Approximation To The Smarandache Curves in the The Null Cone

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Abstract In this paper, we study the Smarandache curves according to the asymptotic orthonormal frame in Null Cone $Q^3$. By using cone frame formulas, we obtain some characterizations of the Smarandache curves and introduce cone frenet invariants of these curves.

1 Introduction

The idea of studying curves has been one of the impressive topic owing to having many application area from mathematics to the diverse branch of science. As a result of this case, many mathematicians have studied different type of curves by using Frenet frame in numerous spaces. Among these, Smarandache curves have attract major attention by investigators for a long while.

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom [4]. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache curve is defined as a regular curve whose position vector is composed by Frenet frame vectors of another regular curve. Smarandache curves in various ambient spaces have been classified in [1]-[8], [14]-[16].

In this study, we give special Smarandache curves such as $x\alpha, xy, x\beta, \alpha\beta, y\beta, \alpha y$ Smarandache curves according to asymptotic orthonormal frame in the Null Cone $Q^3$ and we examine the curvature and the asymptotic orthonormal frame’s vectors of the Smarandache curves. We also present an example related to these curves.

2 Preliminaries

Some basics of the curves in the null cone are provided from, [9]- [10]. Let $E^4_1$ be the 4-dimensional pseudo-Euclidean space with the

$$\tilde{g}(X, Y) = \langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

for all $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in E^4_1$. $E^4_1$ is a flat pseudo-Riemannian manifold of signature $(3, 1)$.

Let $M$ be a submanifold of $E^4_1$. If the pseudo-Riemannian metric $\tilde{g}$ of $E^4_1$ induces a pseudo-Riemannian metric $g$ (respectively, a Riemannian metric, a degenerate quadratic form) on $M$, then $M$ is called a timelike (respectively, spacelike, degenerate) submanifold of $E^3_1$. Let $c$ be a fixed point in $E^4_1$. The pseudo-Riemannian lightlike cone (quadric cone ) is defined by

$$Q^3_1(c) = \left\{ x \in E^4_1 : g(x - c, x - c) = 0 \right\},$$

where the point $c$ is called the center of $Q^3_1(c)$. When $c = 0$, we simply denote $Q^3_1(0)$ by $Q^3$ and call it the null cone.

Let $E^4_1$ be 4-dimensional Minkowski space and $Q^3$ be the lightlike cone in $E^4_1$. A vector $V \neq 0$ in $E^4_1$ is called spacelike, timelike or lightlike, if $\langle V, V \rangle > 0$, $\langle V, V \rangle < 0$ or $\langle V, V \rangle = 0$, respectively. The norm of a vector $x \in E^4_1$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$, [13].
We assume that curve \( x : I \rightarrow Q^3 \subset E^4_1 \) is a regular curve in \( Q^3 \) for \( t \in I \). In the following, we always assume that the curve is regular.

A frame field \( \{x, \alpha, \beta, y\} \) on \( E^4_1 \) is called an asymptotic orthonormal frame field, if

\[
\langle x, x \rangle = \langle y, y \rangle = \langle x, \alpha \rangle = \langle y, \alpha \rangle = \langle \beta, \alpha \rangle = \langle y, \beta \rangle = \langle x, \beta \rangle = 0,
\]

\[
\langle y, x \rangle = \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1.
\]

Using \( x'(s) = \alpha(s) \), we know that \( \{x(s), \alpha(s), \beta(s), y(s)\} \) from an asymptotic orthonormal frame along the curve \( x(s) \) and the cone frenet formulas of \( x(s) \) are given by

\[
x'(s) = \alpha(s) \\
\alpha'(s) = \kappa(s)x(s) - y(s) \\
\beta'(s) = \tau(s)x(s) \\
y'(s) = -\kappa(s)\alpha(s) - \tau(s)\beta(s),
\]

where the functions \( \kappa(s) \) and \( \tau(s) \) are called cone curvature functions of the curve \( x(s) \), [11].

Let \( x : I \rightarrow Q^3 \subset E^4_1 \) be a spacelike curve in \( Q^3 \) with an arc length parameter \( s \). Then \( x = x(s) = (x_1, x_2, x_3, x_4) \) can be written as

\[
x(s) = \frac{1}{2\sqrt{f^2 + g^2}} (2f, 2g, 1 - f^2 - g^2, 1 + f^2 + g^2),
\]

for some non constant function \( f(s) \) and \( g(s) \), [12].

### 3 The Smarandache Curves in The Null Cone \( Q^3 \)

In this section, we define binary Smarandache curves according to the asymptotic orthonormal frame in \( Q^3 \). Also, we obtain the asymptotic orthonormal frame and cone curvature functions of the Smarandache partners lying on \( Q^3 \) using cone frenet formulas.

Smarandache curve \( \gamma = \gamma(s^\ast(s)) \) of the curve \( x \) is a regular unit speed curve lying fully on \( Q^3 \). Let \( \{x, \alpha, \beta, y\} \) and \( \{\gamma, \alpha_{\gamma}, \beta_{\gamma}, y_{\gamma}\} \) be the moving asymptotic orthonormal frames of \( x \) and \( \gamma \), respectively.

**Definition 3.1.** Let \( x \) be unit speed spacelike curve lying on \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, \( x_{\alpha-}\)smarandache curve of \( x \) is defined by

\[
\gamma_{x\alpha}(s^\ast) = \frac{a}{b} x(s) + \alpha(s),
\]

where \( a, b \in \mathbb{R}_0^+ \).

**Theorem 3.2.** Let \( x \) be unit speed spacelike curve in \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvatures \( \kappa(s) \), \( \tau(s) \) and let \( \gamma_{x\alpha} \) be \( x_{\alpha-}\)smarandache curve with asymptotic orthonormal frame \( \{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\} \) of the \( x_{\alpha-}\)smarandache curve \( \gamma_{x\alpha} \) is given as

\[
\begin{bmatrix}
\gamma_{x\alpha} \\
\alpha_{x\alpha} \\
\beta_{x\alpha} \\
y_{x\alpha}
\end{bmatrix} =
\begin{bmatrix}
\frac{a}{b} & \frac{1}{\sqrt{a^2 - 2b\kappa}} & 0 & 0 \\
\frac{1}{\sqrt{a^2 - 2b\kappa}} & \frac{a}{\sqrt{a^2 - 2b\kappa}} & 0 & \frac{-b}{\sqrt{a^2 - 2b\kappa}} \\
B_1 & B_2 & B_3 & B_4 \\
Y_1 & Y_2 & Y_3 & Y_4
\end{bmatrix}
\begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix},
\]

where

\[
\xi = \frac{1}{\sqrt{a^2 - 2b\kappa}}, w = \frac{1}{b} \sqrt{a^2 - 2b^2\kappa},
\]

\[
B_1 = \frac{1}{w} (a\xi + b\kappa\xi + b\kappa\xi'), B_2 = \frac{1}{w} ((a + b\kappa)\xi' + (\kappa' + \kappa)b\xi),
\]

\[
B_3 = \frac{1}{w} (b\xi\tau), B_4 = -\frac{1}{w} (a\xi + b\xi').
\]
and

\[
\begin{align*}
\mathcal{Y}_1 &= -(B_1 + \frac{a}{2b}(2B_1B_4 + B_2^2 + B_3^2)), \\
\mathcal{Y}_2 &= -(B_2 + \frac{1}{2}(2B_1B_4 + B_2^2 + B_3^2)), \\
\mathcal{Y}_3 &= -B_3, \mathcal{Y}_4 = -B_4.
\end{align*}
\] (3.5)

ii) The cone curvatures \( \kappa_{\gamma_{x_0}}(s^*) \) and \( \tau_{\gamma_{x_0}}(s^*) \) of the curve \( \gamma_{x_0} \) is given by

\[
\kappa_{\gamma_{x_0}}(s^*) = -\frac{1}{2}(2B_1B_4 + B_2^2 + B_3^2)
\]

\[
\tau_{\gamma_{x_0}}(s^*) = \sqrt{2(\mathcal{Y}_1 - \kappa')\mathcal{Y}_4 + (\mathcal{Y}_2 - \kappa)^2 + \mathcal{Y}_3^2 - \kappa_{\gamma_{x_0}}^2},
\] (3.6)

where

\[
s^* = \frac{1}{b} \int \sqrt{a^2 - 2b^2\kappa(s)}ds.
\]

**Proof.** i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( [x, \alpha, \beta, y] \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.1) with respect to \( s \) and considering (2.1), we have

\[
\gamma_{x_0}'(s^*) = (a\xi)\alpha(s) + (bs\xi)x(s) + (-b\xi)y(s),
\] (3.7)

where

\[
\frac{ds^*}{ds} = \frac{1}{b} \sqrt{a^2 - 2b^2\kappa(s)},
\]

\[
\xi = \frac{1}{\sqrt{a^2 - 2\kappa(s)b^2}}.
\] (3.8) (3.9)

It can be easily seen that the tangent vector \( \gamma_{x_0}'(s^*) = \alpha_{x_0}(s^*) \) is a unit spacelike vector.

Differentiating (3.7), we obtain equation as follows

\[
\gamma_{x_0}''(s^*) = B_1x(s) + B_2\alpha(s) + B_3\beta(s) + B_4y(s),
\] (3.10)

where

\[
B_1 = \frac{1}{w}(a\xi + b\kappa'\xi + b\kappa\xi'), B_2 = \frac{1}{w}((a + b\kappa)\xi' + (\kappa' + \kappa)b\xi),
\]

\[
B_3 = \frac{1}{w}(b\xi\tau), B_4 = -\frac{1}{w}(a\xi + b\xi').
\]

\[
y_{x_0}(s^*) = -\gamma_{x_0}'' - \frac{1}{2}(\gamma_{x_0}'', \gamma_{x_0}') \gamma_{x_0}.
\] (3.11)

By the help of previous equation (3.11), we obtain

\[
y_{x_0}(s^*) = \mathcal{Y}_1x(s) + \mathcal{Y}_2\alpha(s) + \mathcal{Y}_3\beta(s) + \mathcal{Y}_4y(s),
\] (3.12)

where \( \mathcal{Y}_1 = -(B_1 + \frac{a}{2b}(2B_1B_4 + B_2^2 + B_3^2)), \)

\( \mathcal{Y}_2 = -(B_2 + \frac{1}{2}(2B_1B_4 + B_2^2 + B_3^2)), \)

\( \mathcal{Y}_3 = -B_3, \mathcal{Y}_4 = -B_4. \)

ii) Using equations \( \kappa_{\gamma_{x_0}}(s^*) = -\frac{1}{2}(\gamma_{x_0}'', \gamma_{x_0}') \) and \( \tau_{\gamma_{x_0}}^2(s^*) = (x''\kappa - \kappa'x, x''\kappa - \kappa'x) - \kappa_{\gamma_{x_0}}^2(s^*) \). The curvatures \( \kappa_{\gamma_{x_0}}(s^*) \) and \( \tau_{\gamma_{x_0}}(s^*) \) of the \( \gamma_{x_0}(s^*) \) are explicitly obtained by

\[
\kappa_{\gamma_{x_0}}(s^*) = -\frac{1}{2}(2B_1B_4 + B_2^2 + B_3^2)
\]

\[
\tau_{\gamma_{x_0}}^2(s^*) = 2(\mathcal{Y}_1 - \kappa')\mathcal{Y}_4 + (\mathcal{Y}_2 - \kappa)^2 + \mathcal{Y}_3^2 - \kappa_{\gamma_{x_0}}^2.
\] (3.13)

Thus, the theorem is proved. \( \square \)
Definition 3.3. Let \( x \) be unit speed spacelike curve lying on \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, the \( xy \)-smarandache curve of \( x \) is defined by
\[
\gamma_{xy}(s^*) = \frac{1}{\sqrt{2ab}} (ax(s) + by(s)),
\] (3.14)
where \( a, b \in \mathbb{R}_+^* \).

Theorem 3.4. Let \( x \) be unit speed spacelike curve in \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa \) and let \( \gamma_{xy} \) be \( xy \)-smarandache curve with asymptotic orthonormal frame \( \{\gamma_{xy}, \alpha_{xy}, \beta_{xy}, y_{xy}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{xy}, \alpha_{xy}, \beta_{xy}, y_{xy}\} \) of the \( xy \)-smarandache curve \( \gamma_{xy} \) is given by
\[
\begin{bmatrix}
\gamma_{xy} \\
\alpha_{xy} \\
\beta_{xy} \\
y_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{a}{\sqrt{2ab}} & 0 & 0 & \frac{b}{\sqrt{2ab}} \\
0 & \eta_1 & \eta_2 & 0 \\
\frac{-(\eta_1 \kappa + \eta_2 \tau)}{w} & \frac{\eta_1'}{w} & \frac{\eta_2'}{w} & -\frac{\eta_1}{w} \\
\frac{-aC}{2\sqrt{2ab}} & -\frac{\eta_1'}{w} & -\frac{\eta_2'}{w} & -\frac{aC}{2\sqrt{2ab}}
\end{bmatrix}
\begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix},
\] (3.15)
where
\[
\eta_1 = \frac{a - b\kappa}{w\sqrt{2ab}}, \quad \eta_2 = \frac{-b\tau}{w\sqrt{2ab}},
\]
\[
w = \frac{ds^*}{ds} = \sqrt{\frac{a}{2b} - \kappa + \frac{b}{2a}(\kappa^2 + \tau^2)},
\]
\[
C = \frac{1}{w} \left(-2\eta_1(\eta_1 \kappa + \eta_2 \tau) + (\eta_1')^2 + (\eta_2')^2\right).
\]

ii) The cone curvature \( \kappa_{\gamma_{xy}}(s^*) \) and \( \tau_{\gamma_{xy}}(s^*) \) of the curve \( \gamma_{xy} \) is given by
\[
\kappa_{\gamma_{xy}}(s^*) = -\frac{C^2}{2},
\]
\[
\tau_{\gamma_{xy}}(s^*) = 2(\eta_1 \kappa + \eta_2 \tau) + \frac{aC}{2\sqrt{2ab}} - \kappa' \left(\frac{bC}{2\sqrt{2ab}} - \eta_1\right) + (\eta_1')^2 + (\eta_2')^2 - \frac{C^2}{4},
\] (3.16)
where
\[
s^* = \int \sqrt{\frac{a}{2b} - \kappa + \frac{b}{2a}(\kappa^2 + \tau^2)} ds.
\] (3.17)

Proof. i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.14) with respect to \( s \) and considering (2.1), we have
\[
\gamma_{xy}'(s^*) \frac{ds^*}{ds} = \frac{(a - b\kappa(s))}{\sqrt{2ab}} \alpha(s) - \frac{b\tau}{\sqrt{2ab}} \beta(s),
\] (3.18)
or
\[
\gamma_{xy}'(s^*) = \eta_1 \alpha + \eta_2 \beta.
\]

By considering (3.17), we get
\[
\gamma_{xy}'(s^*) = \alpha(s) = \alpha_{xy}.
\] (3.19)

Here, it can be easily seen that the tangent vector \( \alpha_{xy} \) is a unit spacelike vector. Differentiating (3.19) and using (3.17), we obtain
\[
\gamma_{xy}''(s^*) = \left(\frac{(\eta_1 \kappa + \eta_2 \tau)}{w}\right) x(s) + \frac{\eta_1'}{w} \alpha + \frac{\eta_2'}{w} \beta - \frac{\eta_1}{w} y(s).
\] (3.20)
By the help of equation $y_{xy}(s^*) = -\gamma''_{xy} - \frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle \gamma_{xy}$, we write
\[
y_{xy}(s^*) = \left( -\frac{\eta_2 - \eta_2 \gamma}{w} - \frac{aC}{2\sqrt{2}ab} \right) x(s) - \frac{\eta_1 - \eta_2 \beta}{w} + \left( \frac{\eta_1}{w} - \frac{aC}{2\sqrt{2}ab} \right) y(s). \tag{3.21}
\]

ii)
\[
\kappa_{\tau y}(s^*) = -\frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle,
\]
\[
\tau_{\tau y}^2(s^*) = (\beta - \kappa \alpha - \kappa' x, \beta - \kappa \alpha - \kappa' x) - \kappa^2_{\tau y}.
\tag{3.22}
\]

By using (3.22), the curvatures $\kappa_{\tau y}(s^*)$ and $\tau_{\tau y}(s^*)$ of the $\gamma_{xy}(s^*)$ are explicitly obtained
\[
\kappa_{\tau y}(s^*) = \frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle = -\frac{C}{2},
\]
\[
\tau_{\tau y}^2(s^*) = 2(\eta_1 + \eta_2 \tau + \frac{aC}{2\sqrt{2}ab} - \kappa') \left( \frac{bC}{2\sqrt{2}ab} - \eta_1 \right) + (\eta_1' + \kappa')^2 + (\eta_2')^2 + \frac{C^2}{4}.
\]

\[\square\]

**Definition 3.5.** Let $x$ be unit speed spacelike curve lying on $Q^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, $\alpha y$-smarandache curve of $x$ is defined by
\[
\gamma_{\alpha y}(s^*) = \alpha(s) + \frac{b}{a} y(s), \tag{3.23}
\]
where $a, b \in \mathbb{R}_+^\times$.

**Theorem 3.6.** Let $x$ be unit speed spacelike curve in $Q^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature $\kappa$ and let $\gamma_{\alpha y}$ be $\alpha y$-smarandache curve with asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\}$. Then the following relations hold:

i) The asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\}$ of the $\alpha y$-smarandache curve $\gamma_{\alpha y}$ is given as
\[
\begin{bmatrix}
\gamma_{\alpha y} \\
\alpha_{\alpha y} \\
\beta_{\alpha y} \\
y_{\alpha y}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & \frac{b}{a} \\
\rho_1 & \rho_2 & \rho_3 & -\frac{\rho_3}{M} \\
-\rho_1 & -\rho_2 & \rho_4 & \frac{\rho_4}{M} \\
\frac{M}{\sqrt{2}} & \frac{M}{\sqrt{2}} & \frac{M}{\sqrt{2}} & \frac{D}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix}, \tag{3.24}
\]
where
\[
\rho_1 = \frac{\kappa}{M}, \rho_2 = -\frac{b}{a} \left( \frac{\kappa}{M} \right), \rho_3 = -\frac{b}{a} \left( \frac{\tau}{M} \right), \rho_4 = \frac{1}{M},
\]
\[
M = \sqrt{\frac{b}{a^2}(\kappa^2 + \tau^2) - 2}\kappa.
\tag{3.25}
\]
\[
D = \frac{2}{M^2} \left( \rho_1^2 + \kappa \rho_2 + \rho_2 \tau \right) (-\rho_2 + \rho_4) + \frac{1}{M^2} \left( (\rho_2^2 + \rho_4^2) - (\rho_4 - \tau \rho_1)^2 \right).
\]

ii) The cone curvatures $\kappa_{\alpha y}(s^*)$ and $\tau_{\alpha y}(s^*)$ of the curve $\gamma_{\alpha y}$ is given by
\[
\kappa_{\alpha y}(s^*) = -\frac{D}{2}, \tag{3.26}
\]
\[
\tau_{\alpha y}^2(s^*) = 2\left( \frac{\rho_1^2 + \kappa \rho_2 + \rho_2 \tau}{M} - \kappa' \right) (-\rho_2 + \rho_4) + \frac{bD}{a} + \frac{D}{2} + \kappa)^2 - \left( \frac{\rho_4 - \tau \rho_1}{M} \right)^2 - \frac{D^2}{4}, \tag{3.27}
\]
where
\[
s^* = \int \sqrt{\frac{b}{a^2}(\kappa^2 + \tau^2) - 2\kappa ds}. \tag{3.28}
\]
Proof. i) Let the curve $x$ be a unit speed spacelike curve with the asymptotic orthonormal frame \{x, \alpha, \beta, y\} and cone curvature $\kappa, \tau$. Differentiating the equation (3.23) with respect to $s$ and considering (2.1), we find

$$\gamma'_\alpha(y)(s^*) \frac{ds^*}{ds} = \frac{\kappa}{a}(x(s)) - \frac{b}{c} \alpha(s) - \frac{b}{a} \tau(s) = y(s).$$

This can be written as following

$$\alpha_{\alpha\gamma}(s^*) \frac{ds^*}{ds} = \frac{\kappa}{M}(x(s)) - \frac{b}{c} \alpha(s) - \frac{b}{a} \tau(s) = -\frac{1}{M}(y(s)), \quad \text{(3.29)}$$

where

$$M = \frac{ds^*}{ds} = \sqrt{\frac{b}{a^2}(\kappa^2 + \tau^2) - 2\kappa}.$$ \hspace{1cm} \text{(3.30)}

Differentiating (3.29) and using (3.30), we get

$$\gamma''_{\alpha\gamma} = \left(\frac{\rho_1 + \kappa \rho_2 + \rho_3 \tau}{M} \right) x + \left(\frac{\rho_4 + \rho_1 - \kappa \rho_4}{M} \right) \alpha + \left(\frac{\rho_4' - \tau \rho_4}{M} \right) \beta + \left(\frac{-\rho_2 + \rho_4}{M} \right) y,$$ \hspace{1cm} \text{(3.31)}

where $\rho_1 = \frac{\kappa}{M}, \rho_2 = \frac{-b}{a} \left(\frac{\kappa}{M}\right), \rho_3 = \frac{-b}{a} \left(\frac{\tau}{M}\right), \rho_4 = \frac{1}{M}$.\\

By the help of equation (3.32), we obtain

$$y_{\alpha\gamma}(s^*) = -\gamma''_{\alpha\gamma} - \frac{1}{2} \langle \gamma''_{\alpha\gamma}, \gamma''_{\alpha\gamma} \rangle \gamma_{\alpha\gamma} \quad \text{and} \quad \langle \gamma''_{\alpha\gamma}, \gamma''_{\alpha\gamma} \rangle = D.$$ \hspace{1cm} \text{(3.32)}

ii) Using (3.22), we have (3.26) and (3.27). \hfill $\square$

**Definition 3.7.** Let $x$ be unit speed spacelike curve lying on $Q^3$ with the moving asymptotic orthonormal frame \{x, \alpha, \beta, y\}. Then, $x\beta$-smarandache curve of $x$ is defined by

$$\gamma_{x\beta}(s^*) = \frac{a}{b} x(s) + \beta(s),$$ \hspace{1cm} \text{(3.34)}

where $a, b \in \mathbb{R}^+_0$.

**Theorem 3.8.** Let $x$ be unit speed spacelike curve in $Q^3$ with the moving asymptotic orthonormal frame \{x, \alpha, \beta, y\} and cone curvature $\kappa$ and let $\gamma_{x\beta}$ be $x\beta$-smarandache curve with asymptotic orthonormal frame \{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\}. Then the following relations hold:

i) The asymptotic orthonormal frame \{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\} of the $x\beta$-smarandache curve $\gamma_{x\beta}$ is given as

$$\gamma_{x\beta} = \begin{pmatrix} \frac{a}{b} \\ \frac{\kappa}{a} \tau' \\ \frac{\kappa}{b} \tau' \\ -\frac{\kappa}{M} \end{pmatrix}, \quad \alpha_{x\beta} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_{x\beta} = \begin{pmatrix} \frac{b}{a} \kappa \\ \frac{b}{a} \tau' \\ \frac{a}{b} \tau' \\ 0 \end{pmatrix}, \quad y_{x\beta} = \begin{pmatrix} \frac{b}{a} \kappa \\ \frac{b}{a} \tau' \\ \frac{a}{b} \tau' \\ M \end{pmatrix}, \quad \text{(3.35)}$$

where

$$M = \frac{-b^4}{2a^3} \tau^2 + \frac{b^2}{a^2} \kappa + \frac{b^3}{a^3} \tau'.$$ \hspace{1cm} \text{(3.36)}

ii) The cone curvatures $\kappa_{x\beta}(s^*)$ and $\tau_{x\beta}(s^*)$ of the curve $\gamma_{x\beta}$ is given by

$$\kappa_{x\beta}(s^*) = \frac{b^2}{a^2} \left( \frac{b^2}{a^2} \tau^2 - 2\kappa - 2b \right),$$ \hspace{1cm} \text{(3.37)}

$$\tau_{x\beta}^2(s^*) = M^2 - 2\frac{b}{a} \kappa' + \kappa^2 - 6\frac{b^2}{a^2} \kappa + 4\frac{b^4}{a^4} - 2\frac{b^3}{a^3} \tau',$$ \hspace{1cm} \text{(3.38)}

where

$$s^* = \frac{a}{b} + A; \quad a, b, A \in \mathbb{R}^+_0.$$ \hspace{1cm} \text{(3.39)}
Proof. i) Differentiating the equation (3.44) with respect to \( s \) and considering (2.1), we find

\[
\gamma'_{x\beta}(s^*) \frac{ds^*}{ds} = \frac{a}{b} \alpha(s) + \tau x(s).
\]  

(3.40)

This can be written as follows

\[
\alpha_{x\beta}(s^*) = \frac{b}{a} \frac{\gamma^{'2}}{x(s)} + \alpha(s),
\]

(3.41)

where

\[
\frac{ds^*}{ds} = \frac{a}{b}.
\]

(3.42)

Differentiating (3.41) and using (3.42), we get

\[
\gamma''_{x\beta}(s^*) = \left( \frac{b}{a} \kappa + \left( \frac{b}{a} \right)^3 \tau x(s) + \left( \frac{b}{a} \right)^2 \alpha(s) - \frac{b}{a} y(s)
\]

\[
y_{x\beta}(s^*) = -\gamma''_{x\beta} - \frac{1}{2} \left( \gamma'_{x\beta}, \gamma'_{x\beta} \right) \gamma_{x\beta}.
\]

By the help of equation (3.43), we obtain

\[
y_{x\beta}(s^*) = (- \left( \frac{b}{a} \right)^3 \tau x(s) + \left( \frac{b}{a} \right)^2 \alpha(s) + M \beta(s) + \frac{b}{a} y(s),
\]

(3.44)

where \( M = -\frac{b^4}{2a^2} \tau^2 + \frac{b^2}{a} \kappa + \frac{b^3}{a} \tau' \).

ii) Using (3.22), we have (3.36) and (3.37). \( \square \)

Definition 3.9. Let \( x \) be unit speed spacelike curve lying on \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, \( \alpha \beta \)-smarandache curve of \( x \) is defined by

\[
\gamma_{\alpha\beta}(s^*) = \frac{1}{\sqrt{a^2 + b^2}} (a \alpha(s) + b \beta(s)),
\]

(3.45)

where \( a, b \in \mathbb{R}^+_0 \).

Theorem 3.10. Let \( x \) be unit speed spacelike curve in \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvatures \( \kappa \) and \( \tau \) let \( \gamma_{\alpha\beta} \) be \( \alpha \beta \)-smarandache curve with asymptotic orthonormal frame \( \{\gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta}\} \) of the \( \alpha \beta \)-smarandache curve \( \gamma_{\alpha\beta} \) is given as

\[
\begin{bmatrix}
\gamma_{\alpha\beta} \\
\alpha_{\alpha\beta} \\
\beta_{\alpha\beta} \\
y_{\alpha\beta}
\end{bmatrix} = \begin{bmatrix}
0 & a \kappa + b \tau & 0 & b \\
0 & \sqrt{a^2 + b^2} & 0 & \sqrt{a^2 + b^2} \\
\frac{Y_1}{E} & \frac{Y_1 - \kappa Y_2}{E} & -\frac{Y_2}{E} & \frac{Y_1}{E} \\
\frac{-Y_1}{E} & \frac{-Y_1 - \kappa Y_2}{E} & \frac{Y_2}{E} & \frac{-Y_1}{E}
\end{bmatrix} \begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix},
\]

(3.46)

where

\[
E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2 + b^2}} \left| -a \left( a \kappa + b \tau \right) \right|,
\]

(3.47)

\[
Y_1 = \frac{a \kappa + b \tau}{E \sqrt{a^2 + b^2}}, Y_2 = \frac{-a}{E \sqrt{a^2 + b^2}},
\]

(3.48)

\[
L = \frac{1}{E^2} \left( 2Y_1 Y_2' + (Y_1 - \kappa Y_2)^2 + \tau^2 Y_2^2 \right).
\]

(3.49)

ii) The cone curvatures \( \kappa_{\alpha\beta}(s^*) \) and \( \tau_{\alpha\beta}(s^*) \) of the curve \( \gamma_{\alpha\beta} \) is given by

\[
\kappa_{\alpha\beta}(s^*) = -\frac{L}{2},
\]

(3.50)
\[ \tau^2_{\alpha\beta}(s^*) = 2\left(\frac{Y'_1}{E} + \kappa'\right)\left(-\frac{Y'_2}{E}\right) + \left(\frac{\kappa Y_2 - Y_1}{E} - \frac{a}{2\sqrt{a^2 + b^2}} L - \kappa\right)^2 \]
\[ + \left(\frac{\tau Y_2}{E} - \frac{b}{2\sqrt{a^2 + b^2}} L\right)^2 - \frac{L^2}{4}, \]  
(3.51)

where
\[ s^* = \int \sqrt{\frac{2}{a^2 + b^2}} | -a (\alpha + b\tau)| ds. \]  
(3.52)

**Proof.** i) Let the curve \( x \) be a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.45) with respect to \( s \) and considering (2.1), we find
\[ \gamma'_{\alpha\beta}(s^*) \frac{ds^*}{ds} = \frac{a\kappa + b\tau}{\sqrt{a^2 + b^2}} x(s) - \frac{a}{\sqrt{a^2 + b^2}} y(s), \]  
(3.53)
where
\[ E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2 + b^2}} | -a (\alpha + b\tau) |. \]

Differentiating (3.53) and using (3.47), we get
\[ \gamma''_{\alpha\beta}(s^*) = \left(\frac{Y'_1}{E}\right) x(s) + \left(\frac{Y' - \kappa Y'_1}{E}\right) \alpha(s) + \left(\frac{-\tau Y'_2}{E}\right) \beta(s) + \left(\frac{Y'_2}{E}\right) y(s), \]  
(3.54)
where
\[ Y_1 = \frac{a\kappa + b\tau}{E\sqrt{a^2 + b^2}}, \quad Y_2 = \frac{-a}{E\sqrt{a^2 + b^2}}. \]

By the help of equation (3.55), we obtain
\[ y_{\alpha\beta}(s^*) = -\gamma''_{\alpha\beta} - \frac{1}{2} \langle \gamma''_{\alpha\beta}, \gamma''_{\alpha\beta} \rangle \gamma_{\alpha\beta}, \quad \text{and} \quad \langle \gamma''_{\alpha\beta}, \gamma''_{\alpha\beta} \rangle = L. \]  
(3.55)

ii) Using (3.22), we have (3.50) and (3.51).

**Definition 3.11.** Let \( x \) be unit speed spacelike curve lying on \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, \( \beta y \)-smarandache curve of \( x \) is defined by
\[ \gamma_{\beta y}(s^*) = \beta(s) + \frac{b}{a} y(s), \]  
(3.57)
where \( a, b \in \mathbb{R}_0^+ \).

**Theorem 3.12.** Let \( x \) be unit speed spacelike curve in \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa \) and let \( \gamma_{\beta y} \) be \( \beta y \)-smarandache curve with asymptotic orthonormal frame \( \{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\} \) of the \( \beta y \)-smarandache curve \( \gamma_{\beta y} \) is given as
\[ \begin{bmatrix} \gamma_{\beta y} \\ \alpha_{\beta y} \\ \beta_{\beta y} \\ y_{\beta y} \end{bmatrix} = \begin{bmatrix} 0 & -b \gamma + \kappa^2 + \tau^2 & -\frac{b}{\sqrt{\kappa^2 + \tau^2}} & 0 \\ \frac{b\sqrt{\kappa^2 + \tau^2}}{Z - \omega_1} & -\frac{b}{\sqrt{\kappa^2 + \tau^2}} & -\frac{b}{Z} & 0 \\ \frac{b\sqrt{\kappa^2 + \tau^2}}{Z - \omega_1} & -\frac{b}{\sqrt{\kappa^2 + \tau^2}} & -\frac{b}{Z} & 0 \\ \frac{b\sqrt{\kappa^2 + \tau^2}}{Z - \omega_1} & -\frac{b}{\sqrt{\kappa^2 + \tau^2}} & -\frac{b}{Z} & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \beta \\ y \end{bmatrix}, \]  
(3.58)

where
\[ Z = \frac{ds^*}{ds} = \frac{b}{a} \sqrt{\kappa^2 + \tau^2}, \]  
(3.59)
\[
\omega_1 = \frac{a \tau}{b \sqrt{k^2 + \tau^2}}, \quad \omega_2 = -\frac{\kappa}{\sqrt{k^2 + \tau^2}}, \quad \omega_3 = -\frac{\tau}{\sqrt{k^2 + \tau^2}} \quad (3.60)
\]

ii) The cone curvatures \(\kappa_{\gamma_{by}}(s^*)\) and \(\tau_{\gamma_{by}}(s^*)\) of the curve \(\gamma_{by}\) is given by
\[
\kappa_{\gamma_{by}}(s^*) = -\frac{F}{2} \quad (3.61)
\]
\[
\tau_{\gamma_{by}}^2(s^*) = 2 \left( \frac{\omega_1' + \kappa \omega_1 + \omega_1 \tau}{Z} + \kappa' \right) \left( \frac{bF}{2a} \frac{\omega_1}{Z} \right) + \left( \frac{\omega_1 + \omega_1^2}{Z} + \kappa \right)^2 + \frac{F^2}{4}, \quad (3.62)
\]
where
\[
s^* = \frac{a}{b} \sqrt{k^2 + \tau^2}; \quad a, b \in \mathbb{R}_0^+. \quad (3.63)
\]

**Proof.** i) Differentiating the equation (3.57) with respect to \(s\) and considering (2.1), we find
\[
\gamma_{\beta y}'(s^*) \frac{ds^*}{ds} = \tau x(s) - \frac{b}{a} \alpha(s) - \frac{b}{a} \beta(s). \quad (3.64)
\]
This can be written as follows
\[
\alpha_{by}(s^*) = \frac{a \tau}{b \sqrt{k^2 + \tau^2}} x(s) - \frac{\kappa}{\sqrt{k^2 + \tau^2}} \alpha(s) - \frac{\tau}{\sqrt{k^2 + \tau^2}} \beta(s), \quad (3.65)
\]
where
\[
\frac{ds^*}{ds} = \frac{b}{a} \sqrt{k^2 + \tau^2}. \quad (3.66)
\]
Differentiating (3.65) and using (3.66), we get
\[
\gamma_{\beta y}''(s^*) = \left( \frac{\omega_1' + \kappa \omega_1 + \omega_1 \tau}{Z} \right) x(s) + \left( \frac{\omega_1' + \omega_1^2}{Z} \right) \alpha(s) + \left( \frac{\omega_1'}{Z} \right) \beta(s) + \left( \frac{-\omega_1}{Z} \right) y(s),
\]
\[
y_{\beta y}(s^*) = -\gamma_{\beta y}'' \quad \frac{1}{2} \left( \gamma_{\beta y}, \gamma_{\beta y}, \gamma_{\beta y} \right) \gamma_{\beta y}. \quad (3.67)
\]
By the help of equation (3.67), we obtain
\[
y_{\beta y}(s^*) = \left( -\frac{\omega_1' + \kappa \omega_1 + \omega_1 \tau}{Z} \right) x(s) + \left( -\frac{\omega_1 + \omega_1^2}{Z} \right) \alpha(s)
\]
\[+ \left( \frac{-\omega_1}{Z} \right) \beta(s) + \left( \frac{\omega_2}{Z} - \frac{bF}{2a} \right) y(s), \quad (3.68)
\]
where \(Z = \frac{ds^*}{ds} = \frac{b}{a} \sqrt{k^2 + \tau^2}\).

ii) Using (3.22), we have (3.61) and (3.62). \qed

**Example 3.13.** Let \(x\) be a spacelike curve in \(Q^1\) with arc length parameter \(s\) given by
\[
x(s) = (\sin s, \cos s, 0, 1). \]
Then we can write the smarandache curves of the \(x\)-curve as follows:
1) \(x\alpha\)- smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{a}{\sqrt{ab}} \sin s + \cos s, \frac{a}{\sqrt{ab}} \cos s - \sin s, 0, \frac{a}{\sqrt{ab}})\)
2) \(x\beta\)-smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{a}{\sqrt{ab}} \sin s - \cos s, \frac{a}{\sqrt{ab}} \cos s + \sin s, 0, \frac{a}{\sqrt{ab}})\)
3) \(x\beta\)- smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{a}{\sqrt{ab}} \sin s - \cos s, \frac{a}{\sqrt{ab}} \cos s + \sin s, 0, \frac{a}{\sqrt{ab}})\)
4) \(x\beta\)-smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{\alpha}{\sqrt{ab}} \sin s - \cos s, \frac{\alpha}{\sqrt{ab}} \cos s + \sin s, 0, \frac{\alpha}{\sqrt{ab}})\)
5) \(x\beta\)-smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{\alpha}{\sqrt{ab}} \sin s - \cos s, \frac{\alpha}{\sqrt{ab}} \cos s + \sin s, 0, \frac{\alpha}{\sqrt{ab}})\)
6) \(x\beta\)-smarandache curve \(\gamma_{\alpha x}\) is given by \(\gamma_{\alpha x}(s) = (\frac{\alpha}{\sqrt{ab}} \sin s - \cos s, \frac{\alpha}{\sqrt{ab}} \cos s + \sin s, 0, \frac{\alpha}{\sqrt{ab}})\),
where \(a, b \in \mathbb{R}_0^+\).
References


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