ON THE COSET PRESERVING SUBCENTRAL AUTOMORPHISMS OF FINITE GROUPS

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Abstract. Let G be a group and M be a characteristic subgroup of G. We denote by $\text{Aut}_M^G(G)$ the set of all automorphisms of G which centralize $G/M$ and $M$. In this paper, we give the necessary and sufficient conditions for equality $\text{Aut}_M^G(G)$ with $\text{Aut}_M^G(G)$. Also we study equivalent conditions for equality $\text{Aut}_M^G(G)$ with $\text{Inn}(G)$.

1 Introduction

In this paper p denotes a prime number. Let us denote $\Phi(G)$, $G'$, $Z(G)$, $\text{Aut}(G)$ and $\text{Inn}(G)$, respectively the Frattini subgroup, commutator subgroup, the center, the full automorphism group and the inner automorphisms group of $G$. An automorphism $\alpha$ of $G$ is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for $x \in G$. All the elements of the central automorphism group of $G$, denoted by $\text{Aut}_Z^G(G)$, is a normal subgroup of $\text{Aut}(G)$.

There has been a number of results on the central automorphisms of a group. Curran and McCaughan [4] proved that for any non-abelian finite group $G$, $\text{Aut}_Z^G(G) \cong \text{Hom}(G/G', Z(G), Z(G))$ where $\text{Aut}_Z^G(G)$ is a group of all those central automorphisms which preserve the center $Z(G)$ elementwise. Adney and Yen [1] proved that if a finite group $G$ has no abelian direct factor, then there is a one-one and onto map between $\text{Aut}_Z^G(G)$ and $\text{Hom}(G, Z(G))$. Ghumed and Ghate [5] proved that for a finite group $G$, $\text{Aut}_M^G(M) \cong \text{Hom}(G/KM, M)$. Also he proved that if $G$ is a purely non-abelian finite group, then $|\text{Aut}_M^G(G)| = |\text{Hom}(G, M)|$. Shabani Attar [10] characterized all finite p-groups $G$ for which the equality $\text{Aut}_Z^G(G) = \text{Aut}_Z^G(G)$ holds. Let $\text{IA}(G)$ be the subgroup of $\text{Aut}(G)$ which consists of those automorphisms $\alpha$ for which $g^{-1}\alpha(g) \in G'$ for each $g \in G$. A group $G$ is called semicomplete if $\text{IA}(G) = \text{Inn}(G)$. Shabani Attar [9] gave some necessary conditions for finite p-groups to be semicomplete. In this paper, we give necessary and sufficient conditions for $G$ such that $\text{Aut}_M^G(G) = \text{Aut}_M^G(G)$ and show some equivalent conditions for equality $\text{Aut}_M^G(G)$ with $\text{Inn}(G)$.

2 Preliminaries

Let $M$ be a characteristic subgroup of $G$. By $\text{Aut}_M^G(G)$, we mean the subgroup of $\text{Aut}(G)$ consisting of all automorphisms which induce identity on $G/M$. By $\text{Aut}_M^G(G)$, we mean the subgroup of $\text{Aut}(G)$ consisting of all automorphisms which restrict to the identity on $M$. Let $\text{Aut}_M^G(G) = \text{Aut}_M^G(G) \cap \text{Aut}_M(G)$. From now onwards, $M$ will be a characteristic central subgroup and elements of $\text{Aut}_M^G(G)$ will be called subcentral automorphisms of $G$ (with respect to subcentral subgroup $M$). It can be seen that $\text{Aut}_M^G(G)$ is a normal subgroup of $\text{Aut}_Z^G(G)$. We further let $A^*$ be the set $\{\alpha \in \text{Aut}_M(G) : \alpha \beta = \beta \alpha, \forall \beta \in \text{Aut}_M(G)\}$. Clearly $A^*$ is a normal subgroup of $\text{Aut}(G)$. Since every inner automorphism commutes with elements of $\text{Aut}_Z^G(G)$, $\text{Inn}(G) \leq A^*$. Let $P = \langle [g, \alpha] : g \in G, \alpha \in A^* \rangle$, where $[g, \alpha] = g^{-1}\alpha(g)$.

It is easy to check that $P$ is a characteristic subgroup of $G$. 
We first state some results that will be used in the proof of the main theorems.

**Proposition 2.1.** [5, Proposition 2.1] \( \text{Aut}^M(G) \) acts trivially on \( P \).

Let \( A^{**} \) be any normal subgroup of \( \text{Aut}(G) \) contained in \( A^* \), and \( K = \langle [g, \alpha] : g \in G, \alpha \in A^{**} \rangle \). In particular, when \( A^{**} = \text{Inn}(G) \), we get \( K = G' \). Since \( K \) is a subgroup of \( P \), it is invariant under the action of \( \text{Aut}^M(G) \). It is easy to see that \( K \) is a characteristic subgroup of \( G \), and hence it is a normal subgroup of \( G \).

**Theorem 2.2.** [5, Theorem A] For a finite group \( G \), \( \text{Aut}^M(G) \cong \text{Hom}(G/KM, M) \).

**Theorem 2.3.** [5, Theorem C] If \( G \) is a purely non-abelian finite group, then \( |\text{Aut}^M(G)| = |\text{Hom}(G, M)| \).

**Proposition 2.4.** [5, Proposition 2.4] Let \( G \) be a purely non-abelian finite group, then for each \( \alpha \in \text{Hom}(G, M) \) and each \( x \in K \), we have \( \alpha(x) = 1 \). Further \( \text{Hom}(G/K, M) \cong \text{Hom}(G, M) \).

### 3 Main Results

We note that in this section \( M \) is a central characteristic subgroup.

**Theorem 3.1.** Let \( G \) be a finite group. Then \( G/M \) is abelian if and only if \( \text{Inn}(G) \leq \text{Aut}^M(G) \).

**Proof.** Suppose \( G/M \) is abelian. Thus \( G' \leq M \). Let \( x \in G \). Then for the inner automorphism \( \theta_x \) induced by \( x \) and every \( g \in G \) we have, \( g^{-1}\theta_x(g) = [g, x] \in G' \leq M \). So for every \( \alpha \in \text{Inn}(G) \), \( g^{-1}\alpha(g) \in M \). This means \( \text{Inn}(G) \leq \text{Aut}^M(G) \). Hence \( \text{Inn}(G) \leq \text{Aut}^M(G) \).

Conversely, suppose \( \text{Inn}(G) \leq \text{Aut}^M(G) \). Hence it is clear \( G' \leq M \) and so \( G/M \) is abelian.

Here we give a basic lemmas that will be used in the proof of the results.

**Lemma 3.2.** [3, Lemma E] Suppose \( H \) is an abelian \( p \)-group of exponent \( p^r \), and \( K \) is cyclic group of order divisible by \( p^s \). Then \( \text{Hom}(H, K) \) is isomorphic to \( H \).

**Lemma 3.3.** [3, Lemma H] Let \( G \) be a purely non-abelian \( p \)-group, of nilpotent class 2. Then \( |\text{Hom}(G/Z(G), G')| \geq |G/Z(G)|^{p^{r(s-1)}} \), where \( r = \text{rank}(G/Z(G)) \) and \( s = \text{rank}(G') \).

**Theorem 3.4.** Suppose \( G \) is a non-abelian finite \( p \)-group for which \( G/M \) is abelian. Then \( |\text{Aut}^M(G) : \text{Inn}(G)| \geq p^{r(s-1)} \), where \( r \) and \( s \) are as defined before.

**Proof.** By Theorem 2.3, \( |\text{Aut}^M(G)| = |\text{Hom}(G, M)| \). Now we have

\[
|\text{Hom}(G, M)| = |\text{Hom}(G/Z(G), M)| \geq |\text{Hom}(G/Z(G), G')| \geq |G/Z(G)|^{p^{r(s-1)}}.
\]

Hence \( |\text{Aut}^M(G)| \geq |G/Z(G)|^{p^{r(s-1)}} \), thus \( |\text{Aut}^M(G) : \text{Inn}(G)| = |\text{Aut}^M(G)|/|G/Z(G)| \geq p^{r(s-1)}. \)

Find necessary and sufficient conditions on a finite \( p \)-group \( G \) such that \( \text{Aut}_M^G(G) = \text{Aut}^M(G) \).

Let \( G \) be a non-abelian finite \( p \)-group. Let

\[
G/K = C_{p_1} \times C_{p_2} \times \ldots \times C_{p_k},
\]

where \( C_{p_i} \) is a cyclic group of order \( p^{a_i} \), \( 1 \leq i \leq k \), and \( a_1 \geq a_2 \geq \ldots \geq a_k \geq 1 \). Let

\[
G/KM = C_{p_1} \times C_{p_2} \times \ldots \times C_{p_1}
\]

and

\[
M = C_{p^{b_1}} \times C_{p^{b_2}} \times \ldots \times C_{p^{b_m}},
\]
where \( b_1 \geq b_2 \geq \cdots \geq b_t \geq 1 \) and \( c_1 \geq c_2 \geq \cdots \geq c_m \geq 1 \).

Since \( G/KM \) is a quotient of \( G/K \), by [2, Section 25], we have \( l \leq k \) and \( b_i \leq a_i \) for all \( 1 \leq i \leq l \).

**Theorem 3.5.** Let \( G \) be a purely non-abelian finite \( p \)-group \((p \text{ odd})\). Then \( \text{Aut}^M(G) = \text{Aut}^M(G) \) if and only if \( M \leq K \) or \( M \leq \Phi(G), k = l \) and \( c_1 \leq b_t \) where \( t \) is the largest integer between 1 and \( k \) such that \( a_t > b_t \).

**Proof.** Let \( M \leq K \), by Proposition 2.1 and since \( K \leq P \), every \( \alpha \in \text{Aut}^M(G) \) fixes \( M \) and so \( \text{Aut}^M(G) \leq \text{Aut}^M(G) \), since \( \text{Aut}^M(G) \leq \text{Aut}^M(G) \). Thus \( \text{Aut}^M(G) = \text{Aut}^M(G) \). Now suppose that \( M \leq \Phi(G), k = l \) and \( c_1 \leq b_t \). Since \( G \) is purely non-abelian and by Theorem 2.3 and Proposition 2.4, we have

\[
|\text{Aut}^M(G)| = |\text{Hom}(G, M)| = |\text{Hom}(G/K, M)| = \prod_{1 \leq i \leq k, 1 \leq j \leq m} p^{\min(a_i, c_j)}
\]

On the other hand, by Theorem 2.2, we have

\[
|\text{Aut}^M_M(G)| = |\text{Hom}(G/KM, M)| = \prod_{1 \leq i \leq l, 1 \leq j \leq m} p^{\min(b_i, c_j)}
\]

Since \( b_1 \geq c_1 \), we have

\[
b_1 \geq b_2 \geq \cdots \geq b_{t-1} \geq b_t \geq c_1 \geq c_2 \geq \cdots \geq c_m \geq 1.
\]

Therefore, \( c_j \leq b_i \leq a_i \) for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq t \), whence \( \min(a_i, c_j) = c_j = \min(b_i, c_j) \) for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq t \). Since \( a_i = b_i \) for all \( i > t \), we have \( \min(a_i, c_j) = \min(b_i, c_j) \) for all \( 1 \leq j \leq m \) and \( t + 1 \leq i \leq k \). Thus \( \min(b_i, c_j) = \min(a_i, c_j) \) for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq k \). Therefore, \( \text{Aut}^M_M(G) = \text{Aut}^M(G) \).

Conversely if \( \text{Aut}^M_M(G) = \text{Aut}^M(G) \) and \( M \not\leq K \). We claim that \( M \leq \Phi(G) \). Assume contrarily that \( M \) is not contained in \( \Phi(G) \). Then there exists a maximal subgroup \( D \) of \( G \) such that \( M \not\leq D \). The maximality of \( D \) implies that \( G = DM \) and \( D \leq G \). Hence we assume that \( |G/D| = p \), where \( p \) is a prime number. Now we consider the following two cases.

**Case 1:** \( p \mid |M \cap D| \). Choose \( z \in M \cap D \) such that \( o(z) = p \) and fix \( g \in M \). It is clear that \( G = D < g > \). Then the map \( \alpha \) defined on \( G \) by \( \alpha(dq^i) = dq^iz^k \) for every \( d \in D \) and every \( i \in \{0, 1, 2, ..., p-1\} \), \( \alpha \in \text{Aut}^M(G) \). By the given hypothesis \( g = \alpha(g) = gz \), whence \( z = 1 \), which is a contradiction. Hence \( M \leq \Phi(G) \).

**Case 2:** \( p \not\mid |M \cap D| \). In this case, since

\[
p = |G/D| = |DM/D| = |M/M \cap D|,
\]

we see that \( p \) divides \( |M| \) and we may choose \( z \in M \) such that \( o(z) = p \) and \( z \notin D \). Hence \( G = (D, z) = D \times \langle z \rangle \). Consider the map \( \alpha : G \to G \) where \( \alpha(dz^i) = dz^{2i} \) for every \( d \in D \) and every \( i \in \{0, 1, 2, ..., p-1\} \), \( \alpha \in \text{Aut}^M(G) \). By the given hypothesis and since \( z \in M \) it is clear that \( z = \alpha(z) = \alpha(1.z^{1}) = z^2 \), a contradiction. The proof of the theorem is complete.

**Lemma 3.6.** Let \( G \) be a finite \( p \)-group, then \( \text{Aut}^M_M(G) \cong \text{Hom}(G/M, M) \).

**Proof.** Consider the map \( \phi : \text{Aut}^M_M(G) \to \text{Hom}(G/M, M) \) defined by \( \phi(\alpha) = \alpha^* \), where \( \alpha^*(gM) = g^{-1}\alpha(g) \) for all \( g \in G \). It is clear that \( \alpha^* \) is a well defined homomorphism. We show that \( \phi \) is an isomorphism. If \( \alpha_1, \alpha_2 \in \text{Aut}^M_M(G) \) and \( \alpha_1 = \alpha_2 \), then \( g^{-1}\alpha_1(g) = g^{-1}\alpha_2(g) \), thus \( \alpha_1^*(gM) = \alpha_2^*(gM) \), hence \( \alpha_1^* = \alpha_2^* \), therefore \( \phi(\alpha_1) = \phi(\alpha_2) \), so \( \phi \) is a well defined homomorphism. It is easy to check that \( \phi \) is a monomorphism. We show that \( \phi \) is onto. For given any \( f \in \text{Hom}(G/M, M) \), define \( \alpha : G \to G \) by \( \alpha(g) = gf(gM) \). Clearly \( \alpha \in \text{Aut}(G) \).

We show that \( \alpha \in \text{Aut}^M_M(G) \). Since \( g^{-1}\alpha(g) = f(gM) \in M \), then \( \alpha \in \text{Aut}^M_M(G) \). Also for each \( m \in M \), we have \( \alpha(m) = m \alpha(mM) = m \), thus \( \alpha \in \text{Aut}^M_M(G) \). So \( \alpha \in \text{Aut}^M_M(G) \) and \( \phi(\alpha) = f \). Therefore \( \phi \) is an isomorphism and \( \text{Aut}^M_M(G) \cong \text{Hom}(G/M, M) \).
Theorem 3.7. Let $G$ be a finite $p$-group such that $G/M$ is abelian. Then the following are equivalent:

1. $\text{Aut}_M^G(G) = \text{Inn}(G)$.
2. $\text{Hom}(G/M, M) \cong G/Z(G)$.
3. $G$ is cyclic and $\text{Hom}(G/M, M) \cong \text{Hom}(G/Z(G), M)$.

Proof. (1) $\Rightarrow$ (2) By Lemma 3.6, and since $\text{Aut}_M^G(G) = \text{Inn}(G)$ we have, $\text{Hom}(G/M, M) \cong \text{Inn}(G) \cong G/Z(G)$.

(2) $\Rightarrow$ (1) By Lemma 3.6, and since $\text{Hom}(G/M, M) \cong G/Z(G)$, thus $\text{Aut}_M^G(G) \cong \text{Inn}(G)$, also since $G/M$ is abelian we have $G' \leq M$ and so $\text{Inn}(G) \leq \text{Aut}_M^G(G)$, also for every $\alpha \in \text{Inn}(G)$ and $m \in M$, we have $\alpha(m) = m$, therefore $\text{Inn}(G) \leq \text{Aut}_M^G(G)$ and so $\text{Aut}_M^G(G) = \text{Inn}(G)$.

(1) $\Rightarrow$ (3) Since $\text{Aut}_M^G(G) = \text{Inn}(G)$, every $f \in \text{Aut}_M^G(G)$ is an inner one, and so fixes each element of $Z(G)$. Therefore, for every $f \in \text{Aut}_M^G(G)$, the map $\sigma_f : G/Z(G) \rightarrow M$ defined by $\sigma_f(gZ(G)) = g^{-1}f(g)$ is well defined. Now consider the map $\sigma : f \mapsto \sigma_f$. It is easy to check that $\sigma$ is an isomorphism from $\text{Aut}_M^G(G)$ onto $\text{Hom}(G/Z(G), M)$. Thus $\text{Hom}(G/Z(G), M) \cong G/Z(G)$, so $G$ is nilpotent of class 2, and $\text{exp}(G/Z(G)) = \text{exp}(G)$. Now by Lemma 3.2, $|\text{Hom}(G/Z(G), C_{p^e})| = |G/Z(G)|$. Therefore

$$\text{Hom}(G/Z(G), M) = |G/Z(G)||\text{Hom}(G/Z(G), N)|,$$

which is a contradiction. Hence $M$ is cyclic.

(3) $\Rightarrow$ (1) Since $M$ is cyclic and $G/Z(G)$ is an abelian $p$-group of exponent $|G'|$ and $G'$ is cyclic, it follows that $\text{Hom}(G/Z(G), M) \cong G/Z(G)$. By Lemma 3.6, $\text{Aut}_M^G(G) \cong \text{Hom}(G/M, M)$ and also $\text{Inn}(G) \leq \text{Aut}_M^G(G)$. Thus $\text{Aut}_M^G(G) = \text{Inn}(G)$.

References


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