ell \in E_1^\ell, \ \beta_0 \in L^2(\Gamma_3), \ 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3, \end{gathered} \tag{2.36}$$

where K^ℓ denotes the set of admissible damage functions defined by :

$$K^\ell = \{\alpha \in H^1(\Omega^\ell); \ 0 \leq \alpha \leq 1, \text{ a.e. in } \Omega^\ell\}. \tag{2.37}$$

We remark that if $\beta, \xi, r \in L^2(\Gamma_3)$, and $d : \Gamma_3 \mapsto \mathbb{R}^{d-1}$ is a measurable function, then the conditions (2.31) imply that $\boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x}, \beta(x), \xi(x), r(x), d(x)) \in L^2(\Gamma_3)$.

We define the mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to \boldsymbol{V}$, $q = (q^1, q^2) : [0,T] \to W$, by

$$(\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \boldsymbol{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \boldsymbol{v}^\ell \, da, \tag{2.38}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \zeta^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t) \zeta^\ell \, da \tag{2.39}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$, $\zeta \in W$ and $t \in [0,T]$, and note that conditions (2.33) imply that

$$\mathbf{f} \in C(0,T; \boldsymbol{V}), \quad q \in C(0,T; W). \tag{2.40}$$

We introduce the following continuous functionals $a_0 : E_1 \times E_1 \to \mathbb{R}$, $a : E_1 \times E_1 \to \mathbb{R}$ by

$$a_0(\zeta, \xi) = \sum_{\ell=1}^{2} \kappa_0^\ell \int_{\Omega^\ell} \nabla \zeta^\ell . \nabla \xi^\ell dx + \sum_{\ell=1}^{2} \lambda_0^\ell \int_{\Gamma^\ell} \zeta^\ell \xi^\ell da, \tag{2.41}$$

$$a(\zeta, \xi) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla \zeta^\ell . \nabla \xi^\ell dx. \tag{2.42}$$

Next, we define the four mappings $j_{ad} : L^2(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{fr} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]).[\boldsymbol{v}_\tau] \Big) \, da, \tag{2.43}$$

$$j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da, \tag{2.44}$$

$$j_{fr}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p_\nu([u_\nu]) \big\| [v_\tau] \big\| \, da. \tag{2.45}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)–(2.17).

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential field $\varphi = (\varphi^1, \varphi^2) : [0,T] \to W$, a temperature $\theta = (\theta^1, \theta^2) : [0,T] \to E_1$, a damage field $\alpha = (\alpha^1, \alpha^2) : [0,T] \to E_1$, a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0,T] \to \mathcal{W}$ such that, for a.e. $t \in (0,T)$,

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\big(\varepsilon(\boldsymbol{u}^\ell), \theta^\ell, \alpha^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s), \alpha^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell), \tag{2.46}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\varphi^\ell)\big), \tag{2.47}$$

$$\begin{aligned} &\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)))_{\mathcal{H}^\ell} + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + j_{fr}(\boldsymbol{u}(t), \boldsymbol{v}) \\ &-j_{fr}(\boldsymbol{u}(t), \boldsymbol{u}(t)) + j_{\nu c}(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \geq (\mathbf{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_{\boldsymbol{V}}, \ \forall \boldsymbol{v} \in \boldsymbol{V}, \end{aligned} \tag{2.48}$$

$$\begin{aligned} &\forall \xi \in E_1, \quad \sum_{\ell=1}^{2} (\dot{\theta}^\ell(t) - \rho^\ell(t), \xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta(t), \xi) = \\ &\sum_{\ell=1}^{2} \Big(\Theta^\ell\big(\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \theta^\ell(t), \alpha^\ell(t)\big), \xi^\ell\Big)_{L^2(\Omega^\ell)}, \end{aligned} \tag{2.49}$$

$$\begin{aligned} &\alpha(t) \in K, \ \ \forall \xi \in K, \ \sum_{\ell=1}^{2} (\dot{\alpha}^\ell(t), \xi^\ell - \alpha^\ell(t))_{L^2(\Omega^\ell)} + a(\alpha(t), \xi - \alpha(t)) \geq \\ &\sum_{\ell=1}^{2} \Big(\phi^\ell\big(\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \theta^\ell(t), \alpha^\ell(t)\big), \xi^\ell - \alpha^\ell(t)\Big)_{L^2(\Omega^\ell)}, \end{aligned} \tag{2.50}$$

$$\sum_{\ell=1}^{2} \Big(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)) + \mathcal{G}^\ell\big(E^\ell(\varphi^\ell(t))\big), \ \nabla\phi^\ell\Big)_{H^\ell} = (-q(t), \phi)_W, \ \forall \phi \in W, \tag{2.51}$$

$$\dot{\beta}(t) = H_{ad}(\beta(t), \xi_\beta(t), R_\nu([u_\nu(t)]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)]), \tag{2.52}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \theta(0) = \theta_0, \ \ \alpha(0) = \alpha_0, \ \ \beta(0) = \beta_0. \tag{2.53}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a temperature, a bonding field and a electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 2.1. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, equation (2.52) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (2.36) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (2.52) that $\dot{\beta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\beta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

First, we note that the functional j_{ad} and $j_{\nu c}$ are linear with respect to the last argument and, therefore,

$$\begin{aligned} j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) &= -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}), \\ j_{\nu c}(\boldsymbol{u}, -\boldsymbol{v}) &= -j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{2.54}$$

Next, using (2.44) and (2.32.b) imply

$$j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v}_2) \leq 0. \tag{2.55}$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, $\boldsymbol{R}_\tau$ show that

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v})| \leq c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}}. \tag{2.56}$$

Next, using (2.45), (2.32)(a), keeping in mind (2.22), we obtain

$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_2)-j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_1)+j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_1)-j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_2)\\ &\qquad\le c_0^2 L_\nu\|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{V}}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{\boldsymbol{V}}. \end{aligned} \tag{2.57}$$

3 Main Results

The main results are stated by the following theorems.

Theorem 3.1. *Assume that (2.25)–(2.36) hold. Then, there exists $\mu_0>0$ depending only on Ω^ℓ, Γ_1^ℓ, Γ_2^ℓ, Γ_3, p_ν, p_τ, H_{ad} and $\mathcal{A}^\ell$, $\ell=1,2$ such that, if $\|\mu\|<\mu_0$, then Problem **PV** has a unique solution $\{\boldsymbol{u},\boldsymbol{\sigma},\varphi,\theta,\alpha,\beta,\boldsymbol{D}\}$. Moreover, the solution satisfies*

$$\boldsymbol{u}\in C(0,T;\boldsymbol{V}), \tag{3.1}$$

$$\varphi\in C(0,T;W), \tag{3.2}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap L^\infty(0,T;\mathcal{Z}), \tag{3.3}$$

$$\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1), \tag{3.4}$$

$$\theta\in L^2(0,T;E_1)\cap H^1(0,T;E_0), \tag{3.5}$$

$$\alpha\in L^2(0,T;E_1)\cap H^1(0,T;E_0), \tag{3.6}$$

$$\boldsymbol{D}\in W^{1,\infty}(0,T;\mathcal{W}). \tag{3.7}$$

The proof of Theorem 3.1 is carried out in several steps and is based on the following abstract result for variational inequalities.

Let X be a real Hilbert space, and consider the Problem of finding $\boldsymbol{u}\in X$ such that :

$$(A\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})_X+j(\boldsymbol{u},\boldsymbol{v})-j(\boldsymbol{u},\boldsymbol{u}(t))\ge(f,\boldsymbol{v}-\boldsymbol{u})_X\quad\forall\boldsymbol{v}\in X. \tag{3.8}$$

To study problem (3.8) we need the following assumptions: The operator $A:X\to X$ is Lipschitz continuous and strongly monotone, $i.e.$,

$$\begin{cases} \text{(a) There exists } L_A>0 \text{ such that}\\ \qquad \|A\boldsymbol{u}_1-A\boldsymbol{u}_2\|_X\le L_A\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_X\quad\forall\,\boldsymbol{u}_1,\boldsymbol{u}_2\in X,\\ \text{(b) There exists } m_A>0 \text{ such that}\\ \qquad (A\boldsymbol{u}_1-A\boldsymbol{u}_2,\ \boldsymbol{u}_1-\boldsymbol{u}_2)_X\ge m_A\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_X\quad\forall\,\boldsymbol{u}_1,\boldsymbol{u}_2\in X. \end{cases} \tag{3.9}$$

The functional $j:X\times X\to\mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(\boldsymbol{u},.) \text{ is convex and l.S.C. on } X \text{ for all } u\in X.\\ \text{(b) There exists } m_j>0 \text{ such that}\\ \qquad j(\boldsymbol{u}_1,\boldsymbol{v}_2)-j(\boldsymbol{u}_1,\boldsymbol{v}_1)+j(\boldsymbol{u}_2,\boldsymbol{v}_1)-j(\boldsymbol{u}_2,\boldsymbol{v}_2)\\ \qquad\quad\le m_j\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_X\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_X\quad\forall\,\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2\in X. \end{cases} \tag{3.10}$$

Finally, we assume that

$$f\in X. \tag{3.11}$$

The following existence, uniqueness result and regularity was proved in [27, p.51].

Theorem 3.2. *Let (3.8)–(3.11) hold, and $m_j<m_A$. Then:*

(i) There exists a unique solution $\boldsymbol{u}\in X$ of Problem (3.8).

(ii) If, moreover, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (3.8) corresponding to the data $f_1,f_2\in X$, then there exists $c>0$ such that

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_X\le c\|f_1-f_2\|_X. \tag{3.12}$$

We turn now to the proof of Theorem 3.1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in what follows that (2.25)–(2.36) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{G}^\ell$, $\mathcal{Q}^\ell$, $\mathcal{E}^\ell$, γ_ν, γ_τ, $\mathbf{\Theta}^\ell$, ϕ^ℓ, κ_0^ℓ, κ^ℓ, and T with $\ell = 1, 2$. but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In the first step. Let $(\lambda, \mu) \in C(0, T; E_0 \times E_0)$ and consider the auxiliary problem.

Problem PV$_{(\lambda,\mu)}$. Find $\theta_\lambda : [0, T] \to E_0$, and $\alpha_\mu : [0, T] \to E_0$, such that

$$\sum_{\ell=1}^{2}(\dot\theta_\lambda^\ell(t) - \lambda^\ell(t) - \rho^\ell(t), \xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta_\lambda^\ell(t), \xi) = 0,\ \forall \xi \in E_0, \tag{3.13}$$

$$\alpha_\mu(t) \in K,\ \sum_{\ell=1}^{2}(\dot\alpha_\mu^\ell(t) - \mu^\ell(t), \xi^\ell - \alpha_\mu^\ell(t))_{L^2(\Omega^\ell)} + a(\alpha_\mu(t), \xi - \alpha_\mu(t)) \geq 0,\ \forall \xi \in K, \tag{3.14}$$

$$\theta_\lambda(0) = \theta_0, \quad \alpha_\mu(0) = \alpha_0, \tag{3.15}$$

where $K = K^1 \times K^2$.

Lemma 3.3. *There exists a unique solution $\{\theta_\lambda, \alpha_\mu\}$ to the auxiliary problem **PV**$_{(\lambda,\mu)}$ satisfying* (3.5)–(3.6).

Proof. Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant $c_0 > 0$ such that

$$\int_{\Omega^\ell} |\nabla\xi|^2 dx + \frac{\lambda_0^\ell}{\kappa_0^\ell}\int_{\Gamma^\ell} |\xi|^2 da \geq c_0 \int_{\Omega^\ell} |\xi|^2 dx, \quad \forall \xi \in E_1^\ell,\ \ell = 1, 2.$$

Thus, we obtain

$$a_0(\xi, \xi) \geq c_1 \|\xi\|_{E_1}^2, \quad \forall \xi \in E_1,$$

where $c_1 = \kappa_0 \min(1, c_0)/2$, which implies that a_0 is E_1-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.13) has a unique solution θ_λ satisfying $\theta_\lambda(0) = \theta_0$ and the regularity (3.5).

On the other hand, we know that the form a is not E_1-elliptic. To solve this problem we introduce the functions

$$\tilde\alpha_\mu^\ell(t) = e^{-\kappa^\ell t}\alpha_\mu^\ell(t), \quad \tilde\xi^\ell(t) = e^{-\kappa^\ell t}\xi^\ell(t), \quad \ell = 1, 2.$$

We remark that if α_μ^ℓ, $\xi^\ell \in K^\ell$ then $\tilde\alpha_\mu^\ell$, $\tilde\xi^\ell \in K^\ell$. Consequently, (3.14) is equivalent to the inequality

$$\begin{aligned} &\tilde\alpha_\mu \in K, \quad \sum_{\ell=1}^{2}(\dot{\tilde\alpha}_\mu^\ell(t) - e^{-\kappa^\ell t}\mu^\ell(t), \tilde\xi^\ell - \tilde\alpha_\mu^\ell(t))_{L^2(\Omega^\ell)} + a(\tilde\alpha_\mu(t), \tilde\xi - \tilde\alpha_\mu(t)) + \\ &\quad \sum_{\ell=1}^{2}\kappa^\ell(\tilde\alpha_\mu^\ell, \tilde\xi^\ell - \tilde\alpha_\mu^\ell(t))_{L^2(\Omega^\ell)} \geq 0,\ \forall \tilde\xi \in K,\ \text{a.e. } t \in (0, T). \end{aligned} \tag{3.16}$$

The fact that

$$a(\tilde\xi, \tilde\xi) + \sum_{\ell=1}^{2}\kappa^\ell(\tilde\xi^\ell, \tilde\xi^\ell)_{L^2(\Omega^\ell)} \geq \sum_{\ell=1}^{2}\kappa^\ell\|\tilde\xi^\ell\|_{E_1^\ell}^2 \quad \forall \tilde\xi \in E_1, \tag{3.17}$$

and using classical arguments of functional analysis concerning parabolic inequalities [2, 7], implies that (3.16) has a unique solution $\tilde\alpha_\mu$ having the regularity (3.6). □

In the second step. Let $(\lambda, \mu, \eta) \in C(0, T; E_0 \times E_0 \times \boldsymbol{V})$, we use the $\{\theta_\lambda, \alpha_\mu\}$ obtained in Lemma 3.3 and consider the auxiliary problem.

Problem PV$_{(\lambda,\mu,\eta)}$. Find $\boldsymbol{u}_{\lambda\mu\eta} : [0,T] \to \boldsymbol{V}$, $\varphi_{\lambda\mu\eta} : [0,T] \to W$, and $\beta_{\lambda\mu\eta} : [0,T] \to L^2(\Gamma_3)$ such that

$$\left.\begin{aligned} &\sum_{\ell=1}^{2}\Big(\mathcal{A}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\mu\eta}),\theta^\ell_\lambda,\alpha^\ell_\mu),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\mu\eta}(t))\Big)_{\mathcal{H}^\ell}\\ &+j_{\nu c}(\boldsymbol{u}_{\lambda\mu\eta}(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\mu\eta}(t))+j_{fr}(\boldsymbol{u}_{\lambda\mu\eta}(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}_{\lambda\mu\eta}(t),\boldsymbol{u}_{\lambda\mu\eta}(t))\\ &+(\eta(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\mu\eta}(t))_{\boldsymbol{V}}\geq(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_{\lambda\mu\eta}(t))_{\boldsymbol{V}},\ \ \forall \boldsymbol{v}\in\boldsymbol{V},\end{aligned}\right\} \tag{3.18}$$

$$\textstyle\sum_{\ell=1}^2\left(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(t))+\mathcal{G}^\ell E^\ell(\varphi^\ell_{\lambda\mu\eta}(t)),\nabla\phi^\ell\right)_{H^\ell}=(-q(t),\phi)_W,\quad \forall\phi\in W, \tag{3.19}$$

$$\dot\beta_{\lambda\mu\eta}(t)=H_{ad}\big(\beta_{\lambda\mu\eta}(t),\xi_{\beta_{\lambda\mu\eta}},R_\nu([u_{\lambda\mu\eta\nu}(t)]),\boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\mu\eta\tau}(t)])\big), \tag{3.20}$$

$$\boldsymbol{u}_{\lambda\mu\eta}(0)=\boldsymbol{u}_0,\ \ \beta_{\lambda\mu\eta}(0)=\beta_0. \tag{3.21}$$

We have the following result

Lemma 3.4. *(1) There exists $\mu_0 > 0$ depending only on Ω^ℓ, Γ_1^ℓ, Γ_2^ℓ, Γ_3, p_ν, p_τ, H_{ad} and $\mathcal{A}^\ell$, $\ell = 1,2$ such that, if $\|\mu\| < \mu_0$, then Problem $\boldsymbol{PV}_{(\lambda,\mu,\eta)}$ has a unique solution $\{\boldsymbol{u}_{\lambda\mu\eta},\varphi_{\lambda\mu\eta},\beta_{\lambda\mu\eta}\}$ which satisfies the regularity (3.1)–(3.3).*

(2) If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (3.18) and (3.21) corresponding to the data (λ_1,μ_1,η_1), $(\lambda_2,\mu_2,\eta_2)\in C(0,T;E_0\times E_0\times\boldsymbol{V})$, then there exists $c>0$ such that, for $t\in[0,T]$,

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}\leq c\|\eta_1(t)-\eta_2(t)\|_{\boldsymbol{V}}. \tag{3.22}$$

Proof. We apply Theorem 3.2 where $X=\boldsymbol{V}$, with the inner product $(\cdot,\cdot)_{\boldsymbol{V}}$ and the associated norm $\|.\|_{\boldsymbol{V}}$. Let $t\in[0,T]$. We use the Riesz representation theorem to define the operator $A:\boldsymbol{V}\to\boldsymbol{V}$ by

$$(A\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}}=\sum_{\ell=1}^{2}\left(\mathcal{A}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell),\theta^\ell_\lambda,\alpha^\ell_\mu),\ \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}, \tag{3.23}$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}$, and define $\mathbf{f}_\eta\in X$ and the function $j:\boldsymbol{V}\times\boldsymbol{V}\to\mathbb{R}$ by

$$\mathbf{f}_\eta=\mathbf{f}(t)-\boldsymbol{\eta}(t), \tag{3.24}$$

$$j(\boldsymbol{u},\boldsymbol{v})=j_{\nu c}(\boldsymbol{u},\boldsymbol{v})+j_{fr}(\boldsymbol{u},\boldsymbol{v}),\quad\forall\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}. \tag{3.25}$$

Assumptions (2.25) imply that the operators A satisfy conditions (3.9).
It follows from (2.32), (2.36), (2.44) and (2.45) that the functional j, (3.25), satisfies condition (3.10(a)). We use again (2.55), (2.57) and (3.25) to find

$$\begin{aligned}&j(\boldsymbol{u}_1,\boldsymbol{v}_2)-j(\boldsymbol{u}_1,\boldsymbol{v}_1)+j(\boldsymbol{u}_2,\boldsymbol{v}_1)-j(\boldsymbol{u}_2,\boldsymbol{v}_2)\\ &\quad\leq c_0^2L_\nu\|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{V}}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{\boldsymbol{V}}\quad\forall\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2\in\boldsymbol{V},\end{aligned} \tag{3.26}$$

Using now (3.23)–(3.26) we find that (3.18) and (3.22) is a direct consequence of Theorem 3.2. Let now $t_1,t_2\in[0,T]$, an argument based on (2.25), (2.56) and (2.57) shows that

$$\begin{aligned}\|\boldsymbol{u}_{\lambda\mu\eta}(t_1)-\boldsymbol{u}_{\lambda\mu\eta}(t_2)\|_{\boldsymbol{V}}\leq c\big(&\|\lambda(t_1)-\lambda(t_2)\|_{E_0}+\|\mu(t_1)-\mu(t_2)\|_{E_0}+\\ &\|\eta(t_1)-\eta(t_2)\|_{\boldsymbol{V}}+\|\mathbf{f}(t_1)-\mathbf{f}(t_2)\|_{\boldsymbol{V}}\big).\end{aligned} \tag{3.27}$$

Keeping in mind that $\mathbf{f}\in C(0,T;\boldsymbol{V})$ and recall that $(\mu,\lambda,\eta)\in C(0,T;E_0\times E_0\times\boldsymbol{V})$, it follows now from (3.27) that the mapping $\boldsymbol{u}_{\lambda\mu\eta}$ satisfies the regularity (3.1).

Let us consider the form $G:W\times W\to\mathbb{R}$,

$$G(\varphi,\phi)=\sum_{\ell=1}^{2}(\mathcal{G}^\ell\nabla\varphi^\ell,\nabla\phi^\ell)_{H^\ell}\quad\forall\varphi,\phi\in W. \tag{3.28}$$

We use (2.23), (2.24), (2.30) and (3.28) to show that the form G is bilinear continuous, symmetric and coercive on W, moreover using (2.39) and the Riesz representation Theorem we may define an element $w_{\lambda\mu\eta} : [0,T] \to W$ such that

$$(w_{\lambda\mu\eta}(t), \phi)_W = (q(t), \phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(t)), \nabla\phi^\ell)_{H^\ell} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_{\lambda\mu\eta}(t) \in W$ such that

$$G(\varphi_{\lambda\mu\eta}(t), \phi) = (w_{\lambda\mu\eta}(t), \phi)_W \quad \forall \phi \in W. \tag{3.29}$$

It follows from (3.29) that $\varphi_{\lambda\mu\eta}$ is a solution of the equation (3.19). Let $t_1, t_2 \in [0,T]$, it follows from (3.19) that

$$\|\varphi_{\lambda\mu\eta}(t_1) - \varphi_{\lambda\mu\eta}(t_2)\|_W \leq C\big(\|\boldsymbol{u}_{\lambda\mu\eta}(t_1) - \boldsymbol{u}_{\lambda\mu\eta}(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big). \tag{3.30}$$

Now, from (2.33), (3.30) and $\boldsymbol{u}_{\lambda\mu\eta} \in C(0,T;\boldsymbol{V})$, we obtain that $\varphi_{\lambda\mu\eta} \in C(0,T;W)$.

On the other hand, we consider the mapping $H_{\lambda\mu\eta} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_{\lambda\mu\eta}(t,\beta) = H_{ad}\big(\beta_{\lambda\mu\eta}(t), \xi_{\beta_{\lambda\mu\eta}}, R_\nu([u_{\lambda\mu\eta\nu}(t)]), \boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\mu\eta\tau}(t)])\big),$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that $H_{\lambda\mu\eta}$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to H_{\lambda\mu\eta}(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz theorem (see [26, p.48], we deduce that there exists a unique function $\beta_{\lambda\mu\eta} \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (3.20). Also, the arguments used in Remark 2.1 show that $0 \leq \beta_{\lambda\mu\eta}(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. This completes the proof. □

In the third step, let us consider the element

$$\Lambda(\eta,\lambda,\mu)(t) = \big(\Lambda^1(\eta,\lambda,\mu)(t),\ \Lambda^2(\eta,\lambda,\mu)(t),\ \Lambda^3(\eta,\lambda,\mu)(t)\big) \in \boldsymbol{V} \times E_0 \times E_0, \tag{3.31}$$

defined by the equations

$$\begin{aligned}(\Lambda^1(\eta,\lambda,\mu)(t), \boldsymbol{v})_{\boldsymbol{V}} = &-\sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* E^\ell(\varphi^\ell_{\lambda\mu\eta}),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\beta_{\lambda\mu\eta}(t), \boldsymbol{u}_{\lambda\mu\eta}(t), \boldsymbol{v}) \\ &+ \sum_{\ell=1}^{2} \left(\int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(s)), \theta^\ell_\lambda(s), \alpha^\ell_\mu(s)\big)\, ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{3.32}$$

$$\Lambda^2(\eta,\lambda,\mu) = \Big(\Theta^1\big(\boldsymbol{\sigma}^1_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^1_{\lambda\mu\eta}), \theta^1_\lambda, \alpha^1_\mu\big),\ \Theta^2\big(\boldsymbol{\sigma}^2_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^2_{\lambda\mu\eta}), \theta^2_\lambda, \alpha^2_\mu\big)\Big), \tag{3.33}$$

$$\Lambda^3(\eta,\lambda,\mu) = \Big(\phi^1\big(\boldsymbol{\sigma}^1_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^1_{\lambda\mu\eta}), \theta^1_\lambda, \alpha^1_\mu\big),\ \phi^2\big(\boldsymbol{\sigma}^2_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^2_{\lambda\mu\eta}), \theta^2_\lambda, \alpha^2_\mu\big)\Big), \tag{3.34}$$

where the mapping $\boldsymbol{\sigma}^\ell_{\lambda\mu\eta}$ is given by

$$\boldsymbol{\sigma}^\ell_{\lambda\mu\eta} = \mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\mu\eta}), \theta^\ell_\lambda, \alpha^\ell_\mu\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(s)), \theta^\ell_\lambda(s), \alpha^\ell_\mu(s)\big)ds - (\mathcal{E}^\ell)^* E^\ell(\varphi^\ell_{\lambda\mu\eta}). \tag{3.35}$$

Lemma 3.5. *The mapping Λ has a fixed point $(\eta^*, \lambda^*, \mu^*) \in C(0,T;\boldsymbol{V} \times E_0 \times E_0)$.*

Proof. Let (η_1,λ_1,μ_1), $(\eta_2,\lambda_2,\mu_2)\in C(0,T;\boldsymbol{V}\times E_0\times E_0)$ and denote by θ_i, α_i, $\boldsymbol{u}_i$, φ_i, β_i and $\boldsymbol{\sigma}_i$, the functions obtained in Lemmas 3.3, 3.4 and the relation (3.35), for $(\eta,\lambda,\mu)=(\eta_i,\lambda_i,\mu_i)$, $i=1,2$. Let $t\in[0,T]$. We use (2.28), (2.29), (2.43) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}}\le\sum_{\ell=1}^{2}\|(\mathcal{E}^\ell)^*\nabla\varphi_1^\ell(t)-(\mathcal{E}^\ell)^*\nabla\varphi_2^\ell(t)\|^2_{\mathcal{H}^\ell}+$$

$$\sum_{\ell=1}^{2}\int_0^t\big\|\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_1^\ell(s)),\theta_1^\ell(s),\alpha_1^\ell(s)\big)-\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_2^\ell(s)),\theta_2^\ell(s),\alpha_2^\ell(s)\big)\big\|^2_{\mathcal{H}^\ell}\,ds$$

$$+C\|\beta_1^2(t)R_\nu([u_{1\nu}(t)])-\beta_2^2(t)R_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)}$$
$$+C\|\beta_1^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)])-\beta_2^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|^2_{L^2(\Gamma_3)}.$$

Therefore,

$$\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}}\le C\bigg(\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+$$
$$\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,ds\;+\int_0^t\|\alpha_1(s)-\alpha_2(s))\|^2_{E_0}\,ds+$$
$$\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).\tag{3.36}$$

By similar arguments, from (3.33), (3.35) and (2.27) it follows that

$$\|\Lambda^2(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^2(\eta_2,\lambda_2,\mu_2)(t)\|^2_{E_0}\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\alpha_1(t)-\alpha_2(t))\|^2_{E_0}+\int_0^t\|\alpha_1(s)-\alpha_2(s))\|^2_{E_0}\,ds$$
$$+\|\theta_1(t)-\theta_2(t))\|^2_{E_0}+\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,ds+\|\varphi_1(t)-\varphi_2(t)\|^2_W\bigg).\tag{3.37}$$

Similarly, using (2.28) implies

$$\|\Lambda^3(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^3(\eta_2,\lambda_2,\mu_2)(t)\|^2_{E_0}\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\alpha_1(t)-\alpha_2(t))\|^2_{E_0}+\int_0^t\|\alpha_1(s)-\alpha_2(s))\|^2_{E_0}\,ds$$
$$+\|\theta_1(t)-\theta_2(t))\|^2_{E_0}+\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,ds+\|\varphi_1(t)-\varphi_2(t)\|^2_W\bigg).\tag{3.38}$$

It follows now from (3.36), (3.37) and (3.38) that

$$\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}\times E_0\times E_0}\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\alpha_1(t)-\alpha_2(t))\|^2_{E_0}+\int_0^t\|\alpha_1(s)-\alpha_2(s))\|^2_{E_0}\,ds$$
$$+\|\theta_1(t)-\theta_2(t))\|^2_{E_0}+\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,ds$$
$$+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).\tag{3.39}$$

Also, from the Cauchy problem (3.20) we can write

$$\beta_i(t)=\beta_0-\int_0^t H_{ad}\big(\beta_i(s),\xi_{\beta_i}(s),R_\nu([u_{i\nu}(s)]),\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])\big)ds$$

and, employing (2.18) and (2.31) we obtain that

$$\begin{aligned}\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\leq &C\int_0^t\big\|\beta_1(s)-\beta_2(s)\big\|_{L^2(\Gamma_3)}ds\\&+C\int_0^t\big\|R_\nu([u_{1\nu}(s)])-R_\nu([u_{2\nu}(s)])\big\|_{L^2(\Gamma_3)}ds\\&+C\int_0^t\big\|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)])-\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])\big\|_{L^2(\Gamma_3)}ds.\end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\beta_1=\beta_1-\beta_2+\beta_2$, we get

$$\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\leq C\Big(\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds+\int_0^t\big\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d}ds\Big).\quad(3.40)$$

Next, we apply Gronwall's inequality and from the Sobolev trace theorem we obtain

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\leq C\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V ds.\quad(3.41)$$

We use now (3.19), (2.23), (2.29) and (2.30) to find

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W\leq C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V.\quad(3.42)$$

From (3.13) we deduce that

$$(\dot\theta_1-\dot\theta_2,\theta_1-\theta_2)_{E_0}+a_0(\theta_1-\theta_2,\theta_1-\theta_2)+\big(\lambda_1-\lambda_2,\theta_1-\theta_2\big)_{E_0}=0.$$

We integrate this equality with respect to time, using the initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$ and inequality $a_0(\theta_1-\theta_2,\theta_1-\theta_2)\geq 0$, to find

$$\frac{1}{2}\|\theta_1(t)-\theta_2(t)\|^2_{E_0}\leq\int_0^t\big(\lambda_1(s)-\lambda_2(s),\theta_1(s)-\theta_2(s)\big)_{E_0}ds,$$

which implies that

$$\|\theta_1(t)-\theta_2(t)\|^2_{E_0}\leq\int_0^t\|\lambda_1(s)-\lambda_2(s)\|^2_{E_0}ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{E_0}ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\theta_1(t)-\theta_2(t)\|^2_{E_0}\leq C\int_0^t\|\lambda_1(s)-\lambda_2(s)\|^2_{E_0}ds\quad\forall t\in[0,T].\quad(3.43)$$

Moreover, from (3.14) we deduce that a.e. $t\in(0,T)$,

$$(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{E_0}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\leq\big(\mu_1-\mu_2,\alpha_1-\alpha_2\big)_{E_0},$$

Integrating the previous inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$ and inequality $a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\geq 0$, to find

$$\frac{1}{2}\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\leq\int_0^t\big(\mu_1(s)-\mu_2(s),\alpha_1(s)-\alpha_2(s)\big)_{E_0}ds,$$

which implies that

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\leq\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{E_0}ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{E_0}ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\leq C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{E_0}ds.\quad(3.44)$$

We substitute (3.22), (3.41)-(3.44) in (3.39) to obtain

$$\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}\times E_0\times E_0}\le C\int_0^t \|(\eta_1,\lambda_1,\mu_1)(s)-(\eta_2,\lambda_2,\mu_2)(s)\|^2_{\boldsymbol{V}\times E_0\times E_0}\,ds.$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m(\eta_1,\lambda_1,\mu_1)-\Lambda^m(\eta_2,\lambda_2,\mu_2)\|^2_{C(0,T;\boldsymbol{V}\times E_0\times E_0)}\le \frac{C^mT^m}{m!}\|(\eta_1,\lambda_1,\mu_1)-(\eta_2,\lambda_2,\mu_2)\|^2_{C(0,T;\boldsymbol{V}\times E_0\times E_0)}.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\boldsymbol{V}\times E_0\times E_0)$, and so Λ has a unique fixed point. □

Let $(\eta^*,\lambda^*,\mu^*)\in C(0,T;\boldsymbol{V}\times E_0\times E_0)$, be the fixed point of Λ, and denote

$$\boldsymbol{u}_*=\boldsymbol{u}_{\lambda^*\mu^*\eta^*},\quad \varphi_*=\varphi_{\lambda^*\mu^*\eta^*},\quad \beta_*=\beta_{\lambda^*\mu^*\eta^*},\quad \theta_*=\theta_{\lambda^*},\quad \alpha_*=\alpha_{\mu^*}, \tag{3.45}$$

$$\boldsymbol{\sigma}^\ell_*=\mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*),\theta^\ell_*,\alpha^\ell_*\big)+\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\theta^\ell_*(s),\alpha^\ell_*(s)\big)\,ds-(\mathcal{E}^\ell)^*E^\ell(\varphi^\ell_*), \tag{3.46}$$

$$\boldsymbol{D}^\ell_*=\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_*)+\mathcal{G}^\ell\big(E^\ell(\varphi^\ell_*)\big). \tag{3.47}$$

We use : $\Lambda^1(\eta^*,\lambda^*,\mu^*)=\eta^*$, $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\lambda^*$, and $\Lambda^3(\eta^*,\lambda^*,\mu^*)=\mu^*$, it follows:

$$\begin{aligned}(\eta^*(t),\boldsymbol{v})_{\boldsymbol{V}}=&-\sum_{\ell=1}^2\big((\mathcal{E}^\ell)^*E^\ell(\varphi^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})\\&+\sum_{\ell=1}^2\Big(\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\theta^\ell_*(s),\alpha^\ell_*(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell},\quad\forall\boldsymbol{v}\in\boldsymbol{V},\end{aligned} \tag{3.48}$$

$$\lambda^\ell_*(t)=\Theta^\ell\big(\boldsymbol{\sigma}^\ell_*(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)),\theta^\ell_*(t),\alpha^\ell_*(t)\big),\quad \ell=1,2. \tag{3.49}$$

$$\mu^\ell_*(t)=\phi^\ell\big(\boldsymbol{\sigma}^\ell_*(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)),\theta^\ell_*(t),\alpha^\ell_*(t)\big),\ \ell=1,2. \tag{3.50}$$

Existence. We prove $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\alpha_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (2.46)–(2.53) and the regularites (3.1)–(3.7). Indeed, we write (3.18) for $(\eta,\lambda,\mu)=(\eta^*,\lambda^*,\mu^*)$ and use (3.45) to find

$$\begin{aligned}&\sum_{\ell=1}^2(\mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*),\theta^\ell_*,\alpha^\ell_*\big),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)))_{\mathcal{H}^\ell}\\&+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{u}_*(t))\\&+(\eta^*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}}\ge(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}},\ \forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{3.51}$$

Substitute (3.48) in (3.51) to obtain

$$\begin{aligned}&\sum_{\ell=1}^2(\mathcal{A}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*),\theta^\ell_*,\alpha^\ell_*\big),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)))_{\mathcal{H}^\ell}\\&+\sum_{\ell=1}^2\Big(\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\theta^\ell_*(s),\alpha^\ell_*(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t))\Big)_{\mathcal{H}^\ell}\\&+j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\boldsymbol{u}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})\\&-j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{u}_*(t))-\sum_{\ell=1}^2\big((\mathcal{E}^\ell)^*E^\ell(\varphi^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t))\big)_{\mathcal{H}^\ell}\\&\ge(\mathbf{f}(t),\boldsymbol{v}-\boldsymbol{u}_*(t))_{\boldsymbol{V}}\ \ \forall\boldsymbol{v}\in\boldsymbol{V}\quad a.e.\ t\in[0,T],\end{aligned} \tag{3.52}$$

and we substitute (3.49) in (3.13) to have

$$\sum_{\ell=1}^{2}(\dot\theta_*^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}+a_0(\theta_*^\ell(t),\xi)=\sum_{\ell=1}^{2}\left(\lambda_*^\ell(t)+\rho^\ell(t),\xi^\ell\right)_{L^2(\Omega^\ell)}, \tag{3.53}$$

for all $\xi\in E_0,$ a.e. $t\in(0,T)$.

Next, substitute (3.50) in (2.25) to obtain $\alpha_*(t)\in K$, and

$$\begin{aligned}&\sum_{\ell=1}^{2}(\dot\alpha_*^\ell(t),\xi^\ell-\alpha_*^\ell(t))_{L^2(\Omega^\ell)}+a(\alpha_*(t),\xi-\alpha_*(t))\geq\\&\qquad\sum_{\ell=1}^{2}\left(\phi^\ell(\boldsymbol{\sigma}_*^\ell(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)),\alpha_*^\ell(t)),\xi^\ell-\alpha_*^\ell(t)\right)_{L^2(\Omega^\ell)},\end{aligned} \tag{3.54}$$

for all $\xi\in K,$ a.e. $t\in(0,T)$. We write now (3.20) for $(\eta,\lambda,\mu)=(\eta^*,\lambda^*,\mu^*)$ and use (3.45) to see that

$$\sum_{\ell=1}^{2}(\mathcal{G}^\ell E^\ell(\varphi_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}+\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}=-(q(t),\phi)_W, \tag{3.55}$$

for all $\phi\in W,$ a.e. $t\in(0,T)$. Additionally, we use $\boldsymbol{u}_{\lambda^*\mu^*\eta^*}$ in (3.20) and (3.45) to find

$$\dot\beta_*(t)=H_{ad}\big(\beta_*(t),\xi_{\beta_*}(t),R_\nu([u_{*\nu}(t)]),\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])\big) \tag{3.56}$$

a.e. $t\in[0,T]$. The relations (3.51)–(3.56), allow us to conclude now that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\alpha_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (2.46)–(2.52). Next, (2.53) the regularity (3.1)–(3.3) and (3.5)–(3.6) follow from Lemmas 3.3 and 3.4. Since $\boldsymbol{u}_*$, φ_*, θ_* and α_* satisfies (3.1), (3.2), (3.5) and (3.6), respectively, It follows from (3.46) that

$$\boldsymbol{\sigma}_*\in C(0,T;\mathcal{H}). \tag{3.57}$$

For $\ell=1,2$, we choose $\boldsymbol{v}=\boldsymbol{u}\pm\phi$ in (3.52), with $\phi=(\phi^1,\phi^2)$, $\phi^\ell\in D(\Omega^\ell)^d$ and $\phi^{3-\ell}=0$, to obtain

$$\operatorname{Div}\boldsymbol{\sigma}_*^\ell(t)=-\boldsymbol{f}_0^\ell(t)\quad\forall t\in[0,T],\ \ \ell=1,2, \tag{3.58}$$

where $D(\Omega^\ell)$ is the space of infinitely differentiable real functions with a compact support in Ω^ℓ. The regularity (3.4) follows from (2.33), (3.57) and (3.58). Let now $t_1,t_2\in[0,T]$, from (2.23), (2.29), (2.30) and (3.47), we conclude that there exists a positive constant $C>0$ verifying

$$\|\boldsymbol{D}_*(t_1)-\boldsymbol{D}_*(t_2)\|_H\leq C\left(\|\varphi_*(t_1)-\varphi_*(t_2)\|_W+\|\boldsymbol{u}_*(t_1)-\boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (3.1) and (3.2) implies

$$\boldsymbol{D}_*\in C(0,T;H). \tag{3.59}$$

For $\ell=1,2$, we choose $\phi=(\phi^1,\phi^2)$ with $\phi^\ell\in D(\Omega^\ell)^d$ and $\phi^{3-\ell}=0$ in (3.55) and using (2.39) we find

$$\operatorname{div}\boldsymbol{D}_*^\ell(t)=q_0^\ell(t)\quad\forall t\in[0,T],\ \ \ell=1,2. \tag{3.60}$$

Property (3.7) follows from (2.33), (3.59) and (3.60).

Finally we conclude that the weak solution $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\alpha_*,\beta_*,\boldsymbol{D}_*\}$ of the problem **PV** has the regularity (3.1)–(3.7), which concludes the existence part of Theorem 3.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda(.,.,.)$ defined by (3.32)–(3.33) and the unique solvability of the Problems $\mathrm{PV}_{(\lambda,\mu)}$, and $\mathrm{PV}_{(\lambda,\mu,\eta)}$. □

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