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The Multiplicative Hyper- Zagreb index of Graph Operations

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Abstract Let G be a graph of order n with vertices labeled as $v_1, v_2, \ldots, v_n$. Let d_i be the degree of the vertex v_i, for $i = 1, 2, \ldots, n$. The multiplicative Hyper- Zagreb index, is defined as, $\mathtt{HII}(G) = \prod_{uv\in E(G)}(d_G(u) + d_G(v))^2$. In this paper the upper bounds on the multiplicative Hyper- Zagreb indices of the the Cartesian product, corona product, composition, disjunction, join and symmetric difference of graphs are computed. We apply some of our results to compute the multiplicative Hyper- Zagreb index.

1 Introduction

All graphs considered in this paper are assumed to be simple. Let G be a (molecular) graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set E(G). If v_i and v_j are adjacent vertices of G, then the edge connecting them is denoted by v_iv_j. By d_i we denote the degree of the vertex $v_i \in V(G)$. We consider only simple connected graphs, i.e. connected graphs without loops and multiple edges. A topological index Top(G_1) of a graph G_1, is a number with this property that for every graph G_2 isomorphic to G, Top(G_1) = Top(G_2). The Cartesian product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(a, x)(b, y)$ is an edge of $G_1 \boxtimes G_2$ if $a = b$ and $xy \in E(G_2)$, or $ab \in E(G_1)$ and $x = y$. If (a, x) is a vertex of $G_1 \boxtimes G_2$, then $d_{G_1\boxtimes G_2}((a, x)) = d_{G_1}(a) + d_{G_2}(x)$. The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_1)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (a, x) is adjacent to (b, y) whenever b is adjacent to y or $a = b$ and x is adjacent to y. If (a, x) is a vertex of $G_1[G_2]$, then $d_{G_1[G_2]}((a, x)) = |V(G_2)|d_{G_1}(a) + d_{G_2}(x)$. The corona product $G_1 \circ G_2$ is defined as the graph obtained from G_1 and G_2 by taking one copy of G_1 and $|V(G_2)|$ copies of H and then by joining with an edge each vertex of the i^{th} copy of H which is named (G_2, i) with the i^{th} vertex of G for $i = 1, 2, ..., |V(G_1)|$. If u is a vertex of $G_1 \circ G_2$, then

$$d_{G_1\circ G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \text{if} \quad u \in V(G_1) \\ d_{G_2}(u) + 1 & \text{if} \quad u \in (G_2, i). \end{cases}$$

The join $G_1 + G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \quad \text{and} \quad v \in V(G_2)\}$. The symmetric difference $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \oplus V(G_2)$ and $E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1) \quad \text{or} \quad u_2v_2 \in E(G_2) \quad \text{but not both}\}$. The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G_1), u_2v_2 \in E(G_2)\}$.
This paper is organized as follows. In Section 2, we state some previously known results also we study the multiplicative Hyper- Zagreb index of a graph. In Section 3, we the Hyper- Zagreb index of the Cartesian product, corona product, composition, disjunction, join and symmetric difference of graphs are computed.

2 Preliminaries and known results

In this section, we study the Hyper-multiplicative Zagreb index of a graph and some exact formulae for the Hyper-multiplicative Zagreb index of some well-known graphs are presented. We begin with the definition and crucial theorem related to theorem properties of some graph operations. Let us begin with a few examples, then we will give a crucial theorem related to distance properties of some graph operations. In mathematical chemistry, there is a large number of topological indices of the form

$$TI = TI(G) = \sum_{v_i, v_j \in E(G)} \mathbb{F}(d_i, d_j).$$

The most popular topological indices of this kind are the:

- first Zagreb index, $M_1(G) = \sum_{uv \in E(G)} \big(d_G(u) + d_G(v)\big) = \sum_{u \in V(G)} d_G(u)^2$,
- second Zagreb index, $M_2(G) = \sum_{uv \in E(G)} \Big(d_G(u) d_G(v)\Big)$,
- hyper-Zagreb index, $HM(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big)^2$,
- first multiplicative Zagreb index, $\prod_1(G) = \prod_{u \in V(G)} d_G(u)^2$,
- second multiplicative Zagreb index, $\prod_2(G) = \prod_{uv \in E(G)} \Big(d_G(u) d_G(v)\Big)$,
- hyper-multiplicative Zagreb index, $\mathtt{HII}(G) = \prod_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big)^2$.

Note that there are several more indices, see ([3], [14], [15]) . The $Zagreb$ indices are widely studied degree-based topological indices, and were introduced by $Gutman$ and $Trinajstić$ [5] in 1972, there was a vast research on comparing $Zagreb$ indices see ([9], [10]). A survey on the first $Zagreb$ index see [4]. The Hyper-multiplicative Zagreb index can also be expressed as a sum over edges of G [15],

$$\mathtt{HII}(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$

Readers interested in more information on Hyper-multiplicative Zagreb index can be referred to ([2], [16], [17], [18]). Recently, the analogous concepts of the sigma index of graphs operations [11] and the Nano-Zagreb index and multiplicative Nano-Zagreb index of some graph operations [12] were put forward.

Proposition 2.1. *[15] Let K_n be a complete graph with n vertices. Then*

$$\mathtt{HII}(K_n) = \prod_{uv \in E(K_n)} \Big(d(u) + d(v)\Big)^2 = [2(n-1)]^{n(n-1)}.$$

Proposition 2.2. *[15] Let $K_{m,n}$ be a complete bipartite graph with $1 \leqslant m \leqslant n$. Then*

$$\mathtt{HII}(K_{n,m}) = \prod_{uv \in E(K_{n,m})} \Big(d(u) + d(v)\Big)^2 = (m+n)^{2mn}.$$

Proposition 2.3. *[15] Let $K_{1,n}$ be a star. Then*

$$\mathtt{HII}(K_{1,n}) = \prod_{uv \in E(K_{1,n})} \Big(d(u) + d(v)\Big)^2 = (n+1)^{2n}.$$

Proposition 2.4. *[15] Let C_n be a cycle with $n \geqslant 3$ vertices. Then*

$$\mathtt{HII}(C_n) = \prod_{uv \in E(C_n)} \Big(d(u) + d(v)\Big)^2 = [(2+2)]^n = 4^n.$$

Lemma 2.5. *(AM-GM inequality) Let $x_1, x_2, ..., x_n$ be nonnegative numbers. Then*

$$\frac{x_1 + x_2 + ... + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 ... x_n} \tag{2.1}$$

holds with equality if and only if all the $x_k^,s$ are equal.

3 Multiplicative Hyper- Zagreb indices of Graph Operations

In this section, we the multiplicative Hyper- Zagreb indices of the Cartesian product, composition, join and disjunction of graphs are computed. We apply some of our results to compute the small Zagreb index. We begin this section with standard Lemma as follow.

Lemma 3.1. *Let G_1 and G_2 be two connected graphs, then we have:*

$$
\begin{aligned}
(a)|V(G_1 \times G_2)| &= |V(G_1 \vee G_2)| = |V(G_1[G_2])| = |V(G_1 \oplus G_2)| = |V(G_1)||V(G_2)|,\\
|E(G_1 \times G_2)| &= |E(G_1)||V(G_2)| + |V(G_1)||E(G_2)|,\\
|E(G_1 + G_2)| &= |E(G_1)| + |E(G_2)| + |V(G_1)V(G_2)|,\\
|E(G_1[G_2])| &= |E(G_1)||V(G_2)|^2 + |E(G_1)||V(G_2)|,\\
|E(G_1 \vee G_2)| &= |V(G_1)||V(G_2)|^2 + |E(G_1)||V(G_1)|^2 - 2|E(G_1)||E(G_2)|,\\
|E(G_1 \oplus G_2)| &= |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 42|E(G_1)||E(G_2)|.
\end{aligned}
$$

(b) $G_1 \times G_2$ *is connected if and only if* G_1 *and* G_2 *are connected.*

(c) *If* (a,b) *is a vertex of* $G_1 \times G_2$ *then* $d_{G_1 \times G_2}\big((a,b)\big) = d_{G_1}(a) + d_{G_2}(b)$.

(d) *If* (a,b) *is a vertex of* $G_1[G_2]$ *then* $d_{G_1[G_2]}\big((a,b)\big) = |V(G_1)|d_{G_2}(a) + d_{G_2}(b)$.

(e) *If* (a,b) *is a vertex of* $G_1 \oplus G_2$ *or* $G_1 \otimes G_2$, *we have* :

$d_{G_1 \oplus G_2}\big((a,b)\big) = |V(G_1)|d_{G_1}(a) + |V(G_1)|d_{G_2}(b) - 2d_{G_1}(a)d_{G_2}(b)$.

$d_{G_1 \otimes G_2}\big((a,b)\big) = |V(G_2)|d_{G_1}(a) + |V(G_1)|d_{G_2}(b) - d_{G_1}(a)d_{G_2}(b)$.

(f) *If* u *is a vertex of* $G_1 \vee G_2$ *then we have* :

$$
d_{G_1 \vee G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \textit{if} \quad u \in V(G_1) \\ d_{G_2}(u) + |V(G_1)| & \textit{if} \quad u \in V(G_2). \end{cases}
$$

Proof. The parts (a) and (b) are consequence of definitions and some famous results of the book of Imrich and Klavzar [8]. For the proof of (c-f) we refer to [13]. □

The Cartesian product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \boxtimes G_2$ if
either $u_i = u_k$ and $v_j v_l \in E(G_2)$,
or $u_i u_k \in E(G_1)$ and $v_j = v_l$.

Theorem 3.2. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$
\begin{aligned}
\mathtt{HII}(G_1 \boxtimes G_2) &= \frac{1}{(n_1 m_2)^{n_1 m_2}} \Big(4m_1 M_1(G_1) + n_2 HM(G_1) + 8m_2 M_1(G_2)\Big)^{n_1 m_2} \\
&\times \frac{1}{(n_2 m_1)^{n_2 m_1}} \Big(4m_2 M_1(G_2) + n_1 HM(G_2) + 8m_1 M_1(G_1)\Big)^{n_2 m_1}.
\end{aligned}
$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \boxtimes G_2$, we have

$$
\mathtt{HII}(G_1 \boxtimes G_2) = \prod_{(u_i,v_j)(u_p,v_q) \in E(G_1 \boxtimes G_2)} \Big(d_{G_1 \boxtimes G_2}(u_i, v_j) + d_{G_1 \boxtimes G_2}(u_p, v_q)\Big)^2.
$$

This actually can be written as

$$
\begin{aligned}
&= \prod_{u_i\in V(G_1)} \prod_{(v_j,v_q)\in E(G_2)} \Big(4d^2_{G_1}(u_i) + \big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2 \\
&+ 4d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)\Big) \\
&\times \prod_{v_j\in V(G_2)} \prod_{(u_i,u_p)\in E(G_1)} \Big(4d^2_{G_2}(v_j) + \big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2 \\
&+ 4d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)\Big).
\end{aligned}
$$

However, from the inequality (2.1), we get

$$
\begin{aligned}
&\leqslant \Bigg[\sum_{u_i\in V(G_1)} \sum_{(v_j,v_q)\in E(G_2)} \Big(4d^2_{G_1}(u_i) + \big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2 \\
&+ 4d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)\Big)\Bigg]^{n_1m_2} \\
&\times \Bigg[\sum_{v_j\in V(G_2)} \sum_{(u_i,u_p)\in E(G_1)} \Big(4d^2_{G_2}(v_j) + \big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2 \\
&+ 4d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)\Big)\Bigg]^{n_2m_1}
\end{aligned}
$$

$$
\leqslant \left[\frac{\sum_{(v_j,v_q)\in E(G_2)} \Big(4M_1(G_1) + n_1\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2 + 8m_1\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)\Big)}{n_1m_2}\right]^{n_1m_2}
$$

$$
\times \left[\frac{\sum_{(u_i,u_p)\in E(G_1)} \Big(4M_1(G_2) + n_2\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2 + 8m_2\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)\Big)}{n_2m_1}\right]^{n_2m_1}
$$

$$
\begin{aligned}
&= \frac{1}{(n_1m_2)^{n_1m_2}}\Big(4m_1M_1(G_1) + n_2HM(G_1) + 8m_2M_1(G_2)\Big)^{n_1m_2} \\
&\times \frac{1}{(n_2m_1)^{n_2m_1}}\Big(4m_2M_1(G_2) + n_1HM(G_2) + 8m_1M_1(G_1)\Big)^{n_2m_1}.
\end{aligned}
$$

□

Remark 3.3. [1] For a cycle graph with n vertices, we have, $HM(C_n) = 16n, M_1(C_n) = 4n$.

Example 3.4. Let C_p and C_q be cycles with $n \geqslant 3$ vertices. Then

$$
\mathtt{HII}(C_p \boxtimes C_q) = \frac{1}{(pq)^{pq}}\left[\Big(\big(16p^2+16pq+32q^2\big)\times\big(16q^2+16pq+32p^2\big)\Big)^{pq}\right].
$$

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph Γ obtained by taking one copy of G_1 (which has n_1 vertices) and n_2 copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of $G_2, i = 1, 2, ..., n_1$. Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two graphs such that $V(G) = \{u_1, u_2, ..., u_{n_1}\}, |E(G_1)| = m_1$ and $V(G_2) = \{v_1, v_2, ..., v_{n_2}\}, |E(G_2)| = m_2$. Then it follows from the definition of the corona product that $G_1 \circ G_2$ has $n_1(1+n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges, where $V(G_1 \circ G_2) = \{(u_i, v_j), i = 1, 2, ..., n_1; j = 0, 1, 2, ..., n_2\}$ and $E(G_1 \circ G_2) = \{((u_i, v_0), (u_k, v_0)), (u_i, u_k) \in E(G_1)\} \cup \{((u_i, v_j), (u_i, v_l)), (v_j, v_l) \in E(G_2), i = 1, 2, ..., n_1\} \cup \{((u_i, v_0), (u_i, v_l)), l = 1, 2, ..., n_2, i = 1, 2, ..., n_1\}$. It is clear that if G_1 is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_1 \circ G_2$.

Theorem 3.5. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$\begin{aligned}\mathtt{HII}(G_1 \circ G_2) = &\frac{1}{m_1^{m_1}}\Big(HM(G_1) + 4n_1 n_2^2 + 4n_2 M_1(G_1)\Big)^{m_1}\\ &\times \frac{1}{(n_1 n_2)^{n_1 n_2}}\Big(n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2 + n_1 n_2 (n_2+1)^2\\ &+ 4n_2 m_1 (n_2+1) + 4n_1 m_2 (n_2+1)\Big)^{n_1 n_2}\\ &\times \frac{1}{(n_1 m_2)^{n_1 m_2}}\Big(n_1 HM(G_2) + 4n_1 m_2 + 4n_1 M_1(G_2)\Big)^{n_1 m_1}.\end{aligned}$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \circ G_2$, we have

$$\begin{aligned}\mathtt{HII}(G_1 \circ G_2) = &\prod_{(u_i,v_j)(u_p,v_q)\in E(G_1\circ G_2)}\Big(d_{G_1\circ G_2}(u_i,v_j) + d_{G_1\circ G_2}(u_p,v_q)\Big)^2\\ &\prod_{(u_i,u_p)\in E(G_1)}\Big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big) + 2n_2\Big)^2\\ &\times \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}\big(d_{G_1}(u_i) + d_{G_2}(v_j)\big) + \big(n_2+1\big)\Big)^2\\ &\times \prod_{u_i\in V(G_1)}\prod_{(v_j,v_q)\in E(G_2)}\Big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big) + 2\Big)^2\\ = &\prod_{(u_i,u_p)\in E(G_1)}\Big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4n_2^2 + 4n_2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\Big)\\ &\times \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}\Big(d_{G_1}(u_i) + d_{G_2}(v_j)\big)^2 + \big(n_2+1\big)^2\\ &+ 2\big(n_2+1\big)\big(d_{G_1}(u_i) + d_{G_2}(v_j)\big)\Big)\\ &\times \prod_{u_i\in V(G_1)}\prod_{(v_j,v_q)\in E(G_2)}\Big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2 + 4 + 4(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\Big).\end{aligned}$$

However, from the inequality (2.1), we get

$$\leqslant \left[\frac{\sum_{(u_i,u_p)\in E(G_1)}\Big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4n_2^2 + 4n_2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\Big)}{m_1}\right]^{m_1}$$

$$\times \left[\frac{\sum_{u_i \in V(G_1)} \sum_{v_j \in V(G_2)} \Big(d^2_{G_1}(u_i) + d^2_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + \big(n_2+1\big)^2 + 2\big(n_2+1\big)\big(d_{G_1}(u_i) + d_{G_2}(v_j)\big)\Big)}{n_1 n_2} \right]^{n_1 n_2}$$

$$\times \left[\frac{\sum_{u_i \in V(G_1)} \sum_{(v_j, v_q) \in E(G_2)} \Big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2 + 4 + 4(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\Big)}{n_1 m_2} \right]^{n_1 m_2}$$

$$= \frac{1}{m_1^{m_1}} \Big(HM(G_1) + 4n_1 n_2^2 + 4n_2 M_1(G_1) \Big)^{m_1}$$
$$\times \frac{1}{(n_1 n_2)^{n_1 n_2}} \Big(n_2 M_1(G_1) + n_1 M_1(G_2 + 8m_1 m_2) + n_1 n_2 (n_2+1)^2$$
$$+ 4n_2 m_1 (n_2+1) + 4n_1 m_2 (n_2+1) \Big)^{n_1 n_2}$$
$$\times \frac{1}{(n_1 m_2)^{n_1 m_2}} \Big(n_1 HM(G_2) + 4n_1 m_2 + 4n_1 M_1(G_2) \Big)^{n_1 m_2}.$$

□

Remark 3.6. [1] For a path with n vertices, we have: $HM(P_n) = 16n - 30, M_1(P_n) = 4n - 6.$

Example 3.7. Let C_q and P_n be a cycle and path with $n \geqslant 3$ vertices. Then

$$\mathtt{HII}(C_q \circ P_n) = \frac{1}{q^q} \Big(16n + 4qn^2 + 4n(4n-6) \Big)^q \times \frac{1}{(qn)^{qn}} \Big(16qn + q(4n-6)$$
$$+ 8q(n-1)) + q(n-1)(n)^2 + 4q(n-1)(n) + 4q(n-1)(n) \Big)^{qn}$$
$$\times \frac{1}{(q(n-1))^{q(n-1)}} \Big(q(16n-30) + 4q(n-1) + 4q(4n-6) \Big)^{q(n-1)}.$$

Theorem 3.8. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$\mathtt{HII}(G_1[G_2]) \leqslant \frac{1}{(m_2)^{n_1 m_2}} \left[\frac{\Big(4M_1(G_1) n_2^3 + n_1 HM(G_2) + 4m_1 n_2 M_1(G_2) \Big)}{n_1} \right]^{n_1 m_2}$$
$$\times \frac{1}{(n_2)^{m_1 n_2^2}} \left[\frac{\Big(n_2^3 HM(G_1) + 4m_1 M_1(G_2) + 8n_2 m_2 M_1(G_1)}{m_1} \right]^{n_2^2 m_1}.$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1[G_2]$, we have

$$
\begin{aligned}
\mathtt{HII}(G_1[G_2]) &= \prod_{(u_i,v_j)(u_p,v_q)\in E(G_1[G_2])} \Big(d_{G_1[G_2]}(u_i,v_j) + d_{G_1[G_2]}(u_p,v_q)\Big)^2 \\
&= \prod_{u_i\in V(G_1)} \prod_{(v_j,v_q)\in E(G_2)} \Big(\big(d_{G_1}(u_i)n_2 + d_{G_2}(v_j)\big) + \big(d_{G_1}(u_i)n_2 + d_{G_2}(v_q)\big)\Big)^2 \\
&\times \prod_{(u_i,u_p)\in E(G_1)} \prod_{v_j\in V(G_2)} \left[\Big(\big(d_{G_1}(u_i)n_2 + d_{G_2}(v_j)\big) + \big(d_{G_1}(u_p)n_2 + d_{G_2}(v_j)\big)\Big)^2\right]^{n_2} \\
&= \prod_{u_i\in V(G_1)} \prod_{(v_j,v_q)\in E(G_2)} \Big(4d^2_{G_1}(u_i)n_2^2 + \big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2 \\
&+ 4d_{G_1}(u_i)n_2\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\Big) \\
&\times \prod_{(u_i,u_p)\in E(G_1)} \prod_{v_j\in V(G_2)} \Big(n_2^2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4d^2_dG_2(v_j) \\
&+ 4n_2dG_2(v_j)\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^{n_2}.
\end{aligned}
$$

However, from the inequality (2.1), we get

$$
\begin{aligned}
&\leqslant \prod_{u_i\in V(G_1)} \left[\frac{\Big(4d^2_{G_1}(u_i)n_2^3 + HM(G_2) + 4d_{G_1}(u_i)n_2M_(G_2)\Big)}{m_2}\right]^{m_2} \\
&\times \prod_{(u_i,u_p)\in E(G_1)} \left[\frac{\begin{array}{r}\Big(n_2^3\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4M_1(G_2) \\ +8n_2m_2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\end{array}}{n_2}\right]^{n_2^2} \\
&\leqslant \frac{1}{(m_2)^{n_1m_2}} \left[\frac{\Big(4M_1(G_1)n_2^3 + n_1HM(G_2) + 4m_1n_2M_1(G_2)\Big)}{n_1}\right]^{n_1m_2} \\
&\times \frac{1}{(n_2)^{m_1n_2^2}} \left[\frac{\Big(n_2^3\mathtt{HII}(G_1) + 4m_1M_1(G_2) + 8n_2m_2M_1(G_1)\Big)}{m_1}\right]^{n_2^2m_1}.
\end{aligned}
$$

□

Example 3.9. Let C_p and C_q be cycles with $n \geqslant 3$ vertices. Then

$$
\mathtt{HII}(C_p[C_q]) = \leqslant \frac{1}{(pq)^{pq}}\Big(16pq^3 + 16pq^2 + 16pq\Big)^{pq} \times \frac{1}{(pq)^{pq^2}}\Big(4^{2p}q^3 + 16pq + 32pq^2\Big)^{pq^2}.
$$

The disjunction $G_1 \otimes G_2$ of graphs G_1 and G_2 is the graphwith a vertex set $V(G_1)\times V(G_2)$ and (u_i,v_j) is adjacent to (u_k,v_l) whenever $u_iu_k \in E(G_1)$ or $v_jv_l \in E(G_2)$. The degree of a vertex (u_i,v_j) of $G_1 \otimes G_2$ is given by
$d_{G_1\otimes G_2}(u_i,v_j) = n_2d_{G_1}(u_i) + n_1d_{G_2}(v_j) - d_{G_1}(u_i)d_{G_2}(v_j)$.

Theorem 3.10. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$\begin{aligned}
\mathtt{HII}(G_1 \otimes G_2) \leqslant \frac{1}{(n_1 m_2)^{n_1 m_2}} \Big[& 4m_2 n_2^2 M_1(G_1) + n_1^3 HM(G_2) - 4M_1(G_1)HM(G_2) \\
& + 8n_1 n_2 m_1 M_1(G_2) - 8n_2 M_1(G_1)M_1(G_2) - 8n_1 m_1 HM(G_2)\Big]^{n_1 m_2} \\
& \times \frac{1}{(n_2 m_1)^{n_2 m_1}} \Big[4n_1^2 m_1 M_1(G_2) + n_2^3 HM(G_1) - 4M_1(G_2)HM(G_1) \\
& + 8n_1 n_2 m_2 M_1(G_1) - 8n_1 M_1(G_2)M_1(G_1) - 8n_1 m_2 HM(G_1)]^{n_2^2 m_1}\Big]^{n_2 m_1} \\
& \times \frac{1}{(n_1 n_2)^{n_1 n_2}} \Big[4n_2^3 M_1(G_1) + 4n_1^3 M_1(G_2) - 4M_1(G_1)M_1(G_2) + 32 n_1 n_2 m_1 m_2 \\
& - 8n_2 m_2 M_1(G_1) - 8n_1 m_1 M_1(G_2)]\Big]^{n_1 n_2}.
\end{aligned}$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \otimes G_2$, we have

$$\begin{aligned}
\mathtt{HII}(G_1 \otimes G_2) = & \prod_{u_i \in V(G_1)} \prod_{(v_j, v_q) \in E(G_2)} \Big(\big(2n_2 d_{G_1}(u_i) + n_1\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\big) \\
& - 2d_{G_1}(u_i)\big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\Big)^2 \\
& \times \prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \Big(\big(2n_1 d_{G_2}(v_j) + n_2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\big) \\
& - 2d_{G_2}(v_j)\big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\Big)^2 \\
& \times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \Big(\big(2n_2 d_{G_1}(u_i) + 2n_1 d_{G_2}(v_j) - 2d_{G_1}(u_i) d_{G_2}(v_j)\big)\Big)^2 \\
= & \prod_{u_i \in V(G_1)} \prod_{(v_j, v_q) \in E(G_2)} \Big(\big(4n_2^2 d_{G_1}^2(u_i) + n_1^2\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2\big) \\
& - 4d_{G_1}^2(u_i)\big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2 + 4n_1 n_2 d_{G_1}(u_i)\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big) \\
& - 8n_2 d_{G_1}^2(u_i)\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big) - 4n_1 d_{G_1}(u_i)\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2\Big) \\
& \times \prod_{(u_i, u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \Big(\big(4n_1^2 d_{G_2}^2(v_j) + n_2^2\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2\big) \\
& - 4d_{G_2}^2(v_j)\big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4n_1 n_2 d_{G_1}(v_j)\big(d_{G_1}(u_i) + d_{G_2}(u_p)\big) \\
& - 8n_1 d_{G_2}^2(v_j)\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big) - 4n_2 d_{G_2}(v_j)\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2\Big) \\
& \times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \Big(\big(4n_2^2 d_{G_1}^2(u_i) + 4n_1^2 d_{G_2}^2(v_j) - 4d_{G_1}^2(u_i) d_{G_2}^2(v_j)\big) \\
& + 8n_1 n_2 d_{G_1}(u_i) d_{G_2}(v_j) - 4n_2 d_{G_1}^2(u_i) d_{G_2}(v_j) - 4n_1 d_{G_1}(u_i) d_{G_2}^2(v_j)\Big).
\end{aligned}$$

However, from the inequality (2.1), we get

$$\leqslant \left[\frac{\begin{array}{c}\sum_{u_i\in V(G_1)}\sum_{(v_j,v_q)\in E(G_2)}\Big(\big(4n_2^2d_{G_1}^2(u_i)+n_1^2\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2\big)\\ -4d_{G_1}^2(u_i)\big(\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2+4n_1n_2d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)\\ -8n_2d_{G_1}^2(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)-4n_1d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2\Big)\end{array}}{n_1m_2}\right]^{n_1m_2}$$

$$\times \left[\frac{\begin{array}{c}\sum_{(u_i,u_p)\in E(G_1)}\sum_{v_j\in V(G_2)}\Big(\big(4n_1^2d_{G_2}^2(v_j)+n_2^2\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\big)\\ -4d_{G_2}^2(v_j)\big(\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2+4n_1n_2d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)\\ -8n_1d_{G_2}^2(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)-4n_2d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\Big)\end{array}}{n_2m_1}\right]^{n_2^2m_1}$$

$$\times \left[\frac{\begin{array}{c}\sum_{u_i\in V(G_1)}\sum_{v_j\in V(G_2)}\Big(\big(4n_2^2d_{G_1}^2(u_i)+4n_1^2d_{G_2}^2(v_j)-4d_{G_1}^2(u_i)d_{G_2}^2(v_j)\big)\\ +8n_1n_2d_{G_1}(u_i)d_{G_2}(v_j)-4n_2d_{G_1}^2(u_i)d_{G_2}(v_j)-4n_1d_{G_1}(u_i)d_{G_2}^2(v_j)\Big)\end{array}}{n_1n_2}\right]^{n_1n_2}$$

$$\leqslant \frac{1}{(n_1m_2)^{n_1m_2}}\Big[4m_2n_2^2M_1(G_1)+n_1^3HM(G_2)-4M_1(G_1)HM(G_2)+8n_1n_2m_1M_1(G_2)$$
$$-8n_2M_1(G_1)M_1(G_2)-8n_1m_1HM(G_2)\Big]^{n_1m_2}$$
$$\times \frac{1}{(n_2m_1)^{n_2m_1}}\Big[4n_1^2m_1M_1(G_2)+n_2^3HM(G_1)-4M_1(G_2)HM(G_1)+8n_1n_2m_2M_1(G_1)$$
$$-8n_1M_1(G_2)M_1(G_1)-8n_2m_2HM(G_1)\Big]^{n_2m_1}$$
$$\times \frac{1}{(n_1n_2)^{n_1n_2}}\Big[4n_2^3M_1(G_1)+4n_1^3M_1(G_2)-4M_1(G_1)M_1(G_2)+32n_1n_2m_1m_2$$
$$-8n_2m_2M_1(G_1)-8n_1m_1M_1(G_2)\Big]^{n_1n_2}.$$

□

Example 3.11. Let C_p and C_q be cycles with $n \geqslant 3$ vertices. Then

$$\begin{aligned}\mathtt{HII}(C_p \otimes C_q) \leqslant & \frac{1}{(pq)^{pq}}\left[16pq^3 + 4p^3q - 256pq + 32p^2q^2 - 128q^2p - 128p^2q\right]^{pq} \\ & \times \frac{1}{(pq)^{pq}}\left[16p^3q + 16q^3p - 256pq + 32p^2q^2 - 128p^2q - 128q^2p\right]^{pq} \\ & \times \frac{1}{(pq)^{pq}}\left[16q^3p + 16p^3q - 256pq + 32p^2q^2 - 128q^2p - 128p^2q\right]^{pq}.\end{aligned}$$

Let G_1 and G_2 be two graphs with n_1 and n_2 vertices and m_1 and $_2$ edges, respectively. The join $G_1 \vee G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

Theorem 3.12. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$\begin{aligned}\mathtt{HII}(G_1 \vee G_2) = & \frac{1}{m_1^{m_1}}\left[HM(G_1) + 4n_1^2m_1 + 4n_1M_(G_1)\right]^{m_1} \\ & \times \frac{1}{m_2^{m_2}}\left[HM(G_2) + 4n_2^2m_2 + 4n_1M_(G_1)\right]^{m_1} \\ & \times \frac{1}{(n_1n_2)^{n_1n_2}}\Big[n_2M_(G_1) + n_1M_(G_2) + 8m_1m_2 + n_1n_2(n_1+n_2)^2 \\ & + 4m_1(n_1+n_2) + 4m_2(n_1+n_2)\Big]^{n_1n_2}.\end{aligned}$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \vee G_2$, we have

$$\begin{aligned}\mathtt{HII}(G_1 \vee G_2) = & \prod_{(u_i,v_j)(u_p,v_q)\in E(G_1\vee G_2)} \Big(d_{G_1\vee G_2}(u_i,v_j) + d_{G_1\vee G_2}(u_p,v_q)\Big)^2 \\ = & \prod_{(u_i,u_p)\in E(G_1)} \Big(\big(d_{G_1}(u_i) + n_2\big) + \big(d_{G_1}(u_p) + n_2\big)\Big)^2 \\ & \prod_{(v_j,v_q)\in E(G_2)} \Big(\big(d_{G_2}(v_j) + n_1\big) + \big(d_{G_2}(v_q) + n_1\big)\Big)^2 \\ & \times \prod_{u_i\in V(G_1)} \prod_{v_j\in V(G_2)} \Big(\big(d_{G_1}(u_i) + n_2\big) + \big(d_{G_2}(v_j) + n_1\big)\Big)^2 \\ = & \prod_{(u_i,u_p)\in E(G_1)} \Big(\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)^2 + 4n_1^2 + 4n_1\big(d_{G_1}(u_i) + d_{G_1}(u_p)\big)\Big)\end{aligned}$$

$$\begin{aligned}& \prod_{(v_j,v_q)\in E(G_2)} \Big(\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)^2 + 4n_2^2 + 4n_2\big(d_{G_2}(v_j) + d_{G_2}(v_q)\big)\Big) \\ & \times \prod_{u_i\in V(G_1)} \prod_{v_j\in V(G_2)} \Big(\big(d_{G_1}^2(u_i) + d_{G_2}^2(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + \big(n_1+n_2\big)^2 \\ & + 2\big(n_1+n_2\big)d_{G_1}(u_i) + 2\big(n_1+n_2\big)d_{G_2}(v_j)\Big).\end{aligned}$$

However, from the inequality (2.1), we get

$$= \left[\frac{\sum_{(u_i,u_p)\in E(G_1)} \Big(\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2 + 4n_1^2 + 4n_1\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big) \Big)}{m_1} \right]^{m_1}$$

$$\times \left[\frac{\sum_{(v_j,v_q)\in E(G_2)} \Big(\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2 + 4n_2^2 + 4n_2\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big) \Big)}{m_2} \right]^{m_2}$$

$$\times \left[\frac{\begin{array}{r}\sum_{u_i\in V(G_1)} \sum_{v_j\in V(G_2)} \Big(\big(d_{G_1}^2(u_i) + d_{G_2}^2(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + \big(n_1+n_2\big)^2 \\ +2\big(n_1+n_2\big)d_{G_1}(u_i) + 2\big(n_1+n_2\big)d_{G_2}(v_j)\Big)\end{array}}{n_1n_2} \right]^{n_1n_2}$$

$$= \frac{1}{m_1^{m_1}} \Big[HM(G_1) + 4n_1^2m_1 + 4n_1M_1(G_1)\Big]^{m_1} \times \frac{1}{m_2^{m_2}} \Big[HM(G_2) + 4n_2^2m_2 + 4n_2M_1(G_2)\Big]^{m_1}$$
$$\times \frac{1}{(n_1n_2)^{n_1n_2}} \Big[n_2M_(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_1n_2(n_1+n_2)^2 + 4m_1(n_1+n_2)$$
$$+ 4m_2(n_1+n_2)\Big]^{n_1n_2}.$$

□

Example 3.13. Let C_p and C_q be cycles with $n \geqslant 3$ vertices. Then

$$\mathtt{HII}(C_p \vee C_q) \leqslant \frac{1}{p^p}\Big[16p + 4p^3 + 16p^2\Big]^p \times \frac{1}{q^q}\Big[16q + 4q^3 + 16q^2\Big]^q$$
$$\times \frac{1}{(pq)^{pq}}\Big[16pq + pq(p+q)^2 + 4p(p+q) + 4q(p+q)\Big]^{pq}.$$

The symmetric difference $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with a vertex set $V(G_1)\times V(G_2)$ in which (u_1i, v_j) is adjacent to (u_k, v_l) whenever u_i is adjacent to u_k in G_1 or v_i is adjacent to v_l in G_2, but not both. The degree of a vertex (u_i, v_j) of $G_1 \oplus G_2$ is given by $d_{G_1\oplus G_2}(u_i, v_j) = n_2d_{G_1}(u_i) + n_1d_{G_2}(v_j) - 2d_{G_1}(u_i)d_{G_2}(v_j)$.

Theorem 3.14. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respec-*

tively. Then

$$
\begin{aligned}
\mathtt{HII}(G_1 \oplus G_2) = {} & \frac{1}{(n_1 m_2)^{n_1 m_2}} \Big[4 m_2 n_2^2 M_1(G_1) + n_1^3 HM(G_2) - 16 M_1(G_1) HM(G_2) \\
& + 8 n_1 n_2 m_1 M_1(G_2) - 16 n_2 M_1(G_1) M_1(G_2) - 16 n_1 m_1 HM(G_2) \Big]^{n_1 m_2} \\
& \times \frac{1}{(n_2 m_1)^{n_2^2 m_1}} \Big[4 n_1^2 m_1 M_1(G_2) + n_2^3 HM(G_1) - 16 M_1(G_2) HM(G_1) \\
& + 8 n_1 n_2 m_2 M_1(G_1) - 16 n_1 M_1(G_2) M_1(G_1) - 16 n_1 m_2 HM(G_1) \Big]^{n_2^2 m_1} \\
& \times \frac{1}{(n_1 n_2)^{n_1 n_2}} \Big[4 n_2^3 M_1(G_1) + 4 n_1^3 M_1(G_2) - 16 M_1(G_1) M_1(G_2) + 32 n_1 n_2 m_1 m_2 \\
& - 32 n_2 m_2 M_1(G_1) - 32 n_1 m_1 M_1(G_2)] \Big]^{n_1 n_2}.
\end{aligned}
$$

Proof. By the definition of the multiplicative Hyper-Zagreb index and from the above partition of the edge set in $G_1 \oplus G_2$, we have

$$
\begin{aligned}
\mathtt{HII}(G_1 \oplus G_2) = {} & \prod_{(u_i,v_j)(u_p,v_q) \in E(G_1 \oplus G_2)} \Big(d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_p, v_q) \Big)^2 \\
= {} & \prod_{u_i \in V(G_1)} \prod_{(v_j,v_q) \in E(G_2)} \Big(\big(n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2 d_{G_1}(u_i) d_{G_2}(v_j) \big) \\
& + \big(n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_q) - 2 d_{G_1}(u_i) d_{G_2}(v_q) \big) \Big)^2 \\
& \times \prod_{(u_i,u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \Big[\Big(\big(n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2 d_{G_1}(u_i) d_{G_2}(v_j) \big) \\
& + \big(n_2 d_{G_1}(u_p) + n_1 d_{G_2}(v_j) - 2 d_{G_1}(u_p) d_{G_2}(v_j) \big) \Big)^2 \Big]^{n_2} \\
& \times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} \Big(\big(n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2 d_{G_1}(u_i) d_{G_2}(v_j) \big) \\
& + \big(n_2 d_{G_1}(u_i) + n_1 d_{G_2}(v_j) - 2 d_{G_1}(u_i) d_{G_2}(v_j) \big) \Big)^2 \\
= {} & \prod_{u_i \in V(G_1)} \prod_{(v_j,v_q) \in E(G_2)} \Big(\big(2 n_2 d_{G_1}(u_i) + n_1 \big(d_{G_2}(v_j) + d_{G_2}(v_q) \big) \big) \\
& - 4 d_{G_1}(u_i) \big((d_{G_2}(v_j) + d_{G_2}(v_q) \big) \Big)^2 \\
& \prod_{(u_i,u_p) \in E(G_1)} \prod_{v_j \in V(G_2)} \Big[\Big(\big(2 n_1 d_{G_2}(v_j) + n_2 \big(d_{G_1}(u_i) + d_{G_1}(u_p) \big) \big) \\
& - 4 d_{G_2}(v_j) \big((d_{G_1}(u_i) + d_{G_1}(u_p) \big) \Big)^2 \Big]^{n_2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}\Big(\big(2n_2d_{G_1}(u_i)+2n_1d_{G_2}(v_j)-4d_{G_1}(u_i)d_{G_2}(v_j)\big)\Big)^2\\
&\times \prod_{(u_i,u_p)\in E(G_1)}\prod_{v_j\in V(G_2)}\bigg[\Big(\big(4n_1^2d_{G_2}^2(v_j)+n_2^2\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\big)\\
&-16d_{G_2}^2(v_j)\big(\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2+4n_1n_2d_{G_1}(v_j)\big(d_{G_1}(u_i)+d_{G_2}(u_p)\big)\\
&-16n_1d_{G_2}^2(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)-8n_2d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\Big)\bigg]^{n_2}\\
&\times \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}\Big(\big(4n_2^2d_{G_1}^2(u_i)+4n_1^2d_{G_2}^2(v_j)-16d_{G_1}^2(u_i)d_{G_2}^2(v_j)\big)\\
&+8n_1n_2d_{G_1}(u_i)d_{G_2}(v_j)-16n_2d_{G_1}^2(u_i)d_{G_2}(v_j)-16n_1d_{G_1}(u_i)d_{G_2}^2(v_j)\Big).
\end{aligned}
$$

However, from the inequality (2.1), we get

$$
\leqslant \left[\frac{\begin{array}{c}\sum_{u_i\in V(G_1)}\sum_{(v_j,v_q)\in E(G_2)}\Big(\big(4n_2^2d_{G_1}^2(u_i)+n_1^2\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2\big)\\
-16d_{G_1}^2(u_i)\big(\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2+4n_1n_2d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)\\
-16n_2d_{G_1}^2(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)-8n_1d_{G_1}(u_i)\big(d_{G_2}(v_j)+d_{G_2}(v_q)\big)^2\Big)\end{array}}{n_1m_2}\right]^{n_1m_2}
$$

$$
\times \left[\frac{\begin{array}{c}\sum_{(u_i,u_p)\in E(G_1)}\sum_{v_j\in V(G_2)}\Big(\big(4n_1^2d_{G_2}^2(v_j)+n_2^2\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\big)\\
-16d_{G_2}^2(v_j)\big(\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2+4n_1n_2d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)\\
-16n_1d_{G_2}^2(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)-8n_2d_{G_2}(v_j)\big(d_{G_1}(u_i)+d_{G_1}(u_p)\big)^2\Big)\end{array}}{n_2m_1}\right]^{n_2^2m_1}
$$

$$
\times \left[\frac{\begin{array}{c}\sum_{u_i\in V(G_1)}\sum_{v_j\in V(G_2)}\Big(\big(4n_2^2d_{G_1}^2(u_i)+4n_1^2d_{G_2}^2(v_j)-16d_{G_1}^2(u_i)d_{G_2}^2(v_j)\big)\\
+8n_1n_2d_{G_1}(u_i)d_{G_2}(v_j)-16n_2d_{G_1}^2(u_i)d_{G_2}(v_j)-16n_1d_{G_1}(u_i)d_{G_2}^2(v_j)\Big)\end{array}}{n_1n_2}\right]^{n_1n_2}
$$

$$\leqslant \frac{1}{(n_1m_2)^{n_1m_2}}\Big[4m_2n_2^2M_1(G_1)+n_1^3HM(G_2)-16M_1(G_1)HM(G_2)+8n_1n_2m_1M_1(G_2)$$
$$-16n_2M_1(G_1)M_1(G_2)-16n_1m_1HM(G_2)\Big]^{n_1m_2}$$
$$\times \frac{1}{(n_2m_1)^{n_2^2m_1}}\Big[4n_1^2m_1M_1(G_2)+n_2^3HM(G_1)-16M_1(G_2)HM(G_1)+8n_1n_2m_2M_1(G_1)$$
$$-16n_1M_1(G_2)M_1(G_1)-16n_2m_2HM(G_1)\Big]^{n_2^2m_1}$$
$$\times \frac{1}{(n_1n_2)^{n_1n_2}}\Big[4n_2^3M_1(G_1)+4n_1^3M_1(G_2)-16M_1(G_1)M_1(G_2)+32n_1n_2m_1m_2$$
$$-32n_2m_2M_1(G_1)-32n_1m_1M_1(G_2)]\Big]^{n_1n_2}.$$

□

Example 3.15. Let C_p and C_q be cycles with $n \geqslant 3$ vertices. Then

$$\begin{aligned}\mathtt{HII}(C_p \oplus C_q) \leqslant\ & \frac{1}{(pq)^{pq}}\Big[16q^3p+16p^3q-1024pq+32p^2q^2-256q^2p-256p^2q\Big]^{pq}\\ &\times \frac{1}{(pq)^{q^2p}}\Big[16p^3q+4q^3p-1024pq)+32p^2q^2-256p^2q-256q^2p\Big]^{q^2p}\\ &\times \frac{1}{(pq)^{pq}}\Big[16q^3p+16p^3q-256pq+32p^2q^2-128q^2p-128p^2q]\Big]^{pq}.\end{aligned}$$

References

[1] B. Basavanagou, S. Patil, A note on Hyper-Zagreb index of graph operations, *Iranian Journal of Mathematical Chemistry*. **7**, 89–92 (2016).

[2] M. Eliasi, A. Iranmanesh, I. Gutman, Multiplicative versions of first Zagreb index,*MATCH Commun. Math. Comput. Chem*. **68**, 217–230 (2012) .

[3] W. Gao, W. Wang, M. R. Farahani, Topological Indices Study of Molecular Structure in Anticancer Drugs, *Hindawi Publishing Corporation Journal of Chemistry,* 3216327, (2016), 8 pages.

[4] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50**, 83–92 (2004).

[5] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett*. **17**, 535–538 (1972).

[6] I. Gutman, Degree–based topological indices, *Croat. Chem. Acta*. **86**, 351–361 (2013).

[7] Y. Huang. B. Liu, M. Zhang, On Comparing the variable Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **63**, 453–460 (2010).

[8] W. Imrich, S. Klavzar, Product Graphs: Structure and Recognition,*John Wiley Sons, New York,* USA, (2000).

[9] B. Liu, Z. You, A survey on comparing Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **65**, 581–593 (2011).

[10] N. Jafari. Rad, A. Jahanbani, I. Gutman, Zagreb Energy and Zagreb Estrada Index of Graphs,*MATCH Commun. Math. Comput. Chem*. **79**, 371–386 (2018).

[11] A Jahanbani, S. Ediz, The sigma index of graph operations, *Sigma Journal of Engineering and Natural Sciences*. **37**, 155–162 (2019).

[12] A. Jahanbani, H. Shooshtary, Nano-Zagreb index and multiplicative Nano-Zagreb index of some graph operations, *INTERNATIONAL JOURNAL OF COMPUTING SCIENCE AND APPLIED MATHEMATICS.* **5**, 15–22 (2019).

[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The Hyper-Wiener index of graph operations, *Comput. Math. Appl*. **56**,1402–1407 (2008).

[14] G. H. Shirdel, H. Rezapour, A. M. Sayadi, The Hyper-Zagreb Index of Graph Operations, *Iranian Journal of Mathematical Chemistry*. **2**, 213– 220 (2013).

[15] V. R. Kulli, multplcative Hyper- Zagreb index and coindices of graphs: computing these indices of some nanostructures, *International Research Journal of Pure Algebra*. **7**, 342–347 (2016).

[16] J. Liu, Q. Zhang, Sharp upper bounds on multiplicative Zagreb indices. *MATCH Commun. Math. Comput. Chem*. **68**, 231–240 (2012).

[17] H. Wang, H. Bao, A note on multiplicative sum Zagreb index, *South Asian J. Math.* **6**, 578–583 (2012).

[18] K. Xu, K.C. Das, K. Tang, On the multiplicative Zagreb coindex of graphs, *Opuscula Math.* **1**, 191–204 (2013).

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