The Multiplicative Hyper- Zagreb index of Graph Operations

Akbar. Jahanbani

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Abstract Let $G$ be a graph of order $n$ with vertices labeled as $v_1, v_2, \ldots, v_n$. Let $d_i$ be the degree of the vertex $v_i$, for $i = 1, 2, \ldots, n$. The multiplicative Hyper- Zagreb index, is defined as, $\prod_{i=1}^{n} d_G(v_i)^{d_i}$ where $d_G(v_i)$ is the degree of the vertex $v_i$. A graph is a simple graph if it does not have multiple edges or loops, and is connected if there is a path between any two vertices. In this paper, we study the multiplicative Hyper- Zagreb index and some exact formulas for the Hyper-multiplicative Zagreb index of some well-known graphs. We also apply some of our results to compute the multiplicative Hyper- Zagreb index.

1 Introduction

All graphs considered in this paper are assumed to be simple. Let $G$ be a (molecular) graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. If $v_i$ and $v_j$ are adjacent vertices of $G$, then the edge connecting them is denoted by $v_iv_j$. Let $d_i$ denote the degree of the vertex $v_i \in V(G)$. We consider only simple connected graphs, i.e. connected graphs without loops and multiple edges. A topological index $Top(G)$ of a graph $G$, is a number with this property that for every graph $G_2$ isomorphic to $G$, $Top(G_2) = Top(G)$. The Cartesian product $G_1 \square G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(a, x)(b, y)$ is an edge of $G_1 \square G_2$ if $a = b$ and $xy \in E(G_2)$, or $ab \in E(G_1)$ and $x = y$. The composition $G_1 \circ G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(a, x)$ is adjacent to $(b, y)$ whenever $b$ is adjacent to $x$ and $a \neq b$ and $x$ is adjacent to $y$. The tensor product $G_1 \otimes G_2$ of graphs $G_1$ and $G_2$, $d_{G_1 \circ G_2}(u) = d_{G_1}(u) + d_{G_2}(u)$. The corona product $G_1 \circ G_2$ is defined as the graph obtained from $G_1$ and $G_2$ by taking one copy of $G_1$ and $|V(G_2)|$ copies of $H$ and then by joining with an edge each vertex of the $i$th copy of $H$ which is named $(G_2, i)$ with the $i$th vertex of $G$ for $i = 1, 2, \ldots, |V(G_1)|$. If $u$ is a vertex of $G_1 \circ G_2$, then

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \text{if } u \in V(G_1) \\ d_{G_2}(u) & \text{if } u \in V(G_2). \end{cases}$$

The join $G_1 + G_2$ of graphs $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The symmetric difference $G_1 \oplus G_2$ of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \oplus V(G_2)$ and $E(G_1 \oplus G_2) = \{(u_1, u_2), (v_1, v_2) : u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both}\}$. The tensor product $G_1 \otimes G_2$ of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1, u_2), (v_1, v_2) : u_1v_1 \in E(G_1), u_2v_2 \in E(G_2)\}$.

This paper is organized as follows. In Section 2, we state some previously known results also we study the multiplicative Hyper- Zagreb index of a graph. In Section 3, we the Hyper- Zagreb index of the Cartesian product, corona product, composition, disjunction, join and symmetric difference of graphs are computed.

2 Preliminaries and known results

In this section, we study the Hyper-multiplicative Zagreb index of a graph and some exact formulas for the Hyper-multiplicative Zagreb index of some well-known graphs are presented. We
begin with the definition and crucial theorem related to theorem properties of some graph operations. Let us begin with a few examples, then we will give a crucial theorem related to distance properties of some graph operations. In mathematical chemistry, there is a large number of topological indices of the form
\[ TI = TI(G) = \sum_{v_i, v_j \in E(G)} F(d_i, d_j). \]

The most popular topological indices of this kind are the:

- first Zagreb index, \( M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) = \sum_{u \in V(G)} d_G(u)^2, \)
- second Zagreb index, \( M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v), \)
- hyper-Zagreb index, \( HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2, \)
- first multiplicative Zagreb index, \( \Pi_1(G) = \prod_{u \in V(G)} d_G(u)^2, \)
- second multiplicative Zagreb index, \( \Pi_2(G) = \prod_{uv \in E(G)} (d_G(u)d_G(v)), \)
- hyper-multiplicative Zagreb index, \( HII(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v))^2. \)

Note that there are several more indices, see ([3, 14], [15]). The Zagreb indices are widely studied degree-based topological indices, and were introduced by Gutman and Trinajstić [5] in 1972, there was a vast research on comparing Zagreb indices see ([9], [10]). A survey on the first Zagreb index see [4]. The Hyper-multiplicative Zagreb index can also be expressed as a sum over edges of \( G \) [15],
\[ HII(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v))^2. \]

Readers interested in more information on Hyper-multiplicative Zagreb index can be referred to ([2, 16], [17], [18]). Recently, the analogous concepts of the sigma index of graphs operations [11] and the Nano-Zagreb index and multiplicative Nano-Zagreb index of some graph operations [12] were put forward.

**Proposition 2.1.** [15] Let \( K_n \) be a complete graph with \( n \) vertices. Then
\[ HII(K_n) = \prod_{uv \in E(K_n)} (d(u) + d(v))^2 = [2(n - 1)]^{n(n - 1)}. \]

**Proposition 2.2.** [15] Let \( K_{m,n} \) be a complete bipartite graph with \( 1 \leq m \leq n \). Then
\[ HII(K_{m,n}) = \prod_{uv \in E(K_{m,n})} (d(u) + d(v))^2 = (m + n)^{2mn}. \]

**Proposition 2.3.** [15] Let \( K_{1,n} \) be a star. Then
\[ HII(K_{1,n}) = \prod_{uv \in E(K_{1,n})} (d(u) + d(v))^2 = (n + 1)^{2n}. \]

**Proposition 2.4.** [15] Let \( C_n \) be a cycle with \( n \geq 3 \) vertices. Then
\[ HII(C_n) = \prod_{uv \in E(C_n)} (d(u) + d(v))^2 = [(2 + 2)^n] = 4^n. \]

**Lemma 2.5.** (AM-GM inequality) Let \( x_1, x_2, \ldots, x_n \) be nonnegative numbers. Then
\[ \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1x_2\ldots x_n} \]
(2.1)
holds with equality if and only if all the \( x_i \)'s are equal.
3 Multiplicative Hyper- Zagreb indices of Graph Operations

In this section, we the multiplicative Hyper- Zagreb indices of the Cartesian product, composition, join and disjunction of graphs are computed. We apply some of our results to compute the small Zagreb index. We begin this section with standard Lemma as follow.

Lemma 3.1. Let $G_1$ and $G_2$ be two connected graphs, then we have:

(a) $|V(G_1 \times G_2)| = |V(G_1 \vee G_2)| = |V(G_1[G_2])| = |V(G_1)\times V(G_2)|,$

(b) $G_1 \times G_2$ is connected if and only if $G_1$ and $G_2$ are connected.

(c) If $(a, b)$ is a vertex of $G_1 \times G_2$ then $d_{G_1 \times G_2}((a, b)) = d_{G_1}(a) + d_{G_2}(b).$

(d) If $(a, b)$ is a vertex of $G_1[G_2]$ then $d_{G_1[G_2]}((a, b)) = |V(G_1)|d_{G_2}(a) + d_{G_2}(b).$

(e) If $(a, b)$ is a vertex of $G_1 \oplus_G G_2$ or $G_1 \otimes G_2,$ we have:

$$d_{G_1 \oplus_G G_2}((a, b)) = |V(G_1)|d_{G_1}(a) + |V(G_1)|d_{G_2}(b) - 2d_{G_1}(a)d_{G_2}(b).$$

$$d_{G_1 \otimes G_2}((a, b)) = |V(G_2)|d_{G_1}(a) + |V(G_1)|d_{G_2}(b) - d_{G_1}(a)d_{G_2}(b).$$

(f) If $u$ is a vertex of $G_1 \vee G_2$ then we have:

$$d_{G_1 \vee G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \text{if } u \in V(G_1) \\ d_{G_2}(u) + |V(G_1)| & \text{if } u \in V(G_2). \end{cases}$$

Proof. The parts (a) and (b) are consequence of definitions and some famous results of the book of Imrich and Klavzar [8]. For the proof of (c-f) we refer to [13].

The Cartesian product $G_1 \boxtimes G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \boxtimes G_2$ if either $u_i = u_k$ and $v_jv_l \in E(G_2),$ or $u_iu_k \in E(G_1)$ and $v_j = v_l.$

Theorem 3.2. Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

$$\text{HII}(G_1 \boxtimes G_2) = \frac{1}{(n_1m_2)^{n_1m_2}} \left( 4m_1M_1(G_1) + n_2HM(G_1) + 8m_2M_1(G_2) \right)^{n_1m_2} \times \frac{1}{(n_2m_1)^{n_2m_1}} \left( 4m_2M_1(G_2) + n_1HM(G_2) + 8m_1M_1(G_1) \right)^{n_2m_1}.$$ 

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \boxtimes G_2,$ we have

$$\text{HII}(G_1 \boxtimes G_2) = \prod_{(u_i, v_j)(u_p, v_q) \in E(G_1 \boxtimes G_2)} \left( d_{G_1 \boxtimes G_2}(u_i, v_j) + d_{G_1 \boxtimes G_2}(u_p, v_q) \right)^2.$$
This actually can be written as
\[
= \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 4d_{G_1}^2(u_i) + (d_{G_2}(v_j) + d_{G_2}(v_q))^2 \right)
+ 4d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))
\times \prod_{v_j \in V(G_2)} \prod_{u_i \in V(G_1)} \left( 4d_{G_2}^2(v_j) + (d_{G_1}(u_i) + d_{G_1}(u_p))^2 \right)
+ 4d_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_1}(u_p)).
\]

However, from the inequality (2.1), we get
\[
\leq \left[ \sum_{u_i \in V(G_1)} \sum_{v_j \in V(G_2)} \left( 4d_{G_1}^2(u_i) + (d_{G_2}(v_j) + d_{G_2}(v_q))^2 \right) 
+ 4d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q)) \right]^{n_1m_2}
\times \left[ \sum_{v_j \in V(G_2)} \sum_{u_i \in V(G_1)} \left( 4d_{G_2}^2(v_j) + (d_{G_1}(u_i) + d_{G_1}(u_p))^2 \right) 
+ 4d_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_1}(u_p)) \right]^{n_2m_1}
\times \left[ \sum_{(v_j, v_q) \in E(G_2)} \left( 4M_1(G_1) + n_1(d_{G_2}(v_j) + d_{G_2}(v_q))^2 \right) 
+ 8m_1(d_{G_2}(v_j) + d_{G_2}(v_q)) \right]^{n_1m_2}
\times \left[ \sum_{(u_i, u_p) \in E(G_1)} \left( 4M_1(G_2) + n_2(d_{G_1}(u_i) + d_{G_1}(u_p))^2 \right) 
+ 8m_2(d_{G_1}(u_i) + d_{G_1}(u_p)) \right]^{n_2m_1}
\times \frac{1}{(n_1m_2)^{n_1m_2}} \left( 4m_1M_1(G_1) + n_1HM(G_1) + 8m_2M_1(G_2) \right)^{n_1m_2}
\times \frac{1}{(n_2m_1)^{n_2m_1}} \left( 4m_2M_1(G_2) + n_1HM(G_2) + 8m_1M_1(G_1) \right)^{n_2m_1}.
\]

\[\square\]

**Remark 3.3.** [1] For a cycle graph with $n$ vertices, we have, $HM(C_n) = 16n$, $M_1(C_n) = 4n$.

**Example 3.4.** Let $C_p$ and $C_q$ be cycles with $n \geq 3$ vertices. Then
\[
\text{HII}(C_p \boxtimes C_q) = \frac{1}{(pq)^p} \left[ \left( 16p^2 + 16pq + 32q^2 \right) \times \left( 16q^2 + 16pq + 32p^2 \right) \right]^{pq}.
\]
The corona product $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is defined to be the graph $\Gamma$ obtained by taking one copy of $G_1$ (which has $n_1$ vertices) and $n_2$ copies of $G_2$, and then joining the $i$th vertex of $G_1$ to every vertex in the $i$th copy of $G_2$, $i = 1, 2, \ldots, n_1$. Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two graphs such that $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $|E(G_1)| = m_1$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$, $|E(G_2)| = m_2$. Then it follows from the definition of the corona product that $G_1 \circ G_2$ has $n_1 (1 + n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges, where $V(G_1 \circ G_2) = \{(u_i, v_j) : i = 1, 2, \ldots, n_1; j = 0, 1, 2, \ldots, n_2\}$ and $E(G_1 \circ G_2) = \{((u_i, v_0), (u_k, v_0)), (u_i, u_k) \in E(G_1) \} \cup \{((u_i, v_j), (u_k, v_j)), (v_j, v_j) \in E(G_2) \}$. It is clear that if $G_1$ is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_1 \circ G_2$.

**Theorem 3.5.** Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

$$HII(G_1 \circ G_2) = \frac{1}{m_1 m_2^3} \left( HM(G_1) + 4n_1 n_2^2 + 4n_2 M_1(G_1) \right)^{m_2} \times \frac{1}{(n_1 m_2)^{n_1 n_2}} \left( n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2 + n_1 n_2 (n_2 + 1)^2 \right)^{n_1 n_2} \times \frac{1}{(n_1 m_2)^{n_1 n_2}} \left( n_1 H_M(G_2) + 4n_1 m_2 + 4n_1 M_1(G_2) \right)^{n_1 m_1}.$$ 

**Proof.** By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \circ G_2$, we have

$$HII(G_1 \circ G_2) = \prod_{(u_i, v_j) \in E(G_1 \circ G_2)} \left( d_{G_1 \circ G_2}(u_i, v_j) + d_{G_1 \circ G_2}(u_p, v_q) \right)^2 \prod_{(u_i, u_p) \in E(G_1)} \left( d_{G_1}(u_i) + d_{G_1}(u_p) \right)^2 \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( d_{G_2}(v_j) + d_{G_2}(v_j) \right)^2 = \prod_{(u_i, u_p) \in E(G_1)} \left( d_{G_1}(u_i) + d_{G_1}(u_p) \right)^2 + 4n_2 + 4n_2 (d_{G_1}(u_i) + d_{G_1}(u_p)) \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( d_{G_2}(v_j) + d_{G_2}(v_j) \right)^2 + (n_2 + 1)^2 \times \prod_{u_i \in V(G_1)} \prod_{v_j, v_q \in E(G_2)} \left( d_{G_2}(v_j) + d_{G_2}(v_j) \right)^2 + 4 + 4 \left( d_{G_2}(v_j) + d_{G_2}(v_q) \right) \prod_{u_i \in V(G_1)} \prod_{v_j, v_q \in E(G_2)} \left( d_{G_2}(v_j) + d_{G_2}(v_q) \right)^2 + 4 + 4 \left( d_{G_2}(v_j) + d_{G_2}(v_q) \right).$$

However, from the inequality (2.1), we get

$$\sum_{(u_i, u_p) \in E(G_1)} \frac{\left( d_{G_1}(u_i) + d_{G_1}(u_p) \right)^2 + 4n_2 + 4n_2 (d_{G_1}(u_i) + d_{G_1}(u_p))}{m_1} \left( \sum_{(u_i, u_p) \in E(G_1)} \frac{\left( d_{G_1}(u_i) + d_{G_1}(u_p) \right)^2 + 4n_2 + 4n_2 (d_{G_1}(u_i) + d_{G_1}(u_p))}{m_1} \right).$$
Remark 3.6. [1] For a path with \( n \) vertices, we have: \( \text{HM}(P_n) = 16n - 30, M_1(P_n) = 4n - 6 \).

**Example 3.7.** Let \( C_q \) and \( P_n \) be a cycle and path with \( n \geq 3 \) vertices. Then

\[
\text{HII}(C_q \circ P_n) = \frac{1}{q^q} \left( 16n + 4qn^2 + 4n(4n - 6) \right)^q \times \frac{1}{(qn)^{q^n}} \left( 16qn + q(4n - 6) \right)^{q^n} \times \frac{1}{(q(n - 1))^n} \left( q(16n - 30) + 4q(n - 1) + 4q(4n - 6) \right)^{q(n - 1)}.
\]

**Theorem 3.8.** Let \( G_1 \) and \( G_2 \) be two graphs with \( n_1 \) and \( n_2 \) vertices, \( m_1 \) and \( m_2 \) edges respectively. Then

\[
\text{HII}(G_1 \mid G_2) \leq \frac{1}{(m_2)^{n_1m_2}} \left[ \frac{4M_1(G_1)n_1^2 + n_1HM(G_2) + 4m_1n_2M_1(G_2)}{n_1} \right]^{n_1m_2} \times \frac{1}{(n_2)^{m_1n_2}} \left[ \frac{n_2^3HM(G_1) + 4m_1M_1(G_2) + 8n_2m_2M_1(G_1)}{m_1} \right]^{n_2m_1}.
\]
Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1[G_2]$, we have

$$HII(G_1[G_2]) = \prod_{(u, v_j)(u_p, v_q) \in E(G_1[G_2])} \left( d_{G_1[G_2]}(u, v_j) + d_{G_1[G_2]}(u_p, v_q) \right)^2$$

$$= \prod_{u, v_j \in V(G_1)} \prod_{(u_p, v_q) \in E(G_2)} \left( d_{G_1}(u) n_2 + d_{G_2}(v_j) \right)^2$$

$$\times \prod_{(u, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left[ \left( d_{G_1}(u) n_2 + d_{G_2}(v_j) \right) + d_{G_1}(u_p) n_2 + d_{G_2}(v_j) \right]^{n_2}$$

$$= \prod_{u, v_j \in V(G_1)} \prod_{(u_p, v_q) \in E(G_2)} \left( 4d_{G_1}^2(u_i) n_2^2 + (d_{G_2}(v_j) + d_{G_2}(v_q))^2 \right)$$

$$\times \prod_{(u, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left[ n_2^2(d_{G_1}(u_i) + d_{G_1}(u_p))^2 + 4d_{G_2}^2(v_j) \right.$$

$$\left. + 4n_2d_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_1}(u_p))^2 \right)^{n_2}.$$  

However, from the inequality (2.1), we get

$$\leq \prod_{u, v_j \in V(G_1)} \left[ \left( \frac{4d_{G_1}^2(u_i) n_2^3 + HM(G_2) + 4d_{G_1}(u_i) n_2M(G_2)}{n_2} \right)^{n_2} \right]$$

$$\times \prod_{(u, u_p) \in E(G_1)} \left[ \left( \frac{n_2^3(d_{G_1}(u_i) + d_{G_1}(u_p))^2 + 4M_1(G_2) + 8n_2 M_1(d_{G_1}(u_i) + d_{G_1}(u_p))}{n_2} \right)^{n_2} \right]$$

$$\leq \frac{1}{(n_2)^{m_1 m_2}} \left[ \left( \frac{4M_1(G_1) n_2^3 + n_1 HM(G_2) + 4m_1 n_2 M_1(G_2)}{n_1} \right)^{m_1 m_2} \right]$$

$$\times \frac{1}{(n_2)^{m_1 n_2} m_1} \left[ \left( \frac{n_2^3 HII(G_1) + 4m_1 M_1(G_2) + 8n_2 m_1 M_1(G_1)}{m_1} \right)^{n_2 m_1} \right].$$

\[\square\]

Example 3.9. Let $C_p$ and $C_q$ be cycles with $n \geq 3$ vertices. Then

$$HII(C_p[C_q]) = \leq \frac{1}{(pq)^p} \left( 16pq^3 + 16pq^2 + 16pq \right)^p \times \frac{1}{(pq)^q} \left( 4p^3q + 16pq + 32pq^2 \right)^q.$$  

The disjunction $G_1 \oplus G_2$ of graphs $G_1$ and $G_2$ is the graph with a vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j)$ is adjacent to $(u_k, v_l)$ whenever $u_i u_k \in E(G_1)$ or $v_j v_l \in E(G_2)$. The degree of a vertex $(u_i, v_j)$ of $G_1 \oplus G_2$ is given by

$$d_{G_1 \oplus G_2}(u_i, v_j) = n_2d_{G_1}(u_i) + n_1d_{G_2}(v_j) - d_{G_1}(u_i)d_{G_2}(v_j).$$
**Theorem 3.10.** Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

$$
\text{HII}(G_1 \otimes G_2) \leq \frac{1}{(n_1m_2)^{n_1m_2}} \left[ 4m_2n_2^2M_1(G_1) + n_1^3HM(G_2) - 4M_1(G_1)HM(G_2) \\
+ 8n_1n_2m_1M_1(G_2) - 8n_2M_1(G_1)M_1(G_2) - 8n_1m_1HM(G_2) \right]^{n_1m_2}
$$

$$
\times \frac{1}{(n_2m_1)^{n_2m_1}} \left[ 4n_1^2m_1M_1(G_2) + n_2^3HM(G_1) - 4M_1(G_2)HM(G_1) \\
+ 8n_1n_2m_1M_1(G_1) - 8n_1M_1(G_2)M_1(G_1) - 8n_1m_2HM(G_1) \right]^{n_2m_1}
$$

$$
\times \frac{1}{(n_1n_2)^{n_1n_2}} \left[ 4n_2^2M_1(G_1) + 4n_1^3M_1(G_2) - 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 \\
- 8n_2m_1M_1(G_1) - 8n_1M_1(M_1(G_2)) \right]^{n_1n_2}.
$$

**Proof.** By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \otimes G_2$, we have

$$
\text{HII}(G_1 \otimes G_2) = \prod_{u_i \in V(G_1)} \prod_{(v_j, v_k) \in E(G_2)} \left( 2n_2d_{G_1}(u_i) + n_1(d_{G_1}(v_j) + d_{G_1}(v_k)) \right)^2 \\
- 2d_{G_1}(u_i)((d_{G_2}(v_j) + d_{G_2}(v_k)))^2 \\
\times \prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left( 2n_1d_{G_1}(v_j) + n_2(d_{G_1}(u_i) + d_{G_1}(u_p)) \right)^2 \\
- 2d_{G_2}(v_j)((d_{G_2}(u_i) + d_{G_2}(u_p)))^2 \\
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 4n_2^2d_{G_1}^2(u_i) + n_1^2(d_{G_1}(v_j) + d_{G_1}(v_k))^2 \right) \\
- 4d_{G_1}^2(u_i)((d_{G_2}(v_j) + d_{G_2}(v_k)))^2 \\
- 4n_1n_2d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_k)) - 4n_1d_{G_1}(u_i)(d_{G_1}(v_j) + d_{G_1}(v_k))^2 \\
\times \prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left( 4n_1^2d_{G_1}^2(v_j) + n_2^2(d_{G_1}(u_i) + d_{G_1}(u_p))^2 \right) \\
- 4d_{G_2}^2(v_j)((d_{G_2}(u_i) + d_{G_2}(u_p)))^2 \\
- 4n_1n_2d_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_1}(u_p)) - 4n_2d_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_1}(u_p))^2 \\
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 4n_2^2d_{G_1}^2(u_i) + 4n_1^2d_{G_2}^2(v_j) - 4d_{G_1}^2(u_i)d_{G_2}^2(v_j) \right) \\
+ 8n_1n_2d_{G_1}(u_i)d_{G_2}(v_j) - 4n_2d_{G_1}(u_i)d_{G_2}(v_j) - 4n_1d_{G_1}(u_i)d_{G_2}(v_j).
$$

However, from the inequality (2.1), we get
\[
\sum_{u_i \in V(G_1)} \sum_{(v_j, v_k) \in E(G_2)} \left( 4n_2^2d_{G_2}^2(v_j) + n_2^3(d_{G_2}(v_j) + d_{G_2}(v_k))^2 \right)
- 4d_{G_2}^2(v_j)(d_{G_2}(v_j) + d_{G_2}(v_k))^2 + 4n_1n_2d_{G_2}(v_j)(d_{G_2}(v_j) + d_{G_2}(v_k))
- 8n_2d_{G_2}^2(v_j)(d_{G_2}(v_j) + d_{G_2}(v_k)) - 4n_1d_{G_2}(v_j)(d_{G_2}(v_j) + d_{G_2}(v_k))^2
\]
\[
\leq \frac{1}{(n_1m_2)^{n_1m_2}} \left[ 4m_2n_2^2M_1(G_1) + n_1^3HM(G_2) - 4M_1(G_1)HM(G_2) + 8n_1n_2m_1M_1(G_2) \right.
- 8n_2M_1(G_1)M_1(G_2) - 8n_1m_1HM(G_2)
\]
\[
\times \frac{1}{(n_2m_1)^{n_2m_1}} \left[ 4n_1^2m_1M_1(G_2) + n_1^3HM(G_1) - 4M_1(G_2)HM(G_1) + 8n_1n_2m_2M_1(G_1) \right.
- 8n_1M_1(G_2)M_1(G_1) - 8n_2m_2HM(G_1)
\]
\[
\times \frac{1}{(n_1n_2)^{n_1n_2}} \left[ 4n_1^3M_1(G_1) + 4n_1^3M_1(G_2) - 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 \right.
- 8n_2m_2M_1(G_1) - 8n_1m_1M_1(G_2) \left. \right]^{n_1n_2}.
\]
Example 3.11. Let $C_p$ and $C_q$ be cycles with $n \geq 3$ vertices. Then

\[ \text{HII}(C_p \otimes C_q) = \frac{1}{(pq)^2} \left( 16pq^3 + 4p^3 q - 256pq + 32p^2 q^2 - 128q^2 p - 128p^2 q \right) \]

Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices and $m_1$ and $m_2$ edges, respectively. The join $G_1 \vee G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

Theorem 3.12. Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

\[ \text{HII}(G_1 \vee G_2) = \frac{1}{m_1^2} \left( HM(G_1) + 4n_1^2 m_1 + 4n_1 M_1(G_1) \right) \]

\[ \times \frac{1}{m_2^2} \left( HM(G_2) + 4n_2^2 m_2 + 4n_1 M_1(G_1) \right) \]

\[ \times \frac{1}{n_1 n_2} \left[ n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2 + n_1 n_2 (n_1 + n_2)^2 \right] \]

\[ + 4m_1 (n_1 + n_2) + 4m_2 (n_1 + n_2) \]

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \vee G_2$, we have

\[ \text{HII}(G_1 \vee G_2) = \prod_{(u_i, v_j) \in E(G_1 \vee G_2)} \left( d_{G_1 \vee G_2}(u_i, v_j) + d_{G_1 \vee G_2}(u_p, v_q) \right) \]

\[ = \prod_{(u_i, v_j) \in E(G_1)} \left( d_{G_1}(u_i) + n_2 \right) \left( d_{G_1}(u_p) + n_2 \right) \]

\[ \times \prod_{(v_j, v_k) \in E(G_2)} \left( d_{G_2}(v_j) + n_1 \right) \left( d_{G_2}(v_k) + n_1 \right) \]

\[ \times \prod_{u_i \in V(G_1), v_j \in V(G_2)} \left( d_{G_1}(u_i) + d_{G_1}(u_p) \right) \]

\[ \times \prod_{v_j, v_k \in E(G_2)} \left( d_{G_2}(v_j) + d_{G_2}(v_k) \right) \]

\[ + \frac{1}{n_1 n_2} \left[ n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2 + n_1 n_2 (n_1 + n_2)^2 \right] \]

\[ + 4m_1 (n_1 + n_2) + 4m_2 (n_1 + n_2) \]
However, from the inequality (2.1), we get
\[
= \left[ \frac{\sum_{(u_i, u_p) \in E(G_1)} \left( (d_{G_1}(u_i) + d_{G_1}(u_p))^2 + 4n_1^2 + 4n_1 (d_{G_1}(u_i) + d_{G_1}(u_p)) \right)}{m_1} \right]^{m_1} \times \left[ \frac{\sum_{(v_j, v_q) \in E(G_2)} \left( (d_{G_2}(v_j) + d_{G_2}(v_q))^2 + 4n_2^2 + 4n_2 (d_{G_2}(v_j) + d_{G_2}(v_q)) \right)}{m_2} \right]^{m_2} \times \left[ \frac{\sum_{u_i \in V(G_1)} \sum_{v_j \in V(G_2)} \left( (d_{G_1}^2(u_i) + d_{G_2}^2(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + (n_1 + n_2)^2 + 2(n_1 + n_2)d_{G_1}(u_i) + 2(n_1 + n_2)d_{G_2}(v_j) \right)}{n_1n_2} \right]^{n_1n_2}
\]
\[= \frac{1}{m_1^{m_1}} \left[ HM(G_1) + 4n_1^2m_1 + 4n_1M_1(G_1) \right]^{m_1} \times \frac{1}{m_2^{m_2}} \left[ HM(G_2) + 4n_2^2m_2 + 4n_2M_1(G_2) \right]^{m_2} \times \frac{1}{(n_1n_2)^{n_1n_2}} \left[ n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_1n_2(n_1 + n_2)^2 + 4m_1(n_1 + n_2) + 4n_2(n_1 + n_2) \right]^{n_1n_2}.
\]

\[\Box\]

**Example 3.13.** Let $C_p$ and $C_q$ be cycles with $n \geq 3$ vertices. Then
\[
\text{HII}(C_p \lor C_q) \leq \frac{1}{p^p} \left[ 16p + 4p^2 + 16p^2 \right]^p \times \frac{1}{q^q} \left[ 16q + 4q^3 + 16q^2 \right]^q \times \frac{1}{(pq)^p} \left[ 16pq + pq(p + q)^2 + 4p(p + q) + 4q(p + q) \right]^p.
\]

The symmetric difference $G_1 \oplus G_2$ of two graphs $G_1$ and $G_2$ is the graph with a vertex set $V(G_1) \times V(G_2)$ in which $(u_{ki}, v_{lj})$ is adjacent to $(u_{ki}, v_{lj})$ whenever $u_i$ is adjacent to $u_k$ in $G_1$ or $v_i$ is adjacent to $v_l$ in $G_2$, but not both. The degree of a vertex $(u_i, v_j)$ of $G_1 \oplus G_2$ is given by $d_{G_1 \oplus G_2}(u_i, v_j) = n_2d_{G_1}(u_i) + n_1d_{G_2}(v_j) - 2d_{G_1}(u_i)d_{G_2}(v_j)$.

**Theorem 3.14.** Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respec-
tively. Then

$$
\text{HII}(G_1 \oplus G_2) = \frac{1}{(n_1 n_2)^{m_1 m_2}} \left[ 4m_2 n_2^2 M_1(G_1) + n_1^2 HM(G_2) - 16M_1(G_1)HM(G_2) \\
+ 8n_1 n_2 m_1 M_1(G_2) - 16m_2 M_1(G_1)M_1(G_2) - 16m_1 m_1 HM(G_2) \right]^{n_1 m_2} \\
\times \frac{1}{(n_2 m_1)^{n_2 m_1}} \left[ 4n_1^2 m_1 M_1(G_2) + n_2^2 HM(G_1) - 16M_2(G_2)HM(G_1) \\
+ 8n_1 n_2 m_2 M_1(G_1) - 16m_1 M_1(G_2)M_1(G_1) - 16m_1 m_1 HM(G_1) \right]^{n_2 m_1} \\
\times \frac{1}{(n_1 n_2)^{n_1 n_2}} \left[ 4n_2^2 M_1(G_1) + 4n_1^3 M_1(G_2) - 16M_1(G_1)M_1(G_2) + 32n_1 n_2 m_1 m_2 \\
- 32n_2 m_2 M_1(G_1) - 32m_1 m_1 M_1(G_2) \right]^{n_1 n_2}
$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \oplus G_2$, we have

$$
\text{HII}(G_1 \oplus G_2) = \prod_{(u_i, v_j) \in E(G_1 \oplus G_2)} \left( d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_p, v_q) \right)^2 \\
= \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( (n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2d_{G_1}(u_i)d_{G_2}(v_j)) \\
+ (n_2 d_{G_1}(u_p) + n_1 d_{G_2}(v_q) - 2d_{G_1}(u_p)d_{G_2}(v_q)) \right)^2 \\
\times \prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left[ \left( (n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2d_{G_1}(u_i)d_{G_2}(v_j)) \\
+ (n_2 d_{G_1}(u_p) + n_1 d_{G_2}(v_q) - 2d_{G_1}(u_p)d_{G_2}(v_q)) \right)^2 \right]^{n_2} \\
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( (n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2d_{G_1}(u_i)d_{G_2}(v_j)) \\
+ (n_2 d_{G_1}(u_p) + n_1 d_{G_2}(v_q) - 2d_{G_1}(u_p)d_{G_2}(v_q)) \right)^2 \\
= \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( (2n_2 d_{G_1}(u_i) + n_1 (d_{G_2}(v_j) + d_{G_2}(v_q))) \\
- 4d_{G_1}(u_i) \left( (d_{G_2}(v_j) + d_{G_2}(v_q)) \right)^2 \\
\prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \left[ \left( (2n_1 d_{G_2}(v_j) + n_2 (d_{G_1}(u_i) + d_{G_1}(u_p))) \\
- 4d_{G_2}(v_j) \left( (d_{G_1}(u_i) + d_{G_1}(u_p)) \right)^2 \right]^{n_2} \right]
$$
However, from the inequality (2.1), we get

\[
\sum_{u_i \in V(G_1)} \sum_{v_j \in V(G_2)} \left( \frac{\left( 4n_1^2 d^2_{G_1}(u_i) + n_1^2 (d_{G_1}(u_i) + d_{G_2}(v_j))^2 \right)^2}{n_1 m_2} \right)
- \frac{16d^2_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2 + 4n_1 n_2 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))}{n_1 m_2} 
- \frac{16n_1 d^2_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_2}(v_j))^2 - 8n_1 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2}{n_1 m_2} 
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 2n_2 d_{G_1}(u_i) + 2n_1 d_{G_2}(v_j) - 4d_{G_1}(u_i) d_{G_2}(v_j) \right)^2
\]

\[
+ n_2^2 \frac{16d^2_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2 + 4n_1 n_2 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))}{n_2 m_1} 
- \frac{16n_1 d^2_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_2}(v_j))^2 - 8n_1 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2}{n_2 m_1} 
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 16d^2_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2 + 4n_1 n_2 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q)) 
- \frac{16n_1 d^2_{G_2}(v_j)(d_{G_1}(u_i) + d_{G_2}(v_j))^2 - 8n_1 d_{G_1}(u_i)(d_{G_2}(v_j) + d_{G_2}(v_q))^2}{n_2 m_1} 
\times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \left( 4n_1^2 d^2_{G_1}(u_i) + 4n_1^2 d^2_{G_2}(v_j) - 16d^2_{G_1}(u_i)d^2_{G_2}(v_j) \right)
\]

However, from the inequality (2.1), we get
\[\begin{align*}
&\leq \frac{1}{(n_1m_2)^{n_1m_2}} \left[ 4m_2n_2^2 M_1(G_1) + n_1^3 H M(G_2) - 16M_1(G_1) H M(G_2) + 8n_1n_2m_1M_1(G_2) \\
&- 16n_2M_1(G_1)M_1(G_2) - 16n_1m_1 H M(G_2) \right]^{n_1m_2} \\
&\times \frac{1}{(n_2m_1)^{n_2m_1}} \left[ 4n_1^2m_1M_1(G_2) + n_2^3 H M(G_1) - 16M_1(G_2) H M(G_1) + 8n_1n_2m_2M_1(G_1) \\
&- 16n_1M_1(G_2)M_1(G_1) - 16n_2m_2 H M(G_1) \right]^{n_2m_1} \\
&\times \frac{1}{(n_1n_2)^{n_1n_2}} \left[ 4n_1^2M_1(G_1) + 4n_2^3M_1(G_2) - 16M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 \\
&- 32n_2m_2M_1(G_1) - 32n_1m_1M_1(G_2) \right]^{n_1n_2}.
\end{align*}\]

Example 3.15. Let \(C_p\) and \(C_q\) be cycles with \(n \geq 3\) vertices. Then

\[
\text{HII}(C_p \oplus C_q) \leq \frac{1}{(pq)^{pq}} \left[ 16q^3p + 16p^3q - 1024pq + 32p^2q^2 - 256p^2q - 256p^2q \right]^{pq} \\
\times \frac{1}{(pq)^{pq}} \left[ 16p^3q + 4q^3p - 1024pq \right] + 32p^2q^2 - 256p^2q - 256p^2q \right]^{pq} \\
\times \frac{1}{(pq)^{pq}} \left[ 16q^3p + 16p^3q - 256pq + 32p^2q^2 - 128q^2p - 128p^2q \right]^{pq}.
\]

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**Author information**

Akbar Jahanbani, Department of Mathematics, Shahrood University of Technology, Iran.
E-mail: Akbar . jahanbani92@gmail . com

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