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ON FRAMES IN FINITE DIMENSIONAL QUATERNIONIC HILBERT SPACE

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Abstract. Khokulan et al. [14] introduced frames for finite dimensional quaternionic Hilbert spaces. In this paper, we study frames for quaternonic Hilbert spaces and discuss some properties of a frame operator associated with a frame in a quaternionic Hilbert space.

1 Introduction

Theory of frames was introduced by Duffin and Schaeffer [11] associated to applied harmonic analysis, but its roots are involved in broad areas of functional analysis including operator theory and theory of bases in Hilbert spaces. One may consider, frames as a generalization of bases in sense that frames also provide a strong and healthy representation of vectors in a Hilbert space $\mathcal{H}$. Frames are redundant in nature therefore sometimes they allow easier reconstruction of vectors than bases and that too with some better properties which are not achievable using bases, and therefore these days frames have variety of applications in wide range of area of engineering and sciences [2, 3, 4].

"A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is said to be a *frame* for a Hilbert space $\mathcal{H}$ if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}. \tag{1.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{x_n\}_{n\in\mathbb{N}}$. The inequality (1) is called the *frame inequality* for the frame $\{x_n\}_{n\in\mathbb{N}}$. A frame $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be

- *tight* if it is possible to choose $A = B$.
- *Parseval* if it is a tight frame with $A = B = 1$."

Beside frames one larger class of sequences which played a vital role in the development of frame theory is of Bessel sequences. *Bessel sequences* are the sequences which satisfy only upper condition of frame inequality (1.1). These sequences, in general need not be bases but have some properties similar to that of orthonormal bases. For literature on frame theory in Hilbert spaces and Banach spaces, one may refer to [5, 6, 8, 9, 10, 17].

Recently, Hemmat et al. [13] gave a scheme to form a basis and a frame for a Hilbert space of quaternion valued square integrable function from a basis and a frame, respectively, of a Hilbert space of complex valued square integrable functions. Sharma and Goel [15] introduced and studied frames for separable quaternionic Hilbert spaces and frames for finite dimensional quaternionic Hilbert spaces were introduced and studied by Khokulan et al. [14] . Sharma and Virender [16] discuss some different types of dual frames of a given frame in a finite dimensional quaternionic Hilbert space and gave various types of reconstructions with the help of a dual frame. In this paper, we study frames for finite dimensional quaternonic Hilbert spaces and discussed some properties of a frame operator associated with a frame in finite dimensional quaternionic Hilbert space.

Throughout this paper we denote $\mathbb{H}$ as a non-commutative field of quaternion, i.e.,

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k \;:\; x_0,\ x_1,\ x_2,\ x_3 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1;\ ij = -ji = k;\ jk = -kj = i$ and $ki = -ik = j$, $V_R(\mathbb{H})$ as a finite dimensional right quaternionic Hilbert space and ran T will denote the range of operator T.

For each quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, define conjugate of q denoted by $\overline{q}$ as

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k \in \mathbb{H}.$$

If $q = x_0 + x_1 i + x_2 j + x_3 k$ is a quaternion then x_0 is called the real part of q and $x_1 i + x_2 j + x_3 k$ is called the imaginary part of q. The modulus of quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is defined as

$$|q| = (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

For every non-zero quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, there exists a unique inverse q^{-1} as

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{x_0 - x_1 i - x_2 j - x_3 k}{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

2 Quaternionic Hilbert space

Definition 2.1. ([1]) A right quaternionic Hilbert space $V_R(\mathbb{H})$ is a vector space under right multiplication by quaternionic scalars together with the binary mapping $\langle .|.\rangle : V_R(\mathbb{H}) \times V_R(\mathbb{H}) \to \mathbb{H}$ (called the scalar (quaternion) product) which satisfies following properties:

(a) $\overline{\langle v_1|v_2\rangle} = \langle v_2|v_1\rangle$ for all $v_1, v_2 \in V_R(\mathbb{H})$.

(b) $\langle v|v\rangle > 0$ if $v \neq 0$.

(c) $\langle v|v_1 + v_2\rangle = \langle v|v_1\rangle + \langle v|v_2\rangle$ for all $v, v_1, v_2 \in V_R(\mathbb{H})$

(d) $\langle v|uq\rangle = \langle v|u\rangle q$ for all $v, u \in V_R(\mathbb{H})$ and $q \in \mathbb{H}$.

In view of Definition 2.1, we have following properties of right quaternionic Hilbert space:

(i) $\langle vq|u\rangle = \overline{q}\langle v|u\rangle$ for all $v, u \in V_R(\mathbb{H})$ and $q \in \mathbb{H}$

(ii) $v_1 p + v_2 q \in V_R(\mathbb{H})$, for all $v_1, v_2 \in V_R(\mathbb{H})$ and $p, q \in \mathbb{H}$

One may observe that $\mathbb{H}(= \mathbb{H}(\mathbb{H}))$ is a right quaternionic Hilbert space with respect to the quaternion product

$$\langle p|q\rangle = \overline{p}q,\ p, q \in \mathbb{H}.$$

Also, if we take the vector space

$$\mathbb{H}^m = \{q = (q_1, q_2, \cdots, q_m) : q_i \in \mathbb{H}\}$$

under right multiplication by quaternionic scalars together with the quaternion product on $\mathbb{H}^m$ as

$$\langle p|q\rangle_{\mathbb{H}^m} = \sum_{n=1}^{m} \overline{p_i} q_i,\ \ p = (p_1, \cdots, p_m) \text{ and } q = (q_1, \cdots, q_m) \in V_R(\mathbb{H}).$$

Then $\mathbb{H}^m$ is a right quaternionic Hilbert space with above defined quaternion product.

Definition 2.2 ([1]). Let $V_R(\mathbb{H})$ be a right quaternionic Hilbert space and $T : V_R(\mathbb{H}) \to V_R(\mathbb{H})$ be an operator. Then T is said to be

- *right linear* if $T(v_1 q_1 + v_2 q_2) = T(v_1)q_1 + T(v_2)q_2$ for all $v_1, v_2 \in V_R(\mathbb{H})$ and $q_1,\ q_2 \in \mathbb{H}$.
- *bounded* if there exist $K \geq 0$ such that $\|T(v)\| \leq K\|v\|$ for all $v \in V_R(\mathbb{H})$.

Theorem 2.3 ([18]). *Let $\mathbb{H}$ be a right quaternionic Hilbert space. Then the right dual space of $\mathbb{H}$ is congruent to the space $\mathbb{H}$.*

Definition 2.4 ([1]). Let $V_R(\mathbb{H})$ be a right quaternionic Hilbert space and $T : V_R(\mathbb{H}) \to V_R(\mathbb{H})$ be an operator. Then the adjoint operator T^* of T is defined by

$$\langle v|Tu\rangle = \langle T^* v|u\rangle, \text{ for all } u, v \in V_R(\mathbb{H}).$$

Further, T is said to be *self-adjoint* if $T = T^*$.

Theorem 2.5 ([1]). *Let S and T be two bounded operators on $V_R(\mathbb{H})$. Then*

(a) $\langle Tv|u\rangle = \langle v|S^* u\rangle$.

(b) $(S + T)^* = S^* + T^*$.

(c) $(ST)^* = T^* S^*$.

(d) $(S^*)^* = S$.

(e) *$I^* = I$, where I is the identity operator on $V_R(\mathbb{H})$.*

(f) *If S is an invertible operator then $(S^{-1})^* = (S^*)^{-1}$.*

Definition 2.6. ([12]) Let $V_R(\mathbb{H})$ be a finite dimensional right quaternionic Hilbert space. Then a sequence $\{v_n\}_{n=1}^m$ is said to be right basis for $V_R(\mathbb{H})$ if

(i) $V_R(\mathbb{H}) =$ right span $\{v_n\}_{n=1}^m =$ span $[v_n]_{n=1}^m$.

(ii) $\{v_n\}_{n=1}^m$ is a linearly independent set.

Definition 2.7. A right basis $\{e_n\}_{n=1}^m$ for $V_R(\mathbb{H})$ is said to be right orthonormal basis if

$$\langle e_i|e_j\rangle = \delta_{ij}, \text{ for all } i, j \in \{1, 2, \cdots, m\}.$$

Definition 2.8. ([12]) Let $V_R(\mathbb{H})$ be a finite dimensional right quaternionic Hilbert space. An operator $P : V_R(\mathbb{H}) \to V_R(\mathbb{H})$ is called a projection if $P^2 = P$. It is an orthogonal projection if P is also self adjoint.

Khokulan et al. [14] introduced frames for finite dimensional quaternionic Hilbert spaces and gave the following definition:

Definition 2.9. A sequence $\{v_n\}_{n=1}^m \subset V_R(\mathbb{H})$ is said to be a *frame* for a right quaternionic Hilbert space $V_L(\mathbb{H})$ if there exist positive constants A and B such that

$$A\|v\|^2 \leq \sum_{n=1}^{m} |\langle v_n|v\rangle|^2 \leq B\|v\|^2, \text{ for all } v \in V_L(\mathbb{H}). \tag{2.1}$$

These positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{v_n\}_{n=1}^m$. The inequality (2.1) is called the *frame inequality* for the frame $\{v_n\}_{n=1}^m$. A frame $\{v_n\}_{n=1}^m$ in $V_L(\mathbb{H})$ is said to be

- *tight* if it is possible to choose A, B satisfying inequality (2.1) with $A = B$.
- *Parseval* if it is tight with $A = B = 1$.
- *normalized* if $\|v_n\| = 1$, for all $n = 1, \cdots, m$.

If $\{v_n\}_{n=1}^m$ is a frame for $V_R(\mathbb{H})$, then the bounded linear operator $T : \mathbb{H}^m \to V_R(\mathbb{H})$ given by

$$T(\{q_n\}_{n=1}^m) = \sum_{n=1}^{m} v_n q_n, \text{ for all } \{q_n\}_{n=1}^m \in \mathbb{H}^m \tag{2.2}$$

is called the *pre-frame operator* or *synthesis operator*. The adjoint operator $T^* : V_R(\mathbb{H}) \to \mathbb{H}^m$ of T is given by

$$T^*(v) = \{\langle v_n|v\rangle\}_{n=1}^m, \text{ for all } v \in V_R(\mathbb{H}) \tag{2.3}$$

is called the *analysis operator*. By composing T and T^*, we obtain the frame operator $S_R : V_R(\mathbb{H}) \to V_R(\mathbb{H})$ defined as

$$\begin{aligned} S_R(v) &= TT^*(v) \\ &= \sum_{n=1}^{m} v_n \langle v_n|v\rangle, \text{ for all } v \in V_R(\mathbb{H}). \end{aligned}$$

3 Frames in $\mathbb{H}^n$

In the quaternion domain we may define four different types of subspaces which exist due to the proposed left and right matrix multiplication. If $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{k\times n}$ are two matrices over $\mathbb{H}$, define

$$\begin{aligned} A._L B &= \sum_{r=1}^{k} a_{ir}.b_{rj} \quad \text{and} \\ A._R B &= \sum_{r=1}^{k} b_{rj}.a_{ir}\ . \end{aligned}$$

Due to this the row spaces and column spaces are defined in the order of their multiplication. Hence, the four subspaces corresponding to a matrix $A \in \mathbb{H}^{m\times n}$ are as follows:

$$\begin{aligned} \mathcal{LR}(A) &= \{y \in \mathbb{H}^m : y = A._L x, \quad x \in \mathbb{H}^n\} \ \textbf{Left row space} \\ \mathcal{RR}(A) &= \{y \in \mathbb{H}^m : y = A._R x, \quad x \in \mathbb{H}^n\} \ \textbf{Right row space} \\ \mathcal{LC}(A) &= \{y \in \mathbb{H}^n : y^T = x^T._L A, \ x \in \mathbb{H}^m\} \ \textbf{Left column space} \\ \mathcal{RC}(A) &= \{y \in \mathbb{H}^n : y^T = x^T._R A, \ x \in \mathbb{H}^m\} \ \textbf{Right column space} \end{aligned}$$

Therefore, we propose the following definition of the rank of a matrix over quaternions

Definition 3.1. Let $\mathbb{H}$ be a right quaternionic Hilbert space and A be a $m \times n$ matrix over $\mathbb{H}$. Then, the *right column rank* of a quaternion matrix A is defined as the maximum number of columns of A that are right linearly independent as elements of $\mathbb{H}^m$, and is denoted by rc.$\rho(A)$. Similarly, the *right row rank* of a quaternion matrix A is defined as the maximum number of rows of A that are right linearly independent as elements of $\mathbb{H}^n$, and is denoted by rr.$\rho(A)$. The right rank of A denoted as r.$\rho(A)$ is defined as the rank of the mapping $\mathcal{L}_A : \mathbb{H}^n \to \mathbb{H}^m$ defined by

$$\mathcal{L}_A(v) = A._R v, \quad v \in \mathbb{H}^n.$$

If $A = (a_{ij})$ is a $m \times n$ matrix over $\mathbb{H}$. Then

(i) $\mathcal{L}_A : \mathbb{H}^n \to \mathbb{H}^m$ is right linear. Infact, if $A = (a_{ij})_{m\times n}$, then

$$\begin{aligned} \mathcal{L}_A(vq) &= A(vq) \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} q_1 q \\ q_2 q \\ \vdots \\ q_n q \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} a_{1i} q_i q \\ \sum_{i=1}^{n} a_{2i} q_i q \\ \vdots \\ \sum_{i=1}^{n} a_{mi} q_i q \end{bmatrix} \\ &= (Av)q = \mathcal{L}_A(v)q, \ \text{where } v = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \end{aligned}$$

(ii) r.$\mathcal{L}_{(Ap+Bq)} = ($r.$\mathcal{L}_A)p + ($r.$\mathcal{L}_B)q$, where A & B are $m \times n$ matrices over $\mathbb{H}$ and $p,\ q \in \mathbb{H}$.

(iii) If $m = n$ then r.$\mathcal{L}_{I_n} = I_{\mathbb{H}^n}$.

Further, if β and γ are the standard ordered right orthonormal bases for $\mathbb{H}^n$ and $\mathbb{H}^m$, respectively then $[\mathcal{L}_A]_{\beta\to\gamma} = A$.

Theorem 3.2. *The right rank of a matrix is the dimension of the right subspace generated by its columns.*

Proof. For any $A \in M_{m\times n}(\mathbb{H})$,

$$\text{right rank}(A) = \text{rank}(\mathcal{L}_A) = \dim\,(\text{ran}(\mathcal{L}_A))$$

Let $\beta = \{e_1, e_2, \cdots, e_n\}$ be the standard ordered right orthonormal bases for $\mathbb{H}^n$. Then right span β is $\mathbb{H}^n$. Hence

$$\begin{aligned}\text{ran }(\mathcal{L}_A) &= \text{right-span}(\mathcal{L}_A(\beta))\\ &= \text{right-span}(\{\mathcal{L}_A e_1, \mathcal{L}_A e_2, \cdots, \mathcal{L}_A e_n\})\end{aligned}$$

Also, for any j,

$$\mathcal{L}_A(e_j) = Ae_j = A_j,$$

where A_j is the j^{th} column of A. Hence

$$\text{rank}(A) = \dim(\text{right-span}\{A_1, A_2, \cdots, A_n\})$$

□

In view of above theorem, if A is a $m \times n$ matrix over $\mathbb{H}$, then we have following:

- The right rank is the dimension of the subspace (right) spanned by the columns of A as a subset of $m-$dimensional right quaternionic Hilbert space over $\mathbb{H}$.
- The right row rank is the dimension of the subspace (right) spanned by the rows of A as a subset of $n-$dimensional right quaternionic Hilbert space over $\mathbb{H}$.

On the similar lines, if we consider $\mathbb{H}$ as a left quaternionic Hilbert space then two more other types of numbers (ranks) we can associate with A:

- The left column rank is the dimension of the subspace (left) spanned by the columns of A as a subset of $n-$dimensional left quaternionic Hilbert space over $\mathbb{H}$, denoted it as lc.$\rho(A)$.
- The left row rank is the dimension of the subspace (left) spanned by the rows of A as a subset of $m-$dimensional left quaternionic Hilbert space over $\mathbb{H}$, denoted it as lr.$\rho(A)$.

Since the columns of A are the rows of its transpose A^T, therefore we have

$$\begin{aligned}\text{rc}.\rho(A) &= \text{rr}.\rho(A^T)\\ \text{lc}.\rho(A) &= \text{lr}.\rho(A^T)\end{aligned}$$

Further, one may observe that

$$\begin{aligned}\text{lc}.\rho(A) &= \text{rr}.\rho(A)\\ \text{rc}.\rho(A) &= \text{lr}.\rho(A^T)\end{aligned}$$

If $\{v_k\}_{k=1}^m$ is a frame for $\mathbb{H}^n$, then the matrix of pre-frame operator $T : \mathbb{H}^m \to \mathbb{H}^n$ (there is an ambiguity of notation here, $\mathbb{H}^m$ is a right vector space, but $\mathbb{H}^n$ is a left vector space when considered as the set of linear transformations $T : \mathbb{H}^m \to \mathbb{H}^n$) with respect to the canonical bases β in $\mathbb{H}^n$ and γ in $\mathbb{H}^m$ is given by

$$[T]_{\beta\to\gamma} = \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \dots & v_m \\ | & | & | & & | \end{bmatrix}$$

Since minimum m vectors are required to span an $m-$dimensional quaternion Hilbert space, therefore we must have $m \geq n$. Thus, in case $\{v_k\}_{k=1}^m$ is a frame for $\mathbb{H}^n$ then the matrix $[T]_{\beta\to\gamma}$ has at least as many columns as rows. Further, we have a following result

Theorem 3.3. *Let $\{v_k\}_{k=1}^m$ be a frame for $\mathbb{H}^n$ with pre-frame operator T. Then the followings hold:*

(i) Each vector v_k can be considered as the first n coordinate of some vectors w_k in $\mathbb{H}^m$ constituting a left basis for $\mathbb{H}^m$.

(ii) If $\{v_k\}_{k=1}^m$ is a tight frame, then each vector v_k is the first n coordinate of some vector w_k in $\mathbb{H}^m$ constituting an orthogonal basis for $\mathbb{H}^m$.

Proof. Let $T^* : \mathbb{H}^n \to \mathbb{H}^m$ be the adjoint of pre-frame operator T. Then

$$T^*(q) = \{\langle v_k|q\rangle\}_{k=1}^m, q \in \mathbb{H}^n.$$

The matrix for T^* with respect to the canonical bases is the $m \times n$ matrix, where the k^{th} row is the quaternionic conjugate of v_k, i.e.,

$$[T^*]_{\gamma\to\beta} = \begin{bmatrix} \text{———} & \overline{v_1} & \text{———} \\ \text{———} & \overline{v_2} & \text{———} \\ & \vdots & \\ \text{———} & \overline{v_m} & \text{———} \end{bmatrix}$$

If $T^*(q) = 0$, then $0 = \|T^*(q)\| = \sum_{k=1}^m |\langle v_k|q\rangle|^2$. Since right-span $\{v_k\}_{k=1}^m = \mathbb{H}^n$, therefore $q = 0$. Thus T^* is an injective map. Extending T^* to a bijection $\mathcal{T}$ of $\mathbb{H}^m$ to $\mathbb{H}^m$ by the setting

$$\mathcal{T}(e_k) = v'_k, \quad k = n+1,\ n+2, \cdots, m,$$

where $\{v'_k\}$ is a basis for the orthogonal complement of ran T^* in $\mathbb{H}^m$. Therefore, the matrix for $\mathcal{T}$ is a $m \times m$ matrix, whose first n columns are of T^*:

$$[T^*]_{\gamma\to\beta} = \left[\begin{array}{ccc|ccc} \text{———} & \overline{v_1} & \text{———} & | & & | \\ & \vdots & & v'_{n+1} & \cdots & v'_m \\ \text{———} & \overline{v_m} & \text{———} & | & & | \end{array}\right]$$

Since $\mathcal{T}$ is a surjective map, the columns right span $\mathbb{H}^m$. Therefore, the rows in $[T^*]_{\gamma\to\beta}$ are left linearly independent and so they are hence constitute a left basis of $\mathbb{H}^m$. □

4 Frame bounds and Frame Algorithm

We begin this section, by giving a relationship between the frame elements and frame bounds.

Theorem 4.1. *Let $\{v_i\}_{i=1}^m$ be a frame for finite dimensional right quaternionic Hilbert space $\mathbb{H}^n$ with frame bounds A and B. Then, $\|v_i\|^2 \le B$ for all $1 \le i \le m$ and if $\|v_i\|^2 = B$ holds for some i, then $v_i \perp$ span $\{v_j\}_{j\neq i}$. If $\|v_i\|^2 < A$, then $v_i \in$ span $\{v_j\}_{j\neq i}$.*

Proof. In particular, for each $1 \le i \le m$, taking v_i in place of v in frame inequality (2.1), we have

$$A\|v_i\|^2 \le \|v_i\|^4 + \sum_{\substack{j=1 \\ j\neq i}}^m |\langle v_j|v_i\rangle|^2 \le B\|v_i\|^2, \quad \text{for all } v \in \mathbb{H}^n. \tag{4.1}$$

Then, we have $\|v_i\|^2 \le B$ for all $1 \le i \le m$. Further, let $\|v_i\|^2 = B$ holds for some i, then we have $\sum_{j=1,\ j\neq i}^m |\langle v_j|v_i\rangle|^2 = 0$. This gives $v_i \perp \text{span}\{v_j\}_{j\neq i}$. Assume to the contrary that $W = \overline{\text{span}}\{v_j\}_{j\neq i}$ is a proper subspace of $\mathbb{H}^n$. Now, replacing v_i in the inequality (4.1) by $P_{W^\perp}(v_i)$ and using the left hand side of the inequality we obtain a contradiction. □

Theorem 4.2. *Let $\{v_i\}_{i=1}^m$ be a frame for finite dimensional right quaternionic Hilbert space $\mathbb{H}^n$ with frame operator S and $v \in \mathbb{H}^n$. If $\{q_i\}_{i=1}^m \subset \mathbb{H}$ is a sequence such that*

$$v = \sum_{i=1} v_i q_i$$

then

$$\sum_{i=1}^m |q_i|^2 = \sum_{i=1}^m |\langle v_i|S^{-1}v\rangle|^2 + \sum_{i=1}^m |\langle v_i|S^{-1}v\rangle - q_i|^2.$$

Proof. As

$$\sum_{i=1}^{m} v_i \langle v_i | S^{-1} v \rangle = v = \sum_{i=1}^{m} v_i q_i.$$

This gives $\sum_{i=1}^{\infty} v_i(\langle v_i|S^{-1}v\rangle - q_i) = 0$. Therefore, $\langle v_i|S^{-1}v\rangle - q_i = \ker T^* = (\text{ran } T)^{\perp}$, where T is the analysis operator of $\{v_i\}_{i=1}^{m}$. Therefore

$$q_i = \langle v_i|S^{-1}v\rangle - (\langle v_i|S^{-1}v\rangle - q_i).$$

Since the sequence $\{\langle v_i|S^{-1}v\rangle\} \in \text{ran } T$, and therefore orthogonal to $\{\langle v_i|S^{-1}v\rangle - q_i\}$, so the result holds. □

Next, we investigate few properties of the frame operator of a frame in quaternionic Hilbert space. If T is a linear transformation on the real vector space $\mathbb{H}^n$. Then,

$$T(q_1\xi_1 + q_2\xi_2) = q_1 T(\xi_1) + q_2 T(\xi_2), \;\; q_1, \; q_2 \in \mathbb{R} \text{ and } \xi_1, \xi_2 \in \mathbb{H}^n. \tag{4.2}$$

However, if we take right quaternionic Hilbert space $\mathbb{H}$ in place of $\mathbb{R}$ then it is non-commutative, therefore equation (4.2) will not hold in general. Thus, the analogue definitions of eigenvalues and eigenvectors in the quaternion context is as follows:

Definition 4.3. Let $T_R : \mathbb{H}^n \to \mathbb{H}^n$ be a right linear operator on right quaternionic Hilbert space $\mathbb{H}^n(\mathbb{H})$. Then, a non zero vector $v \in \mathbb{H}^n$ is said to be an right eigenvector of T_R with right eigenvalue q, if

$$T_R(v) = vq. \tag{4.3}$$

In other words, any solution $(q, v) \in \mathbb{H} \times (\mathbb{H}_n \backslash \{0\})$ to (4.3) is called a right eigenpair for T_R. Further, if T_R has a non-real right eigenvalue, then T_R has infinitely many non-real right eigenvalues. Infact, If (q, v) is a right eigenpair for T_R, then $(r^{-1}qr, vr)$ is also a right eigenpair for T_R, for all nonzero $r \in \mathbb{H}$. Therefore, one should consider the whole (similarity) orbit $\sigma(q)$ of $q \in \mathbb{H}$, $\sigma(q) = \{r^{-1}qr : r \in \mathbb{H}, \; r \neq 0\}$. The orbit of q is a singleton if and only if $q \in \mathbb{R}$. In all other cases, $\sigma(q)$ contains infinitely many elements. For further details one may refer to [7].

Definition 4.4. Let $T_R : \mathbb{H}^n \to \mathbb{H}^n$ be an invertible positive right linear operator on right quaternionic Hilbert space $\mathbb{H}^n(\mathbb{H})$ with right eigenvalues $q_1 \geq q_2 \geq q_3 \cdots \geq q_n$. Then its (right) conditional number is defined by $\dfrac{q_1}{q_n}$.

In the next result we show that the largest and the smallest eigenvalue of frame operator corresponds to optimal frame bounds.

Theorem 4.5. *Let $\{v_i\}_{i=1}^{m}$ be a frame for the right quaternionic Hilbert space $\mathbb{H}^n$ with the frame operator S_R with the right eigenvalues $q_1 \geq q_2 \geq q_3 \cdots \geq q_n$. Then q_1 coincides with the optimal upper frame bound and q_n coincides with the optimal lower frame bound.*

Proof. Let $\{e_i\}_{i=1}^{n}$ be the right eigenvectors of the frame operator S_R with respect to right eigenvalues $\{\lambda_i\}_{i=1}^{n}$ expressed in the decreasing order. As for each $x \in \mathbb{H}^n$, $x = \sum_{j=1}^{n} e_j \langle e_j|x\rangle$, therefore

$$\begin{aligned} S_R(x) &= \sum_{j=1}^{n} S_R(e_j)\langle e_j|x\rangle \\ &= \sum_{j=1}^{n} e_j q_j \langle e_j|x\rangle, \; x \in \mathbb{H}^n. \end{aligned}$$

This gives

$$\begin{aligned}
\sum_{j=1}^{m}|\langle v_i|x\rangle|^2 &= \langle S_R(x)|x\rangle \\
&= \left\langle \sum_{j=1}^{n} e_j q_j \langle e_j|x\rangle, \sum_{j=1}^{n} e_j \langle e_j|x\rangle \right\rangle \\
&= \sum_{j=1}^{n} q_j |\langle e_j|x\rangle|^2 \\
&\leq q_1 \sum_{j=1}^{n} |e_j \langle e_j|x\rangle|^2 \\
&= q_1 \|x\|^2.
\end{aligned}$$

Thus $B_{op} \leq q_1$, where B_{op} denotes the optimal upper bound of the frame $\{v_i\}_{i=1}^{m}$. Also

$$\begin{aligned}
\sum_{j=1}^{m}|\langle v_j|e_1\rangle|^2 &= \langle S_R(e_1)|e_1\rangle \\
&= \langle e_1 q_1|e_1\rangle \\
&= \overline{q_1}\langle e_1|e_1\rangle \\
&= q_1.
\end{aligned}$$

Similarly, we can show for lower bound. □

Theorem 4.6. *Let $\{v_i\}_{i=1}^{m}$ be a frame for the right quaternionic Hilbert space $\mathbb{H}^n$ with the frame operator S_R and $\{q_i\}_{i=1}^{n}$ denotes the right eigenvalues for S_R. If each eigenvalue appears in the list corresponding to algebraic multiplicity. Then*

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \|v_i\|^2.$$

Proof. Let $\{e_i\}_{i=1}^{n}$ be the right orthonormal basis of right eigenvectors such that

$$S_R(e_i) = e_i q_i, \ i = 1, 2, \cdots, n.$$

Then, we have

$$\begin{aligned}
\sum_{i=1}^{n} q_i &= \sum_{i=1}^{n} q_i \|e_i\|^2 \\
&= \sum_{i=1}^{n} q_i \langle e_i|e_i\rangle \\
&= \sum_{i=1}^{n} \langle e_i|S_R(e_i)\rangle \\
&= \sum_{i=1}^{n} \left\langle e_i \middle| \sum_{j=1}^{m} v_j \langle v_j|e_i\rangle \right\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle e_i|v_j\rangle\langle v_j|e_i\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} |\langle e_i|v_j\rangle|^2 = \sum_{j=1}^{m} \|v_j\|^2.
\end{aligned}$$

In view of above theorems, we have a following corollary:

Corollary 4.7. *Let $\{v_i\}_{i=1}^m$ be a normalized tight frame for a n dimensional right quaternionic Hilbert space $V_R(\mathbb{H})$. Then the frame bound is $\frac{m}{n}$.*

Proof. As the set of right eigenvalues $\{q_i\}_{i=1}^n$ consists of the frame bound A repeated n times, thus the result follows. □

If we want to find an element $v \in V_R(\mathbb{H})$ in terms of fram cofficients then we have a reconstruction formula

$$\begin{aligned} v &= \sum_{i=1}^m S_R^{-1} v_i \langle v_i | v \rangle \\ &= S_R^{-1} T(\{\langle f, f_k \rangle\}_{k=1}^m) \end{aligned}$$

In case dimension $V_R(\mathbb{H})$ is large it would be complicated to invert frame operator. Therefore now we discuss an iterative method to derive a converging sequence of approximations of v from the knowledge of $\{\langle v_i|v\rangle\}_{i=1}^m$, called the frame algorithm.

Theorem 4.8. *Let $\{v_i\}_{i=1}^m$ be a frame for the right quaternionic Hilbert space $V_R(\mathbb{H})$ with frame operators A,B. Given $v \in V_R(\mathbb{H})$, define a sequence $\{g_k\}_{k=0}^\infty$ in $V_R(\mathbb{H})$ by*

$$g_0 = 0,\ g_k = g_{k-1} + \frac{2}{A+B} S(f - g_{k-1}),\ \ k \geq 1.$$

Then

$$\|f - g_k\| \leq \left(\frac{B-A}{B+A}\right)^k \|f\|.$$

Proof. Let I denotes the identity operator on V and S be the frame operator of the frame $\{v_i\}_{i=1}^m$. Then

$$\begin{aligned} \langle Sv|v\rangle &= \left\langle \sum_{i=1}^m v_i \langle v_i|v\rangle \middle| v \right\rangle \\ &= \sum_{i=1}^m |\langle v_i|v\rangle|^2. \end{aligned}$$

This gives

$$\begin{aligned} \left\langle \left(I - \frac{2}{A+B} S\right) v \middle| v \right\rangle &= \left\langle v - \frac{2}{A+B} Sv \middle| v \right\rangle \\ &= \|v\|^2 - \frac{2}{A+B} \sum_{i=1}^m |\langle v_i|v\rangle|^2 \\ &\leq \frac{B-A}{B+A} \|v\|^2. \end{aligned}$$

Similarly, we have

$$-\frac{B-A}{B+A}\|v\|^2 \leq \left\langle \left(I - \frac{2}{A+B} S\right) v \middle| v \right\rangle.$$

As $\left(I - \frac{2}{A+B} S\right)$ is a self-adjoint operator, therefore

$$\left\| I - \frac{2}{A+B} S \right\| \leq \frac{B-A}{B+A}.$$

Using the definition of $\{g_k\}_{k=0}^{\infty}$, we have

$$f - g_k = f - g_{k-1} - \frac{2}{A+B}\, S(f - g_{k-1}) = \left(I - \frac{2}{A+B} S\right)(f - g_{k-1}).$$

Therefore, by repeated use of above argument, we have

$$f - g_k = \left(I - \frac{2}{A+B} S\right)^k (f - g_0).$$

Thus

$$\begin{aligned} \|f_g - k\| &= \left\| \left(I - \frac{2}{A+B} S\right)^k (f - g_0). \right\| \\ &\leq \left\| I - \frac{2}{A+B} S \right\|^k \|f - g_0\| \\ &\leq \left(\frac{B-A}{B+A}\right)^k \|f\|. \end{aligned}$$

□

Definition 4.9. Let $T : \mathbb{H}^n \to \mathbb{H}^n$ be a right linear operator. Then, the trace (right) of T is given by

$$\text{Tr}_R\, T = \sum_{i=1}^{n} \langle e_i | T(e_i) \rangle,$$

where $\{e_i\}_{i=1}^{n}$ is an arbitrary right orthonormal basis for $\mathbb{H}^n$.

In the next, we give a relation between the frame vectors, eigenvalues and eigenvectors of the associated frame operator.

Theorem 4.10. *Let $\{v_i\}_{i=1}^{m}$ be a frame for the right quaternionic Hilbert space $\mathbb{H}^n$ with the frame operator S_R having normalized right eigenvectors $\{e_i\}_{i=1}^{n}$ and respective right eigenvalues $\{q_i\}_{i=1}^{n}$. Then*

$$q_j = \sum_{i=1}^{m} |\langle v_i | e_j \rangle|^2.$$

In particular,

$$Tr_R(S_R) = \sum_{i=1}^{n} q_i = \sum_{i=1}^{m} \|v_i\|^2.$$

Proof. As $q_i = \langle e_i | S_R(e_i) \rangle$ for all $i = 1, 2, \cdots, n$ and for each $x \in \mathbb{H}^n$, we have

$$\langle x, S_R(x) \rangle = \sum_{i=1}^{m} |\langle v_i | x \rangle|^2.$$

Therefore result follows. Further

$$\begin{aligned} \text{Tr}_R\, S_R &= \sum_{i=1}^{n} \langle e_i | S_R(e_i) \rangle \\ &= \sum_{i=1}^{n} q_i = \sum_{i=1}^{m} \|v_i\|^2. \end{aligned}$$

Theorem 4.11. *If $\{v_n\}_{n=1}^{m}$ be a frame for $\mathbb{H}^n$ with frame operator S having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. Then*

(a) $\sum\limits_{i=1}^{n} |\langle v_i | e_n \rangle|^2 = \lambda_j$, *for all* $1 \leq j \leq n$.

(b) $\sum_{i=1}^{m} \overline{\langle v_i|e_r\rangle}\langle v_i|e_j\rangle = 0$, *for all* $1 \leq r \neq j \leq n$.

where $\{e_i\}_{i=1}^{n}$ *is the standard orthonormal basis of* $\mathbb{H}^n$.

Proof. **(a)** For each $j = 1, 2, \cdots n$, we have

$$\begin{aligned} \lambda_j &= \langle e_j|\lambda_j e_j\rangle \\ &= \langle e_j|S(e_j)\rangle \\ &= \left\langle e_j \middle| \sum_{i=1}^{m} v_i\langle v_i|e_j\rangle \right\rangle \\ &= \sum_{i=1}^{m}\langle e_j|v_i\rangle\langle v_i|e_j\rangle = \sum_{i=1}^{m}|\langle v_i|e_j\rangle|^2. \end{aligned}$$

(b) As for $1 \leq r \neq j \leq n$, we have

$$\begin{aligned} & \langle e_r|S(e_j)\rangle = 0 \\ \Rightarrow\quad & \left\langle e_r \middle| \sum_{i=1}^{m} v_i\langle v_i|e_j\rangle \right\rangle = 0 \\ \Rightarrow\quad & \sum_{i=1}^{m}\langle e_r|v_i\rangle\langle v_i|e_j\rangle = 0 \\ \Rightarrow\quad & \sum_{i=1}^{m}\overline{\langle v_i|e_r\rangle}\langle v_i|e_j\rangle = 0. \end{aligned}$$

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