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COLORING OF PRIME GRAPH $PG_1(R)$ and $PG_2(R)$ OF A RING

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Abstract The Prime graph associated to rings denoted by $PG(R)$. $PG_1(R)$ is a graph with all the elements of a ring R as vertices, and two distinct vertices x, y are adjacent if and only if either $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in U(R)$, the set of all units of R and $PG_2(R)$ is a graph with all the elements of a ring R as vertices, and two distinct vertices x, y are adjacent if and only if either $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of R (including zero). In this paper the chromatic number of prime graphs $PG_1(R)$ and $PG_2(R)$ of ring $\mathbb{Z}_n$, where n is power of a prime number, are studied.

1 Introduction

The concept of graph for a commutative ring was began when Beck in [1] introduced the notion of zero divisor of graph. The graph $\Gamma_1(R)$ defined by R. Sen Gupta et al. [7] as: Let R be a ring with unity. Let $G = (V, E)$ be an undirected graph in which $V = R$ - $\{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit. Also, the graph $\Gamma_2(R)$ defined by R. Sen Gupta et al. [7] as: Let R be a ring with unity. Let $G = (V, E)$ be an undirected graph in which $V = R$ - $\{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero). Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [3]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Significant results obtained from above motivated us to introduce new graphs $PG_1(R)$ and $PG_2(R)$ in [9] and [10] respectively. From the results on chromatic number in [8] in this paper we studied the results on chromatic number of $PG_1(R)$ and $PG_2(R)$ of ring $\mathbb{Z}_n$, where n is power of a prime number.

2 Preliminary Definitions

Here we are listing some preliminary definitions of graph theory and Algebra. For more details the reader is referred to [2]-[6].

Definition 2.1. [4] A ring R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:

(i) $(R, +)$ is an abelian group.

(ii) $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in R$.

(iii) The distributive law holds in R : for all $a, b, c \in R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in R$.

Moreover, if a ring R satisfies the condition $a \cdot b = b \cdot a$, for all $a, b \in R$, then we say that R is a commutative ring. If R contains the multiplicative identity i.e. $(1 \cdot a = a \cdot 1 = a, for\ all\ a \in R)$ then we say that R is a ring with identity.

Definition 2.2. [4] Let R be a ring. A non-zero element a of R is called a zero-divisor if there is a non-zero element b in R such that $a \cdot b = 0$ or $b \cdot a = 0$. The set of all zero-divisors in a ring R is denoted by $Z(R)$.

Definition 2.3. [4] The elements which are not zero-divisors are called units. The set of all units in a ring R is denoted by $U(R)$.

Definition 2.4. [8] A graph G consists of a set V called vertex set together with a set E of unordered pairs of distinct elements of V called edge set.

Definition 2.5. A graph that has neither self-loops nor parallel edges is called a simple graph.

Definition 2.6. A simple graph in which every pair of vertices is adjacent is called as a complete graph. Complete graph on n vertices is denoted by K_n.

Definition 2.7. [8] A Graph H is said to be a subgraph of a graph G if all the vertices and edges of H are also the vertices and edges of G.

Definition 2.8. [8] A star graph is a graph with n vertices such that exactly one vertex has degree $n-1$ and the remaining $n-1$ vertices have degree 1.

Definition 2.9. [8] A clique is a complete subgraph of a graph.

Definition 2.10. [8] A coloring of a graph is an assignment of $k-$colors to the vertices of G such that no two adjacent vertices are assigned the same color. The chromatic number $\chi(G)$ of a graph G is the minimum value of k for which G has a $k-$coloring.

Definition 2.11. [7] For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_1(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit.

Definition 2.12. [7] For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_2(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero).

3 The Prime Graph of $PG_1(\mathbb{Z}_n)$

In this section we investigate the chromatic number of $PG_1(\mathbb{Z}_n)$ for some particular values of n.

Theorem 3.1. *For the ring $\mathbb{Z}_p$,*

$$\begin{aligned} \chi PG_1(\mathbb{Z}_p) &= \frac{p+1}{2}, && \textit{if } p \textit{ is odd prime.} \\ &= 2 = p, && \textit{if } p = 2. \end{aligned}$$

Proof. Let a and b be any two elements of $\mathbb{Z}_p - \{0\}$, then a and b are not adjacent if and only if $p|a+b$. In $PG_1(\mathbb{Z}_p - \{0\})$ there are $p-1$ elements which contains two sets of non-adjacent elements. So these non-adjacent elements can be colored with same $\frac{p-1}{2}$ colors.

Also, the vertex 0 is adjacent to all the vertices of $PG_1(\mathbb{Z}_p)$. So, the vertex 0 can be colored with a single color except the color assigned to the $\frac{p-1}{2}$ vertices. Thus, the graph $PG_1(\mathbb{Z}_p)$ can be properly colored with $\frac{p-1}{2} + 1$ equals to $\frac{p+1}{2}$ colors. Therefore, $\chi PG_1(\mathbb{Z}_p) = \frac{p+1}{2}$, if p is an odd prime.

In case when $p = 2$, $\mathbb{Z}_2 = \{0, 1\}$. As the graph $PG_1(\mathbb{Z}_n)$ we are considering a simple graph. So, the graph $PG_1(\mathbb{Z}_2)$ is a star graph and hence $\chi PG_1(\mathbb{Z}_2) = 2 = p$.

□

Theorem 3.2. *For the ring $\mathbb{Z}_{p^2}$,*

$$\begin{aligned} \chi PG_1(\mathbb{Z}_{p^2}) &= \frac{p(p+1)}{2}, && \textit{if } p \textit{ is odd prime.} \\ &= 3 = p+1, && \textit{if } p = 2. \end{aligned}$$

Proof. In $PG_1(\mathbb{Z}_{p^2} - \{0\})$ there are $p-1$ elements which are divisible by p and all are adjacent to each other. So, these elements induces a complete subgraph k_{p-1}. All the vertices of this clique can be colored with $p-1$ colors.

The remaining $p(p-1)$ elements which are not divisible by p, are adjacent to $p-1$ elements form two complete subgraphs having $\frac{p(p-1)}{2}$ elements in each subgraph. So, the vertices in these two complete subgraphs can be properly colored with $\frac{p(p-1)}{2}$ colors except the colors assigned to the vertices of the clique k_{p-1}.

Also, the vertex 0 is adjacent to all the other vertices of $PG_1(\mathbb{Z}_{p^2})$. So, the vertex 0 can be colored with a single color except the colors assigned to the vertices of the cliques k_{p-1} and $k_{\frac{p(p-1)}{2}}$. Thus, the graph $PG_1(\mathbb{Z}_{p^2})$ can be properly colored with $(p-1) + \frac{p(p-1)}{2} + 1$ colors. Therefore, $\chi PG_1(\mathbb{Z}_{p^2}) = \frac{p(p+1)}{2}$, if p is an odd prime.

In case when $p = 2$, $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. The elements 0 and 2 are adjacent to every element in $PG_1(\mathbb{Z}_4)$. So, the vertices 0 and 2 are colored by two different colors. Also, the elements 1 and 3 are adjacent to elements 0 and 2 but not adjacent to each other. So, the vertices 1 and 3 can be colored with a single color except the colors assigned to the vertices 0 and 2. Thus, the graph $PG_1(\mathbb{Z}_4)$ can be properly colored with 3 colors. Hence, $\chi PG_1(\mathbb{Z}_4) = 3 = p + 1$.

□

Theorem 3.3. *For the ring* $\mathbb{Z}_{p^3}$,

$$\begin{aligned} \chi PG_1(\mathbb{Z}_{p^3}) &= \frac{p^3 - p^2 + 2(p+1)}{2}, && \text{if } p \text{ is odd prime.} \\ &= 4 = p + 2, && \text{if } p = 2. \end{aligned}$$

Proof. Any two non-zero elements in $PG_1(\mathbb{Z}_{p^3})$ are adjacent if and only if both these elements are divisible by p^2. There are $p-1$ elements divisible by p^2 and all are adjacent to each other. So, these vertices induces a complete subgraph k_{p-1} and the vertices of this clique can be colored with $p-1$ colors.

There are $p(p-1)$ elements divisible by p but not p^2. These elements are not adjacent to each other but are adjacent to all other elements. So, all these vertices can be colored with a single color.

The remaining $p^2(p-1)$ elements which are not divisible by p and p^2 are adjacent to the elements divisible by p, p^2 and the vertex 0 and form two complete subgraphs having $\frac{p^2(p-1)}{2}$ elements in each subgraph. So, the vertices in these two complete subgraphs can be properly colored with $\frac{p^2(p-1)}{2}$ colors except the colors assigned to the vertices of the clique k_{p-1} and the vertices divisible by p but not p^2.

Also, the vertex 0 is adjacent to all the other vertices of $PG_1(\mathbb{Z}_{p^3})$. So, the vertex 0 can be colored with a single color except the colors assigned to the vertices of the cliques k_{p-1}, $k_{\frac{p^2(p-1)}{2}}$ and the vertices divisible by p but not p^2. Thus, the graph $PG_1(\mathbb{Z}_{p^3})$ can be properly colored with $(p-1) + 1 + \frac{p^2(p-1)}{2} + 1$ colors. Therefore, $\chi PG_1(\mathbb{Z}_{p^3}) = \frac{p^3 - p^2 + 2(p+1)}{2}$, if p is an odd prime.

In case when $p = 2$, $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Here, the elements 0 and 4 are adjacent to all the elements in $PG_1(\mathbb{Z}_8)$. So, the vertices 0 and 4 can be colored with two distinct colors. Also, the elements 2 and 6 are adjacent to all the elements except themselves. So, the vertices 2 and 6 can be colored with a single color except the colors assigned to the vertices 0 and 4. Also, the elements $\{1, 3, 5, 7\}$ are not adjacent to each other but are adjacent to other elements in $PG_1(\mathbb{Z}_8)$. So, the vertices $\{1, 3, 5, 7\}$ are colored with a single color except the colors assigned to the vertices $\{0, 2, 4, 6\}$. Thus, the graph $PG_1(\mathbb{Z}_8)$ can be properly colored with 4 colors. Hence, $\chi PG_1(\mathbb{Z}_8) = 4 = p + 2$.

□

Theorem 3.4. *For the ring $\mathbb{Z}_{p^4}$,*

$$\chi PG_1(\mathbb{Z}_{p^4}) = \frac{p^4 - p^3 + 2p^2}{2}, \quad \text{if } p \text{ is odd prime.}$$
$$= 5 = p + 3, \quad \text{if } p = 2.$$

Proof. Any two non-zero elements of $PG_1(\mathbb{Z}_{p^4})$ are adjacent if and only if both these elements are divisible by p^2 and p^3. There are $p^2 - 1$ elements divisible by p^2 and p^3 and all are adjacent to each other. So, these vertices induces a complete subgraph k_{p^2-1} and the vertices of this clique can be colored with $p^2 - 1$ colors.

There are $p^2(p-1)$ elements which are divisible by p but not p^2 and p^3. These elements are not adjacent to each other but are adjacent to the elements divisible by p^3 and the elements in a set of units. So, the vertices divisible by p can be colored with any one color assigned to the vertices divisible by p^2.

The remaining $p^3(p-1)$ elements which are not divisible by p, p^2 and p^3 are adjacent to all the elements in a set of non-zero zero divisors and form two complete subgraphs having $\frac{p^3(p-1)}{2}$ elements in each subgraph. So, the vertices in these two complete subgraphs can be properly colored with $\frac{p^3(p-1)}{2}$ colors except the colors assigned to the vertices of the clique k_{p^2-1}.

Also, the vertex 0 is adjacent to all other vertices in $PG_1(\mathbb{Z}_{p^4})$. So, the vertex 0 can be colored with a single color except the colors assigned to the vertices of the cliques cliques k_{p^2-1}, $k_{\frac{p^3(p-1)}{2}}$. Thus, the graph $PG_1(\mathbb{Z}_{p^4})$ can be properly colored with $(p^2 - 1) + \frac{p^3(p-1)}{2} + 1$ colors. Therefore, $\chi PG_1(\mathbb{Z}_{p^4}) = \frac{p^4-p^3+2p^2}{2}$, if p is an odd prime.

In case when $p = 2$, $\mathbb{Z}_{16} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Here, the elements $\{0, 4, 8, 12\}$ forms a complete subgraph k_4 and the vertices of this clique can be colored with 4 colors. The elements $\{2, 6, 10, 14\}$ are not adjacent to each other and the elements 4 and 12. So, these vertices can be colored with any one color assigned to the vertices 4 and 12. The elements $\{1, 3, 5, 7, 9, 11, 13, 15\}$ are not adjacent to each other but are adjacent to all the elements in a set $\{0, 2, 4, 6, 8, 10, 12, 14\}$. So, these vertices can be colored with a single color except the colors assigned to the vertices of the clique k_4. Thus, the graph $PG_1(\mathbb{Z}_{16})$ can be properly colored by 5 colors. Hence $\chi PG_1(\mathbb{Z}_{16}) = 5 = p + 3$.

□

Theorem 3.5. *For the ring $\mathbb{Z}_{p^5}$,*

$$\chi PG_1(\mathbb{Z}_{p^5}) = \frac{p^5 - p^4 + 2(p^2 + 1)}{2}, \quad \text{if } p \text{ is odd prime.}$$
$$= 6 = p + 4, \quad \text{if } p = 2.$$

Proof. Any two non-zero elements of $PG_1(\mathbb{Z}_{p^5})$ are adjacent if and only if both these elements are divisible by p^4 and p^3. There are $p^2 - 1$ elements divisible by p^4 and p^3 and all are adjacent to each other. So, these vertices induces a complete subgraph k_{p^2-1} and the vertices of this clique can be colored with $p^2 - 1$ colors.

There are $p^2(p-1)$ elements which are divisible by p^2. These elements are not adjacent to each other and the elements divisible by p but are adjacent to the elements divisible by p^4 and p^3. So, these vertices can be colored with a single color except the colors assigned to the vertices divisible by p^4 and p^3.

There are $p^3(p-1)$ elements which are divisible by p. These elements are not adjacent to each other and the elements divisible by p^3 and p^2, but are adjacent to every element from the set of units in a graph. So, these vertices can be colored with any one color assigned to the vertices divisible by p^3 and p^2.

The remaining $p^4(p-1)$ elements which are not divisible by p, p^2, p^3 and p^4 are adjacent to all the elements in a set of non-zero zero divisors and form two complete subgraphs having $\frac{p^4(p-1)}{2}$ elements in each subgraph. So, the vertices in these two complete subgraphs can be properly colored with $\frac{p^4(p-1)}{2}$ colors except the colors assigned to the vertices of the clique k_{p^2-1} and the vertices divisible by p^2.

Also, the vertex 0 is adjacent to all other vertices in $PG_1(\mathbb{Z}_{p^5})$. So, the vertex 0 can be colored with a single color except the colors assigned to the vertices of the cliques k_{p^2-1}, $k_{\frac{p^4(p-1)}{2}}$ and the vertices divisible by p^2. Thus, the graph $PG_1(\mathbb{Z}_{p^5})$ can be properly colored with $(p^2-1)+1+\frac{p^4(p-1)}{2}+1$. Therefore, $\chi PG_1(\mathbb{Z}_{p^5}) = \frac{p^5-p^4+2(p^2+1)}{2}$, if p is an odd prime.

In case when $p=2$, In $PG_1(\mathbb{Z}_{32})$ the elements $\{0,8,16,24\}$ form a complete subgraph k_4. So, the vertices of this clique can be colored with 4 colors. The elements $\{4,12,20,28\}$ are not adjacent to each other but are adjacent to all the elements in a set $\{0,8,16,24\}$. So, these vertices can be colored with a single color except the colors assigned to a clique k_4. The elements $\{2,6,10,14,18,22,26,30\}$ are not adjacent to each other and the elements divisible by 2^3 and 2^2. So, these vertices can be colored by any one color assigned to the vertices divisible by 2^3 and 2^2. Also, The elements in a set of units are not adjacent to each other but are adjacent to all the elements in a set of zero-divisors. So, these elements can be colored by a single color except the colors assigned to the clique k_4 and the elements divisible by 2^2. Thus, the graph $PG_1(\mathbb{Z}_{32})$ can be properly colored with 6 colors. Hence $\chi PG_1(\mathbb{Z}_{32}) = 6 = p+4$.

□

Theorem 3.6. *From the above discussion we conclude that*
For the ring $\mathbb{Z}_p$,

$$\chi PG_1(\mathbb{Z}_p) = \frac{p+1}{2}, \qquad \text{if } p \text{ is odd prime.}$$

For the ring $\mathbb{Z}_{p^{2n}}$,

$$\chi PG_1(\mathbb{Z}_{p^{2n}}) = \frac{p^{2n}-p^{2n-1}+2p^n}{2}, \qquad \text{if } p \text{ is odd prime and } n = 1,2,3,...$$

For the ring $\mathbb{Z}_{p^{2n+1}}$,

$$\chi PG_1(\mathbb{Z}_{p^{2n+1}}) = \frac{p^{2n+1}-p^{2n}+2(p^n+1)}{2}, \qquad \text{if } p \text{ is odd prime and } n = 1,2,3,...$$

For the ring $\mathbb{Z}_{p^n}$,

$$\chi PG_1(\mathbb{Z}_{p^n}) = p+(n-1), \qquad \text{if } p = 2 \text{ and } n = 1,2,3,...$$

4 The Prime Graph of $PG_2(\mathbb{Z}_n)$

In this section we investigate the chromatic number of $PG_2(\mathbb{Z}_n)$ for some particular values of n.

Theorem 4.1. *[10] $PG_2(\mathbb{Z}_{2^r})$, where $r \in \mathbb{N}-\{1\}$, has two components consisting of zero divisors and units of $(\mathbb{Z}_{2^r})$ respectively. The first is $k_{2^{r-1}}$ consists of all zero divisors and the other is $k_{2^{r-1}+1}$ consists of all the units and the element zero.*

Theorem 4.2. *[10] Let $n = p^r$, where p is an odd prime and $r \in \mathbb{N}-\{1\}$ then $PG_2(\mathbb{Z}_n)$ has $(p+1)/2$ components, one is $k_{p^{r-1}}$ consisting of the zero divisors and $(p-1)/2$ copies of $k_{p^{r-1},p^{r-1}} \bigcup \{0\}$ for the units and the element zero.*

Theorem 4.3. *[10] Let F be a finite field with $|F| = p^n, p \geq 3$ for some prime p and $n \in \mathbb{N}$, then $PG_2(F)$ is a union of $(p^n-1)/2$ copies of k_3 in which the element zero is adjacent to all the vertices.*

Theorem 4.4. *For the ring $\mathbb{Z}_p$,*

$$\chi PG_2(\mathbb{Z}_p) = 3, \quad \text{if } p \text{ is odd prime.}$$
$$= 2 = p, \quad \text{if } p = 2.$$

Proof. The graph $PG_2(\mathbb{Z}_p)$ is a union of $\frac{p-1}{2}$ copies of k_3 in which the element zero is adjacent to all the vertices. Hence, $\chi PG_2(\mathbb{Z}_p) = 3$, if p is odd prime. In case when $p = 2$, $\mathbb{Z}_2 = \{0, 1\}$. $PG_2(\mathbb{Z}_2)$ is a star graph hence, $\chi PG_2(\mathbb{Z}_2) = 2 = p$. □

Theorem 4.5. *For the ring $\mathbb{Z}_{p^2}$,*

$$\chi PG_2(\mathbb{Z}_{p^2}) = p, \quad \text{if } p \text{ is odd prime.}$$
$$= 3 = p + 1, \quad \text{if } p = 2.$$

Proof. The graph $PG_2(\mathbb{Z}_{p^2})$ has $\frac{p+1}{2}$ components. One is k_p consisting of all the zero divisors and an element zero. So, the vertices of this complete subgraph can be colored with p colors. Second component contains $\frac{p-1}{2}$ copies of $k_{p,p} \bigcup \{0\}$ consisting of all the units and the element zero. So, the vertices of this complete bipartite subgraph can be colored with any two colors assigned to the vertices of the clique k_p except the color assigned to the vertex 0. Hence, $PG_2(\mathbb{Z}_{p^2})$ can be properly colored with p colors. Therefore, $\chi PG_2(\mathbb{Z}_{p^2}) = p$, if p is odd prime.

In case when $p = 2$, $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Here, the elements $\{0, 1, 3\}$ form a clique k_3. So, the vertices of this clique can be colored with 3 colors. An element 2 is adjacent to the vertex 0 only. So, the vertex 2 can be colored with any one color assigned to the vertices of the clique k_3 except the vertex 0. Thus, $\chi PG_2(\mathbb{Z}_4) = 3 = p + 1$.

□

Theorem 4.6. *For the ring $\mathbb{Z}_{p^3}$,*

$$\chi PG_2(\mathbb{Z}_{p^3}) = p^2, \quad \text{if } p \text{ is odd prime.}$$
$$= 5 = p^2 + 1, \quad \text{if } p = 2.$$

Proof. The graph $PG_2(\mathbb{Z}_{p^3})$ has $\frac{p+1}{2}$ components. One is k_{p^2} consisting of all the zero divisors and an element zero. So, the vertices of this complete subgraph can be colored with p^2 colors. Second component contains $\frac{p-1}{2}$ copies of $k_{p^2,p^2} \bigcup \{0\}$ consisting of all the units and the element zero. So, the vertices of this complete bipartite subgraph can be colored with any two colors assigned to the vertices of the clique k_{p^2} except the color assigned to the vertex 0. Hence, $PG_2(\mathbb{Z}_{p^3})$ can be properly colored with p^2 colors. Therefore, $\chi PG_2(\mathbb{Z}_{p^3}) = p^2$, if p is odd prime.

In case when $p = 2$, $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Here, the elements $\{0, 1, 3, 5, 7\}$ form a clique k_5. So, all the vertices of this clique can be colored with 5 colors. The elements $\{2, 4, 6\}$ are adjacent to each other and form a clique k_3 but are not adjacent to the elements in a clique k_5 except the vertex zero. So, the vertices of this clique can be colored with any three colors assigned to the vertices of the clique k_5 except the vertex 0. Thus, $\chi PG_2(\mathbb{Z}_8) = 5 = p^2 + 1$.

□

Theorem 4.7. *For the ring $\mathbb{Z}_{p^4}$,*

$$\chi PG_2(\mathbb{Z}_{p^4}) = p^3, \quad \text{if } p \text{ is odd prime.}$$
$$= p^3 + 1, \quad \text{if } p = 2.$$

Proof. The graph $PG_2(\mathbb{Z}_{p^4})$ has $\frac{p+1}{2}$ components. One is k_{p^3} consisting of all the zero divisors and an element zero. So, the vertices of this complete subgraph can be colored with p^3 colors.

Second component contains $\frac{p-1}{2}$ copies of $k_{p^3,p^3}\bigcup\{0\}$ consisting of all the units and the element zero. So, the vertices of this complete bipartite subgraph can be colored with any two colors assigned to the vertices of the clique k_{p^3} except the color assigned to the vertex 0. Hence, $PG_2(\mathbb{Z}_{p^4})$ can be properly colored with p^3 colors. Therefore, $\chi PG_2(\mathbb{Z}_{p^4}) = p^3$, if p is odd prime.

In case when $p = 2$, $\mathbb{Z}_{16} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$. Here, the elements $\{0,1,3,5,7,9,11,13,15\}$ form a clique k_9. So, all the vertices of this clique can be colored with 9 colors. The elements $\{2,4,6,8,10,12,14\}$ are adjacent to each other and form a clique k_7 but are not adjacent to the elements in a clique k_9 except the vertex zero. So, the vertices of this clique can be colored with any seven colors assigned to the vertices of the clique k_9 except the vertex 0. Thus, $\chi PG_2(\mathbb{Z}_{16}) = 9 = p^3 + 1$.

□

Theorem 4.8. *For the ring $\mathbb{Z}_{p^5}$,*

$$\chi PG_2(\mathbb{Z}_{p^5}) = p^4, \quad \text{if } p \text{ is odd prime.}$$
$$= p^4 + 1, \quad \text{if } p = 2.$$

Proof. The graph $PG_2(\mathbb{Z}_{p^5})$ has $\frac{p+1}{2}$ components. One is k_{p^4} consisting of all the zero divisors and an element zero. So, the vertices of this complete subgraph can be colored with p^4 colors. Second component contains $\frac{p-1}{2}$ copies of $k_{p^4,p^4}\bigcup\{0\}$ consisting of all the units and the element zero. So, the vertices of this complete bipartite subgraph can be colored with any two colors assigned to the vertices of the clique k_{p^4} except the color assigned to the vertex 0. Hence, $PG_2(\mathbb{Z}_{p^5})$ can be properly colored with p^4 colors. Therefore, $\chi PG_2(\mathbb{Z}_{p^5}) = p^4$, if p is odd prime.

In case when $p = 2$, In $\mathbb{Z}_{32}$ the elements $\{0,1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31\}$ form a clique k_{17}. So, all the vertices of this clique can be colored with 17 colors. The elements $\{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30\}$ are adjacent to each other and form a clique k_{15} but are not adjacent to the elements in a clique k_{17} except the vertex zero. So, the vertices of this clique can be colored with any fifteen colors assigned to the vertices of the clique k_{17} except the vertex 0. Thus, $\chi PG_2(\mathbb{Z}_{32}) = 17 = p^4 + 1$.

□

Theorem 4.9. *From the above discussion we conclude that,For the ring $\mathbb{Z}_{p^n}$, where $n \geq 2$*

$$\chi PG_2(\mathbb{Z}_{p^n}) = p^{n-1}, \quad \text{if } p \text{ is odd prime.}$$
$$= p^{n-1} + 1, \quad \text{if } p = 2.$$

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